Homework 2 (Class Test on 02 September)

1. Consider a system of m simultaneous linear equations in n unknowns, Ax = c, where

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \ x_{n \times 1} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ c_{m \times 1} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

Prove that the system of equations will have a solution for every choice of right-hand side $(c_1, c_2, ..., c_m)$ if and only if

rank
$$(A) =$$
 number of rows of A.

- 2. For the same system of m simultaneous linear equations in n unknowns, Ax = c, as in Problem 1 above, suppose that the number of equations < the number of unknowns. Prove that
 - (a) Ax = 0 has infinitely many solutions;
 - (b) for any given c, Ax = c has 0 or infinitely many solutions;
 - (c) if rank (A) = number of equations, Ax = c has infinitely many solutions for every choice of right-hand side $(c_1, c_2, ..., c_m)$.
- 3. Consider a system of m simultaneous linear equations in n unknowns, Ax = c, as in the above two problems.
 - (a) Prove that the system of equations must have either no solution, one solution, or infinitely many solutions.
 - (b) Prove that the system of equations will have at most one solution for every choice of right-hand side $(c_1, c_2, ..., c_m)$ if and only if

rank (A) = number of columns of A.

(c) Prove that the system of equations has one and only one solution for every choice of right-hand side $(c_1, c_2, ..., c_m)$ if and only if

number of rows of A = number of columns of A = rank (A).

• A general linear model will have m equations in n variables:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = c_1,$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = c_m.$$
(1)

The variables whose values are determined by the system of equations (1) are called endogenous variables. On the other hand, the variables whose values are determined outside of system (1) are called exogenous variables. The division of the *n* variables into endogenous and exogenous variables will be successful only if, after choosing values for the exogenous variables and plugging them into system (1), one can then uniquely solve the system for the endogenous variables.

- (d) Let $x_1, x_2, ..., x_k$ and $x_{k+1}, x_{k+2}, ..., x_n$ be a partition of the *n* variables in (1) into endogenous and exogenous variables, respectively. Provide, with clear explanations, the necessary and sufficient conditions so that there is, for each choice of values $x_{k+1}^0, x_{k+2}^0, ..., x_n^0$ for the exogenous variables, a *unique* set of values $x_1^0, x_2^0, ..., x_k^0$ for the endogenous variables which solves (1).
- 4. Let A be an $m \times n$ matrix, and let c be a vector in \mathbb{R}^m . Prove that exactly one of the following two alternatives holds.

Either the system of equations Ax = c has a solution,

or, the system of equations yA = 0 and yc = 1 has a solution.

5. Consider the matrix

$$A = \left(\begin{array}{cc} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{array}\right)$$

- (a) Find the eigenvalues and the normalized eigenvectors of A.
- (b) Use this example to verify the Spectral Decomposition Theorem, that is, matrix A can be decomposed into a matrix L consisting of its eigenvalues on the diagonal and the matrices B and B^T consisting of its eigenvectors.

- 6. We have so far studied the Spectral Decomposition Theorem for symmetric matrices with *distinct* eigenvalues. The theorem can be extended to accommodate *non-distinct* eigenvalues as follows.
 - Let A be an $n \times n$ symmetric matrix with eigenvalues, $\lambda_1, \lambda_2, ..., \lambda_n$. In this listing, an eigenvalue is repeated a number of times equal to its multiplicity. Thus, if one eigenvalue has multiplicity k, there will be k eigenvalues with the same numerical value.

• Theorem (Spectral Decomposition):

Let A be an $n \times n$ symmetric matrix with eigenvalues, $\lambda_1, \lambda_2, ..., \lambda_n$. Even if A has multiple eigenvalues, there exists a nonsingular matrix B whose columns $y^1, y^2, ..., y^n$ are eigenvectors of A such that

- (i) $y^1, y^2, ..., y^n$ are mutually orthogonal to each other,
- (ii) $B^{-1} = B^T$, and
- (iii) $B^T A B = L$, where L is the diagonal matrix with the eigenvalues of A $(\lambda_1, \lambda_2, ..., \lambda_n)$ on its diagonal.

• Question:

Let A be an $n \times n$ symmetric matrix. Show that if 0 is an eigenvalue of A of multiplicity k, then rank (A) = n - k.

7. In this problem you will prove, using the method of induction, the following theorem:

Let $\lambda_1, \lambda_2, ..., \lambda_k$ be k **distinct** eigenvalues of the $n \times n$ matrix A. Let $x^1, x^2, ..., x^k$ be the corresponding eigenvectors. Then, $x^1, x^2, ..., x^k$ are linearly independent vectors. [Note that A may not be a symmetric matrix.]

- (a) Initial step: Prove that the theorem is true for k = 2.
- (b) Inductive step: Define the inductive step carefully and then prove it to complete the proof of the theorem.
- 8. Theorem: Let A be a symmetric $n \times n$ matrix. A is negative definite if and only if all its n leading principal minors alternate in sign, starting with negative. (That is, the r-th leading principal minor, A_r , r = 1, 2, ..., n, has the same sign as $(-1)^r$.)

In the following steps we will prove this theorem. The proof has two major ingredi-

ents: the principal of induction and the theory of partitioned matrices. First a brief introduction to the theory of partitioned matrices.

