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Homework 3 (Class Test on 23 September)

1. Suppose that $\{x_n\}$ is a sequence of real numbers that converges to x_0 and that all x_n and x_0 are nonzero.

   (a) Prove that there is a positive number B such that $|x_n| \geq B$ for all n.

   (b) Using (a), prove that $\left\{\frac{1}{x_n}\right\}$ converges to $\left\{\frac{1}{x_0}\right\}$.

2. Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences of real numbers with limits x and y, respectively. Suppose that all y_n and y are nonzero. Show that the sequence $\left\{\frac{x_n}{y_n}\right\}$ converges to $\left\{\frac{x}{y}\right\}$.

3. A sequence of real numbers $\{x_n\}$ is *increasing* if $x_{n+1} \geq x_n$ for all n. It is *bounded above* if there exists $B \in \mathbb{R}$ such that $x_n \leq B$ for all n. Prove that if $\{x_n\}$ is increasing and bounded above, then it converges to its *smallest upper bound*.

4. **Cauchy Sequence:** A sequence of real numbers $\{x_n\}$ is a Cauchy sequence if for any $\epsilon > 0$ there is an integer N such that for all $i, j \geq N$, $|x_i - x_j| < \epsilon$.

   (a) Prove that any convergent sequence of real numbers is a Cauchy sequence.

   (b) Suppose $\{y_n\}$ is a Cauchy sequence of real numbers. Prove that there exists a real number $B > 0$ such that $|y_n| \leq B$, for all n.

   (c) Prove that if a Cauchy sequence $\{x_n\}$ has a subsequence converging to y, then the whole sequence converges to y.

5. Show that a convergent sequence in $\mathbb{R}^n$ can have only one accumulation point, and therefore only one limit.

6. For each of the following subsets in $\mathbb{R}^2$, draw the set, state whether it is open, closed, or neither, and justify your answer briefly:

   (a) $\{(x, y) \in \Re^2 : -1 < x < 1,\ y = 0\}$;

   (b) $\{(x, y) \in \Re^2 : x \text{ and } y \text{ are integers}\}$;

   (c) $\{(x, y) \in \Re^2 : x + y = 1\}$;

   (d) $\{(x, y) \in \Re^2 : x + y < 1\}$;

   (e) $\{(x, y) \in \Re^2 : x = 0 \text{ or } y = 0\}$.

7. Show that if $x \in \mathit{int}\ S$ and $\{x_n\}$ is a sequence converging to x, then $x_n \in S$ for all sufficiently large n.

8. (a) Prove that any finite set of real numbers is a closed set.

   (b) Prove that the set of integers is a closed set.

9. Consider the set in $\mathbb{R}$ consisting of all numbers of the form $\frac{1}{n}$, where $n = 1, 2, 3, \ldots$. Explain clearly whether this set is closed or open or neither.