Math 271: Mathematical Methods | Tridip Ray
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Homework 5 (Class Test on 16 October)

1. Let f be a real-valued function defined on an open interval I in $\mathbb{R}$ which contains the point x_0. Prove rigorously that if f is differentiable at x_0, then f is continuous at x_0.

2. Prove that the product of homogeneous functions is homogeneous.

3. If $y = f(x_1, x_2)$ is C^2 and homogeneous of degree r, show that
$$x_1^2 \frac{\partial^2 f}{\partial x_1^2} + 2x_1 x_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + x_2^2 \frac{\partial^2 f}{\partial x_2^2} = r(r-1) f.$$

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x + e^y,\ y + e^{-x})$ for $x, y \in \mathbb{R}^2$. Show that f is everywhere locally invertible.

5. One solution of the system
$$\begin{aligned} x^3 y - z &= 1, \\ x + y^2 + z^3 &= 6, \end{aligned}$$
is $x = 1, y = 2, z = 1$. Use calculus to estimate the corresponding x and y when $z = 1.1$.

6. Consider the system of equations
$$\begin{aligned} y^2 + 2u^2 + v^2 - xy &= 15, \\ 2y^2 + u^2 + v^2 + xy &= 38, \end{aligned}$$
at the solution $x = 1, y = 4, u = 1, v = -1$. Think of u and v as exogenous and x and y as endogenous. Use calculus to estimate the values of x and y that correspond to $u = 0.9$ and $v = -1.1$.

7. Does the system
$$\begin{aligned} xz^3 + y^2 v^4 &= 2, \\ xz + yvz^2 &= 2, \end{aligned}$$
define v and z as C^1 functions of x and y around the point $(1, 1, 1, 1)$? If so, find $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ there.

8. Consider the system of equations

$$\begin{aligned} x + 2y + z &= 5, \\ 3x^2yz &= 12, \end{aligned}$$

as defining some endogenous variables in terms of some exogenous variables.

   (a) Divide the three variables into exogenous ones and endogenous ones in a neighbourhood of $x = 2, y = 1, z = 1$ so that the Implicit Function Theorem applies.

   (b) If each of the exogenous variables in your answer to (a) increase by 0.25, use calculus to estimate how each of the endogenous variables will change.

9. A firm uses two inputs to produce its output via the Cobb-Douglas production function $z = x^a y^b$, where $a = b = 0.5$. Its current level of inputs is $x = 25, y = 100$. The firm will introduce a new technology that will change the b-exponent on its production function to $b = 0.504$, with no change in a.

   Use calculus to estimate the input combination which will keep the total output the same and the sum of inputs the same.

10. Consider the linear system of equations $Ax = b$ where A is an $m \times n$ matrix, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. We have the following two results.

   - The system of equations has a solution for every right-hand side b if and only if $m \leq n$ and the rank of A is m.
   - The system of equations has at most one solution for every right-hand side b if and only if $m \geq n$ and the rank of A is n.

- **Definitions:** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if for *every* b in $\mathbb{R}^m$ there is *at least one* x in $\mathbb{R}^n$ such that $f(x) = b$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if for *every* b in $\mathbb{R}^m$ there is *at most one* x in $\mathbb{R}^n$ such that $f(x) = b$.

- **Definitions:** Let x_0 be a point in the domain of $f : \mathbb{R}^n \to \mathbb{R}^m$ with $f(x_0) = b_0$.

   - f is **locally onto** at x_0 if, given any open ball $B_r(x_0)$ about x_0 in $\mathbb{R}^n$, there is an open ball $B_s(b_0)$ about b_0 in $\mathbb{R}^m$ such that for *every* b in $B_s(b_0)$ there is *at least one* x in $B_r(x_0)$ such that $f(x) = b$.

– f is **locally one-to-one** at x_0 if, given any open ball $B_r(x_0)$ about x_0 in $\mathbb{R}^n$, there is an open ball $B_s(b_0)$ about b_0 in $\mathbb{R}^m$ such that for *every* b in $B_s(b_0)$ there is *at most one* x in $B_r(x_0)$ such that $f(x) = b$.

(a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 function with $f(x^*) = b^*$. Let $Df(x^*)$ denote the $m \times n$ Jacobian matrix of f at x^*.

(i) Prove that if rank $(Df(x^*)) = m \leq n$, then f is locally onto at x^*.

(ii) Prove that if rank $(Df(x^*)) = n \leq m$, then f is locally one-to-one at x^*.

(b) Prove the following theorem.

(Inverse Function Theorem) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function with $f(x^*) = y^*$. If $Df(x^*)$ is nonsingular then there exists an open ball $B_r(x^*)$ about x^* and an open set V about y^* such that f is a one-to-one and onto map from $B_r(x^*)$ to V. The inverse map $f^{-1} : V \to B_r(x^*)$ is also C^1 and $Df^{-1}(f(x^*)) = (Df(x^*))^{-1}$.