

## Homework 6 (Class Test on 21 October)

### 1. [Proof of Intermediate Value Theorem]

Consider the following version of the intermediate value theorem:

*Let  $f$  be a real-valued continuous function defined on an interval containing the real numbers  $a$  and  $b$ , with say  $f(a) < f(b)$ . Then, given any  $y \in \mathbb{R}$  such that  $f(a) < y < f(b)$ , there exists  $x \in (a, b)$  such that  $f(x) = y$ .*

**Proof:** We use the divide and conquer method. We repeatedly bisect the interval, retaining the half that might contain a solution.

- We consider a sequence of intervals  $[a_1, b_1], [a_2, b_2], \dots$  where
    - $[a_1, b_1]$  is the original interval  $[a, b]$ ,
    - $[a_2, b_2]$  is one of the halves  $\left[a, \frac{a+b}{2}\right]$  or  $\left[\frac{a+b}{2}, b\right]$  – chosen so that  $f(a_2) < y < f(b_2)$ . Of course, if  $f\left(\frac{a+b}{2}\right) = y$ , then we are done. Otherwise either  $f\left(\frac{a+b}{2}\right) < y$ , in which case we choose  $\left[\frac{a+b}{2}, b\right]$ , or  $f\left(\frac{a+b}{2}\right) > y$ , in which case we choose  $\left[a, \frac{a+b}{2}\right]$ .
    - Iterating this process we obtain either the solution we seek or a pair of sequences  $a_1, a_2, \dots$  and  $b_1, b_2, \dots$  such that  $f(a_n) < y < f(b_n)$ , where  $b_n - a_n = 2^{1-n}(b - a)$ .
  - **Definition (Cauchy Sequence):** A sequence of real numbers  $\{x_n\}$  is a Cauchy sequence if for any  $\epsilon > 0$ , there exists an integer  $N$  such that, for all  $i, j \geq N$ ,  $|x_i - x_j| < \epsilon$ .
  - **Theorem:** Any Cauchy sequence of real numbers converges.
- [If you are interested, read section 29.1 (pages 803 – 806) of the textbook.]
- (a) Prove that both the sequences  $a_1, a_2, \dots$  and  $b_1, b_2, \dots$  constructed above are Cauchy sequences and converge to a common limit, say  $x$ .
  - (b) Use part (a) and the continuity of  $f$  to prove the intermediate value theorem.

2. [Alternative Proof of Intermediate Value Theorem]

(a) Prove the following proposition:

Let  $f$  be a real-valued function defined on an interval  $S$  in  $\mathbb{R}$ . Assume that  $f$  is continuous at a point  $c$  in  $S$  and that  $f(c) \neq 0$ . Then there is an open ball  $B_\delta(c)$  such that  $f(x)$  has the same sign as  $f(c)$  in  $B_\delta(c) \cap S$ .

(b) Prove the following theorem:

Let  $f$  be a real-valued and continuous function on a compact interval  $[\alpha, \beta]$  in  $\mathbb{R}$ , and suppose that  $f(\alpha)$  and  $f(\beta)$  have opposite signs. Then there is at least one point  $\gamma$  in the open interval  $(\alpha, \beta)$  such that  $f(\gamma) = 0$ .

- **Hints:** For definiteness, assume  $f(\alpha) > 0$  and  $f(\beta) < 0$ . Define the set

$$A = \{x: x \in [\alpha, \beta] \text{ and } f(x) \geq 0\}.$$

Then  $A$  is nonempty since  $\alpha \in A$ , and  $A$  is bounded above by  $\beta$ . Let  $\gamma = \sup A$ . Then  $\alpha < \gamma < \beta$ . Prove that  $f(\gamma) = 0$ . [Suppose not, and then use (a) to come up with a contradiction.]

