Math 271: Mathematical Methods — Tridip Ray
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Homework 4 (Class Test on 06 October)

1. In this question we will prove the following version of the Bolzano-Weierstrass Theorem:

   Any sequence contained in a compact interval of $\mathbb{R}$, $[a,b]$, *has a convergent subsequence whose limit lies in* $[a,b]$.

   – Let $\{x_n\}_{n=1}^{\infty}$ be a sequence contained in $[a,b]$. Divide $[a,b]$ into two equal halves: $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$.

   (a) Argue that infinitely many elements of the sequence $\{x_n\}_{n=1}^{\infty}$ must lie in one (or both) of these halves.

   – Let I_1 denote a half which contains infinitely many members of the sequence. Now divide subinterval I_1 into two equal halves. Call the half which contains infinitely many elements of the sequence I_2. Continue dividing the interval into halves; each time choose a half which contains infinitely many elements of the sequence. The result of continuing this process indefinitely is a sequence of intervals $\{I_k\}_{k=1}^{\infty}$, with $I_{k+1} \subset I_k$. Construct a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ by choosing $x_{n_k} \in I_k$. Since each I_k contains infinitely many x_n's, we can ensure that $\{x_{n_k}\}_{k=1}^{\infty}$ really is an (infinite) subsequence of $\{x_n\}_{n=1}^{\infty}$.

   (b) Prove that the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ has a limit.

   (c) Prove that this limit lies in $[a,b]$.

2. The Bolzano-Weierstrass Theorem has a converse.

   – Let S be a subset of $\mathbb{R}^n$ with the property that any sequence in S has a convergent subsequence with limit in S. Prove that S is closed and bounded.

3. Suppose $f : \Re_+^2 \to \Re$ is defined by

$$f(x,y) = \begin{cases} 0, & \text{when } (x,y) = (0,0) \\ \dfrac{2xy}{(x^2+y^2)}, & \text{otherwise.} \end{cases}$$

   Show that f is *not* continuous at $(0,0)$.

4. Let $f = (f^1, f^2, ..., f^m)$ be a function from $\Re^n$ to $\Re^m$. Then f is continuous at x if and only if each of its component functions $f^i : \Re^n \rightarrow \Re$ is continuous at x.

5. (a) Prove carefully that $h : \Re^k \rightarrow \Re$ defined by

$$h(x_1, x_2, ..., x_k) = x_i$$

   is continuous on $\Re^k$.

   (b) Use (a) to prove that any monomial

$$g(x_1, x_2, ..., x_k) = c x_1^{n_1} x_2^{n_2} ... x_k^{n_k}$$

   is continuous on $\Re^k$, and that any polynomial from $\Re^k$ to $\Re^m$ is continuous on $\Re^k$.

6. Suppose that $f : \Re^k \rightarrow \Re$ is a continuous function and that $f(x^*) > 0$. Show that there is a ball $B = B_\delta(x^*)$ such that $f(x) > 0$ for all $x \in B$.

7. Consider the following constrained maximization problem:

$$\left.\begin{array}{ll} \underset{x \in \Re^n_+}{\text{Maximize}} & \|x\| \\ \text{subject to} & \sum_{i=1}^{n} x_i \leq 1. \end{array}\right\}$$

   Does this problem have a solution? Provide a rigorous argument for your answer.

8.
   - A set S of real numbers is *bounded* if and only if there exists a real number K such that $|x| \leq K$ for any $x \in S$.
   - *Continuum Property*: Every non-empty set of real numbers which is bounded above has a smallest upper bound. Every non-empty set of real numbers which is bounded below has a largest lower bound.
     - The smallest upper bound is called the *supremum* of the set.
     - The largest lower bound is called the *infimum* of the set.
   - We say that a sequence $\{x_n\}$ *diverges* to $+\infty$ and write $\{x_n\} \rightarrow +\infty$ as $n \rightarrow \infty$ if, for any $H > 0$, we can find an N such that, for any $n > N$, $x_n > H$.
   - Recall the *Bolzano-Weierstrass Theorem*: Let C be a compact subset in $\Re$ and let $\{x_n\}$ be any sequence in C. Then $\{x_n\}$ has a convergent subsequence whose limit lies in C.

(a) Let S be a non-empty set of real numbers which is bounded above. Show that a sequence of points of S can be found which converges to its supremum.

(b) Let S be a non-empty set of real numbers which is unbounded above. Show that a sequence of points of S can be found which diverges to $+\infty$.

(c) Let f be a continuous function on the compact interval $[a, b]$. Prove that f is bounded on $[a, b]$.

(d) Let f be a continuous function on the compact interval $[a, b]$. Prove that there exist points x_m and x_M in $[a, b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$, for all $x \in [a, b]$.