• Partitioned Matrices: Let A be a $m \times n$ matrix. A submatrix of A is a matrix formed by discarding some entire rows and/or columns of A. A partitioned matrix is a matrix which has been partitioned into submatrices by horizontal and/or vertical lines which extend along entire rows or columns of A. For example,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{pmatrix}$$

which we can write as

$$A = \left(\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array} \right).$$

Each submatrix A_{ij} is called a *block* of A.

Suppose that A and B are two $m \times n$ matrices which are partitioned in the same way, that is,

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{pmatrix} \text{ and } B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ \hline B_{21} & B_{22} & B_{23} \end{pmatrix}$$

where A_{11} and B_{11} have the same dimensions, A_{12} and B_{12} have the same dimensions, and so on. Then A and B can be added as if the blocks are scalar entries:

$$A + B = \left(\begin{array}{c|c|c} A_{11} + B_{11} & A_{12} + B_{12} & A_{13} + B_{13} \\ \hline A_{21} + B_{21} & A_{22} + B_{22} & A_{23} + B_{23} \end{array}\right).$$

Similarly, two partitioned matrices A and C can be multiplied, treating the blocks as scalars, if the blocks are all of a size such that the matrix multiplication of blocks can be done. For example, if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ \hline B_{21} & B_{22} & B_{23} \end{pmatrix},$$

then

$$AB = \left(\begin{array}{c|c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} & A_{11}B_{13} + A_{12}B_{23} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{23} \end{array}\right)$$

so long as the various matrix products $A_{ij}B_{jk}$ can be formed. For example, A_{11} must have as many columns as B_{11} has rows, and so on.

- Next we need two lemmas.
 - Lemma 1: If A is a positive or negative definite matrix, then A is nonsingular.
 - Lemma 2: Suppose that A is a symmetric matrix and that Q is a nonsingular matrix. Then, Q^TAQ is a symmetric matrix, and A is positive (negative) definite if and only if Q^TAQ is positive (negative) definite.
- Now we proceed to prove the theorem by using induction on the size n of A. The result
 is trivially true for 1 × 1 matrices. It is straightforward to verify the theorem (you do
 not have to do it) directly for 2 × 2 symmetric matrices by completing the square in
 the corresponding quadratic form on R²:

$$f(x_1, x_2) = (x_1, x_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2.$$

So we will suppose that the theorem is true for $n \times n$ matrices and prove it to be true for $(n+1) \times (n+1)$ matrices.

- Let A be an $(n + 1) \times (n + 1)$ symmetric matrix. Write A_j for the $j \times j$ leading principal submatrix of A for j = 1, 2, ..., n + 1. By the inductive hypothesis the theorem is true for $n \times n$ matrices. In part (a) we will prove that if sign of $|A_r|$ is the same as $(-1)^r$, r = 1, 2, ..., n + 1, then A is negative definite. In part (b) we will prove the converse: A is negative definite implies that sign of $|A_r|$ is the same as $(-1)^r$, r = 1, 2, ..., n + 1, then A is negative definite. In part (b) we will prove the converse:
- (a) The inductive hypothesis is given, and assume that sign of $|A_r|$ is the same as $(-1)^r$, r = 1, 2, ..., n + 1.
 - (i) Argue that A_n is invertible.
 - Partition A as

$$A = \left(\begin{array}{c|c} A_n & a \\ \hline a^T & a_{n+1,n+1} \end{array}\right), \text{ where } a = \left(\begin{array}{c|c} a_{1,n+1} \\ \vdots \\ a_{n,n+1} \end{array}\right)$$

Let $d = a_{n+1,n+1} - a^T (A_n)^{-1} a$, let I_n denote the $n \times n$ identity matrix, and let 0_n denote the $n \times 1$ column vector of all 0s.

(ii) Verify that

$$A = \left(\frac{I_n}{\left(A_n^{-1}a\right)^T} \begin{vmatrix} 0_n \\ 1 \end{vmatrix}\right) \left(\frac{A_n}{0_n^T} \begin{vmatrix} 0_n \\ d \end{vmatrix}\right) \left(\frac{I_n}{0_n^T} \begin{vmatrix} A_n^{-1}a \\ 0_n^T \end{vmatrix}\right) \equiv Q^T B Q.$$

- (iii) Show that $|A| = d \cdot |A_n|$, and argue that d < 0.
- (iv) Let X be an arbitrary (n+1)-vector. Write $X = \begin{pmatrix} x \\ x_{n+1} \end{pmatrix}$, where x is an *n*-vector. Argue that $X^T B X = x^T A_n x + d \cdot (x_{n+1})^2 < 0.$
- (v) Conclude that A is negative definite.
- (b) The inductive hypothesis is given, and assume that A is negative definite. (Note that A is $(n+1) \times (n+1)$).
 - (i) Prove that A_n is negative definite.
 - (ii) Argue that sign of $|A_r|$ is the same as $(-1)^r$, r = 1, 2, ..., n.
 - So we need to prove only that the sign of determinant of A itself is $(-1)^{n+1}$.
 - Since A_n is invertible, we can once again write A as $Q^T B Q$ as in part (a) (ii) and conclude that $|A| = d \cdot |A_n|$ still holds.
 - (iii) Argue that B is negative definite.
 - (iv) Choose X suitably in part (a) (iv) to show that d < 0.
 - (v) Conclude that the sign of |A| is $(-1)^{n+1}$, that is, sign of $|A_r|$ is the same as $(-1)^r$, r = 1, 2, ..., n+1.