Math 271: Mathematical Methods Tridip Ray
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**Homework 2 (Class Test on 04 September)**

1. Consider a system of $m$ simultaneous linear equations in $n$ unknowns, $Ax = c$, where

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \; x_{n\times 1} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; c_{m\times 1} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}.$$

   Prove that the system of equations will have a solution for every choice of right-hand side $(c_1, c_2, ..., c_m)$ if and only if

$$\text{rank } (A) = \text{ number of rows of } A.$$

2. For the same system of $m$ simultaneous linear equations in $n$ unknowns, $Ax = c$, as in Problem 1 above, suppose that the number of equations $<$ the number of unknowns. Prove that

   (a) $Ax = 0$ has infinitely many solutions;

   (b) for any given $c$, $Ax = c$ has 0 or infinitely many solutions;

   (c) if *rank* $(A) =$ number of equations, $Ax = c$ has infinitely many solutions for every choice of right-hand side $(c_1, c_2, ..., c_m)$.

3. Consider a system of $m$ simultaneous linear equations in $n$ unknowns, $Ax = c$, as in the above two problems.

   (a) Prove that the system of equations must have either no solution, one solution, or infinitely many solutions.

   (b) Prove that the system of equations will have at most one solution for every choice of right-hand side $(c_1, c_2, ..., c_m)$ if and only if

$$\text{rank } (A) = \text{ number of columns of } A.$$

(c) Prove that the system of equations has one and only one solution for every choice of right-hand side $(c_1, c_2, ..., c_m)$ if and only if

$$\text{number of rows of } A = \text{ number of columns of } A = \text{ rank } (A)\,.$$

- A general linear model will have $m$ equations in $n$ variables:

$$\begin{array}{ccc} a_{11}x_1 + a_{12}x_2 + & \cdots & +a_{1n}x_n = c_1, \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + & \cdots & +a_{mn}x_n = c_m. \end{array} \tag{1}$$

The variables whose values are determined by the system of equations (1) are called *endogenous variables*. On the other hand, the variables whose values are determined outside of system (1) are called *exogenous variables*. The division of the $n$ variables into endogenous and exogenous variables will be successful only if, after choosing values for the exogenous variables and plugging them into system (1), one can then uniquely solve the system for the endogenous variables.

(d) Let $x_1, x_2, ..., x_k$ and $x_{k+1}, x_{k+2}, ..., x_n$ be a partition of the $n$ variables in (1) into endogenous and exogenous variables, respectively. Provide, with clear explanations, the necessary and sufficient conditions so that there is, for each choice of values $x^0_{k+1}, x^0_{k+2}, ..., x^0_n$ for the exogenous variables, a *unique* set of values $x^0_1, x^0_2, ..., x^0_k$ for the endogenous variables which solves (1).

4. Let $A$ be an $m \times n$ matrix, and let $c$ be a vector in $\mathbb{R}^m$. Prove that exactly one of the following two alternatives holds.

   Either the system of equations $Ax = c$ has a solution,

   or, the system of equations $yA = 0$ and $yc = 1$ has a solution.

5. Consider the matrix

$$A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}.$$

   (a) Find the eigenvalues and the normalized eigenvectors of $A$.

   (b) Use this example to verify the Spectral Decomposition Theorem, that is, matrix $A$ can be decomposed into a matrix $L$ consisting of its eigenvalues on the diagonal and the matrices $B$ and $B^T$ consisting of its eigenvectors.

6. We have so far studied the Spectral Decomposition Theorem for symmetric matrices with *distinct* eigenvalues. The theorem can be extended to accommodate *non-distinct* eigenvalues as follows.

   - Let $A$ be an $n \times n$ symmetric matrix with eigenvalues, $\lambda_1, \lambda_2, ..., \lambda_n$. In this listing, an eigenvalue is repeated a number of times equal to its multiplicity. Thus, if one eigenvalue has multiplicity $k$, there will be $k$ eigenvalues with the same numerical value.
   - **Theorem (Spectral Decomposition):**
     *Let $A$ be an $n \times n$ symmetric matrix with eigenvalues, $\lambda_1, \lambda_2, ..., \lambda_n$. Even if $A$ has multiple eigenvalues, there exists a nonsingular matrix $B$ whose columns $y^1, y^2, ..., y^n$ are eigenvectors of $A$ such that*
     (i) *$y^1, y^2, ..., y^n$ are mutually orthogonal to each other,*
     (ii) *$B^{-1} = B^T$, and*
     (iii) *$B^T AB = L$, where $L$ is the diagonal matrix with the eigenvalues of $A$ $(\lambda_1, \lambda_2, ..., \lambda_n)$ on its diagonal.*

   - **Question:**

     Let $A$ be an $n \times n$ symmetric matrix. Show that if 0 is an eigenvalue of $A$ of multiplicity $k$, then rank $(A) = n - k$.

7. Let $A$ be a *symmetric* matrix. Prove that $A$ is *positive definite* if and only if there exists a *nonsingular* matrix $B$ so that $A = B^T B$.

8. (a) For any two matrices, $A$ of order $m \times n$ and $B$ of order $n \times m$, prove that *trace* $(AB) =$ *trace* $(BA)$.
   (b) Suppose $A$ is an $n \times n$ real symmetric matrix.
      (i) Use the part (a) above and Spectral Decomposition Theorem to prove that the sum of all the eigenvalues of $A$ is equal to the *trace* of $A$.
      (ii) Use Spectral Decomposition Theorem to prove that the product of all the eigenvalues of $A$ is equal to the *determinant* of $A$.
      (iii) Prove that the rank of $A$ is equal to the number of nonzero eigenvalues.
      (iv) Prove that the eigenvalues of $A^2$ (defined as $A^2 \equiv AA$), are the squares of the eigenvalues of $A$, but the eigenvectors of both matrices are the same.

(v) A matrix $A$ is called an *idempotent* matrix if $A = A^2 = A^3 = ...$, that is, multiplying $A$ by itself, however many times, simply reproduces the original matrix.

- Prove that each eigenvalue of an idempotent matrix is either 0 or 1.
- Prove that the rank of an idempotent matrix is equal to the sum of its diagonal elements.

9. In this problem you will prove, using the method of induction, the following theorem:

*Let $\lambda_1$, $\lambda_2$, ..., $\lambda_k$ be $k$* ***distinct*** *eigenvalues of the $n \times n$ matrix $A$. Let $x^1$, $x^2$, ..., $x^k$ be the corresponding eigenvectors. Then, $x^1$, $x^2$, ..., $x^k$ are linearly independent vectors.* [Note that $A$ may not be a symmetric matrix.]

(a) Initial step: Prove that the theorem is true for $k = 2$.

(b) Inductive step: Define the inductive step carefully and then prove it to complete the proof of the theorem.