Homework 5 (Class Test on 11 October)

- 1. Let f be a real-valued function defined on an open interval I in \mathbb{R} which contains the point x_0 . Prove rigorously that if f is differentiable at x_0 , then f is continuous at x_0 .
- 2. Prove that the product of homogeneous functions is homogeneous.
- 3. If $y = f(x_1, x_2)$ is C^2 and homogeneous of degree r, show that

$$x_1^2 \frac{\partial^2 f}{\partial x_1^2} + 2x_1 x_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + x_2^2 \frac{\partial^2 f}{\partial x_2^2} = r \left(r - 1\right) f.$$

- 4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x + e^y, y + e^{-x})$ for $x, y \in \mathbb{R}^2$. Show that f is everywhere locally invertible.
- 5. One solution of the system

$$x^{3}y - z = 1,$$

 $x + y^{2} + z^{3} = 6,$

is x = 1, y = 2, z = 1. Use calculus to estimate the corresponding x and y when z = 1.1.

6. Consider the system of equations

$$y^{2} + 2u^{2} + v^{2} - xy = 15,$$

$$2y^{2} + u^{2} + v^{2} + xy = 38,$$

at the solution x = 1, y = 4, u = 1, v = -1. Think of u and v as exogenous and x and y as endogenous. Use calculus to estimate the values of x and y that correspond to u = 0.9 and v = -1.1.

7. Does the system

$$xz^3 + y^2v^4 = 2,$$

$$xz + yvz^2 = 2,$$

define v and z as C^1 functions of x and y around the point (1,1,1,1)? If so, find $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ there.

8. Consider the system of equations

$$\begin{aligned} x + 2y + z &= 5, \\ 3x^2yz &= 12, \end{aligned}$$

as defining some endogenous variables in terms of some exogenous variables.

- (a) Divide the three variables into exogenous ones and endogenous ones in a neighbourhood of x = 2, y = 1, z = 1 so that the Implicit Function Theorem applies.
- (b) If each of the exogenous variables in your answer to (a) increase by 0.25, use calculus to estimate how each of the endogenous variables will change.
- 9. A firm uses two inputs to produce its output via the Cobb-Douglas production function $z = x^a y^b$, where a = b = 0.5. Its current level of inputs is x = 25, y = 100. The firm will introduce a new technology that will change the *b*-exponent on its production function to b = 0.504, with no change in *a*.

Use calculus to estimate the input combination which will keep the total output the same and the sum of inputs the same.

- 10. Consider the linear system of equations Ax = b where A is an $m \times n$ matrix, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. We have the following two results.
 - The system of equations has a solution for every right-hand side b if and only if $m \leq n$ and the rank of A is m.
 - The system of equations has at most one solution for every right-hand side b if and only if $m \ge n$ and the rank of A is n.
 - Definitions: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is onto if for every b in \mathbb{R}^m there is at least one x in \mathbb{R}^n such that f(x) = b. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if for every bin \mathbb{R}^m there is at most one x in \mathbb{R}^n such that f(x) = b.
 - **Definitions:** Let x_0 be a point in the domain of $f : \mathbb{R}^n \to \mathbb{R}^m$ with $f(x_0) = b_0$.
 - f is **locally onto** at x_0 if, given any open ball $B_r(x_0)$ about x_0 in \mathbb{R}^n , there is an open ball $B_s(b_0)$ about b_0 in \mathbb{R}^m such that for every b in $B_s(b_0)$ there is at least one x in $B_r(x_0)$ such that f(x) = b.

- f is **locally one-to-one** at x_0 if, given any open ball $B_r(x_0)$ about x_0 in \mathbb{R}^n , there is an open ball $B_s(b_0)$ about b_0 in \mathbb{R}^m such that for every b in $B_s(b_0)$ there is at most one x in $B_r(x_0)$ such that f(x) = b.
- (a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 function with $f(x^*) = b^*$. Let $Df(x^*)$ denote the $m \times n$ Jacobian matrix of f at x^* .
 - (i) Prove that if rank $(Df(x^*)) = m \le n$, then f is locally onto at x^* .
 - (ii) Prove that if rank $(Df(x^*)) = n \le m$, then f is locally one-to-one at x^* .
- (b) Prove the following theorem.

(Inverse Function Theorem) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function with $f(x^*) = y^*$. If $Df(x^*)$ is nonsingular then there exists an open ball $B_r(x^*)$ about x^* and an open set V about y^* such that f is a one-to-one and onto map from $B_r(x^*)$ to V. The inverse map $f^{-1}: V \to B_r(x^*)$ is also C^1 and $Df^{-1}(f(x^*)) = (Df(x^*))^{-1}$.