Homework 4 (Class Test on 30 September)

1. In this question we will prove the following version of the Bolzano-Weierstrass Theorem:

Any sequence contained in a compact interval of \mathbb{R} , [a, b], has a convergent subsequence whose limit lies in [a, b].

- Let $\{x_n\}_{n=1}^{\infty}$ be a sequence contained in [a, b]. Divide [a, b] into two equal halves: $\begin{bmatrix} a, \frac{a+b}{2} \end{bmatrix}$ and $\begin{bmatrix} \frac{a+b}{2}, b \end{bmatrix}$.
- (a) Argue that infinitely many elements of the sequence $\{x_n\}_{n=1}^{\infty}$ must lie in one (or both) of these halves.
 - Let I_1 denote a half which contains infinitely many members of the sequence. Now divide subinterval I_1 into two equal halves. Call the half which contains infinitely many elements of the sequence I_2 . Continue dividing the interval into halves; each time choose a half which contains infinitely many elements of the sequence. The result of continuing this process indefinitely is a sequence of intervals $\{I_k\}_{k=1}^{\infty}$, with $I_{k+1} \subset I_k$. Construct a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ by choosing $x_{n_k} \in I_k$. Since each I_k contains infinitely many x_n 's, we can ensure that $\{x_{n_k}\}_{k=1}^{\infty}$ really is an (infinite) subsequence of $\{x_n\}_{n=1}^{\infty}$.
- (b) Prove that the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ has a limit.
- (c) Prove that this limit lies in [a, b].
- 2. The Bolzano-Weierstrass Theorem has a converse.
 - Let S be a subset of \mathbb{R}^n with the property that any sequence in S has a convergent subsequence with limit in S. Prove that S is closed and bounded.
- 3. Suppose $f : \mathfrak{R}^2_+ \to \mathfrak{R}$ is defined by

$$f(x,y) = \begin{cases} 0, & \text{when } (x,y) = (0,0) \\ \frac{2xy}{(x^2 + y^2)}, & \text{otherwise.} \end{cases}$$

Show that f is not continuous at (0,0).

- 4. Let $f = (f^1, f^2, ..., f^m)$ be a function from \Re^n to \Re^m . Then f is continuous at x if and only if each of its component functions $f^i : \Re^n \to \Re$ is continuous at x.
- 5. (a) Prove carefully that $h: \Re^k \to \Re$ defined by

$$h\left(x_1, x_2, \dots, x_k\right) = x_i$$

is continuous on \Re^k .

(b) Use (a) to prove that any monomial

$$g(x_1, x_2, ..., x_k) = cx_1^{n_1} x_2^{n_2} ... x_k^{n_k}$$

is continuous on \mathbb{R}^k , and that any polynomial from \mathbb{R}^k to \mathbb{R}^m is continuous on \mathbb{R}^k .

- 6. Suppose that $f : \Re^k \to \Re$ is a continuous function and that $f(x^*) > 0$. Show that there is a ball $B = B_{\delta}(x^*)$ such that f(x) > 0 for all $x \in B$.
- 7. Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$.

Continuity: f is continuous at every point $y \in A$ if given any $\epsilon > 0$ and any $y \in A$, there is a $\delta(\epsilon, y) > 0$ such that for all x if $x \in A$ and $|x - y| < \delta(\epsilon, y)$, then $|f(x) - f(y)| < \epsilon$. By writing δ as a function of ϵ and y, $\delta(\epsilon, y)$, it is emphasized that, in general, δ depends on both $\epsilon > 0$ and $y \in A$.

Now it often happens that the function f is such that the number δ can be chosen to be independent of the point $y \in A$ and to depend only on ϵ . For example, if f(x) = 2xfor all $x \in \mathbb{R}$, then

$$|f(x) - f(y)| = 2|x - y|,$$

and so we can choose $\delta(\epsilon, y) = \frac{\epsilon}{2}$ for all $\epsilon > 0$ and all $y \in \mathbb{R}$.

Uniform Continuity: We say that f is uniformly continuous on A if given any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $x, y \in A$ are any numbers satisfying $|x - y| < \delta(\epsilon)$, then $|f(x) - f(y)| < \epsilon$.

- (a) Argue that if f is not uniformly continuous on A, then there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ there are points x_{δ} and y_{δ} in A such that $|x_{\delta} y_{\delta}| < \delta$ and $|f(x_{\delta}) f(y_{\delta})| \ge \epsilon_0$.
- (b) Argue that if f is not uniformly continuous on A, then there exists an $\epsilon_0 > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ in A such that $|x_n y_n| < \frac{1}{n}$ and $|f(x_n) f(y_n)| \ge \epsilon_0$ for all $n \in \mathbb{N}$.

(c) Prove the following theorem:

Let I be a closed and bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

- 8. A set S of real numbers is *bounded* if and only if there exists a real number K such that $|x| \leq K$ for any $x \in S$.
 - Continuum Property: Every non-empty set of real numbers which is bounded above has a smallest upper bound. Every non-empty set of real numbers which is bounded below has a largest lower bound.
 - The smallest upper bound is called the *supremum* of the set.
 - The largest lower bound is called the *infimum* of the set.
 - We say that a sequence $\{x_n\}$ diverges to $+\infty$ and write $\{x_n\} \to +\infty$ as $n \to \infty$ if, for any H > 0, we can find an N such that, for any n > N, $x_n > H$.
 - Recall the *Bolzano-Weierstrass Theorem*: Let C be a compact subset in \Re and let $\{x_n\}$ be any sequence in C. Then $\{x_n\}$ has a convergent subsequence whose limit lies in C.
 - (a) Let S be a non-empty set of real numbers which is bounded above. Show that a sequence of points of S can be found which converges to its supremum.
 - (b) Let S be a non-empty set of real numbers which is unbounded above. Show that a sequence of points of S can be found which diverges to $+\infty$.
 - (c) Let f be a continuous function on the compact interval [a, b]. Prove that f is bounded on [a, b].
 - (d) Let f be a continuous function on the compact interval [a, b]. Prove that there exist points x_m and x_M in [a, b] such that $f(x_m) \leq f(x) \leq f(x_M)$, for all $x \in [a, b]$.