(c) Use (b) to prove the following version of the intermediate value theorem:

*Let  $f$  be a real-valued continuous function defined on an interval containing the real numbers  $a$  and  $b$ , with say  $f(a) < f(b)$ . Then, given any  $y \in \mathbb{R}$  such that  $f(a) < y < f(b)$ , there exists  $x \in (a, b)$  such that  $f(x) = y$ .*

3. Use the statement of the intermediate value theorem given in questions 1 and 2 above to prove the following version of the intermediate value theorem:

*Suppose  $A$  is a convex subset of  $\mathbb{R}^n$ , and  $f : A \rightarrow \mathbb{R}$  is a continuous function on  $A$ . Suppose  $x^1$  and  $x^2$  are in  $A$ , and  $f(x^1) > f(x^2)$ . Then, given any  $c \in \mathbb{R}$  such that  $f(x^1) > c > f(x^2)$ , there exists  $0 < \theta < 1$  such that  $f(\theta x^1 + (1 - \theta)x^2) = c$ .*

4. Let  $f : [a, b] \rightarrow [a, b]$  be a continuous function on  $[a, b]$ . Prove that there exists some  $\beta \in [a, b]$  such that  $f(\beta) = \beta$ .

5. Let  $f$  be a convex function on the interval  $I$  and  $x_1, x_2$  and  $x_3$  be points of  $I$  which satisfy  $x_1 < x_2 < x_3$ .

Prove that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

6. If  $a$  and  $b$  are arbitrary positive real numbers, show that

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b, \text{ for every } 0 < \theta < 1.$$

7. Prove the following proposition (*Jensen's Inequality*):

Suppose  $A$  is a convex set in  $\mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}$  is a concave function. Then, for any integer  $m > 1$ ,

$$f\left(\sum_{i=1}^m \theta_i x^i\right) \geq \sum_{i=1}^m \theta_i f(x^i)$$

whenever  $x^1, x^2, \dots, x^m \in A$ ,  $(\theta_1, \theta_2, \dots, \theta_m) \in \mathbb{R}_+^m$  and  $\sum_{i=1}^m \theta_i = 1$ .

8. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a quasi-concave function. Let  $I$  be the closed interval  $[0, 1]$ . Let  $a, b$  be two arbitrary given vectors in  $\mathbb{R}^2$ . Define a function  $g : I \rightarrow \mathbb{R}$  by  $g(t) = f(ta + (1-t)b)$ . Prove that  $g$  is a quasi-concave function on  $I$ .

9. Suppose  $A$  is a convex set in  $\mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}$ . Prove that the following two statements are equivalent to each other:

(a)  $f(x^2) \geq f(x^1)$  implies  $f(\theta x^1 + (1-\theta)x^2) \geq f(x^1)$  whenever  $x^1, x^2 \in A$ , and  $0 \leq \theta \leq 1$ ;

(b) For every  $\alpha \in \mathbb{R}$ , the set  $S(\alpha) = \{x \in A : f(x) \geq \alpha\}$  is a convex set in  $\mathbb{R}^n$ .

10. Recall the following characterization of twice differentiable quasi-concave functions.

Let  $A \subset \mathbb{R}^n$  be an open convex set, and  $f : A \rightarrow \mathbb{R}$  is twice continuously differentiable on  $A$ . The *bordered Hessian matrix* of  $f$  at  $x \in A$  is denoted by  $G_f(x)$  and is defined by the following  $(n+1) \times (n+1)$  matrix:

$$G_f(x) = \begin{pmatrix} 0 & \nabla f(x) \\ \nabla f(x) & H_f(x) \end{pmatrix}.$$

We denote the  $(k+1)$ th leading principal minor of  $G_f(x)$  by  $|G_f(x; k)|$ , where  $k = 1, 2, \dots, n$ .

*Theorem:* Suppose  $A \subset \mathbb{R}^n$  is an open convex set, and  $f : A \rightarrow \mathbb{R}$  is twice continuously differentiable on  $A$ .

- (i) If  $f$  is quasi-concave on  $A$ , then  $(-1)^k |G_f(x; k)| \geq 0$  for  $x \in A$ , and  $k = 1, 2, \dots, n$ .
- (ii) If  $(-1)^k |G_f(x; k)| > 0$  for  $x \in A$ , and  $k = 1, 2, \dots, n$ , then  $f$  is quasi-concave on  $A$ .

• **Question:** Let  $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  be given by

$$f(x, y) = (x - 1)^2 (y - 1)^2, \quad (x, y) \in \mathbb{R}_+^2.$$

In light of this example explain whether the ‘strict inequalities’ ( $> 0$ ) in condition (ii) of the theorem can be replaced with ‘weak inequalities’ ( $\geq 0$ ).

11. (a) Let  $A \subset \mathbb{R}^n$  and  $g : A \rightarrow \mathbb{R}$ .

(i) Define  $V(a_1, a_2, \dots, a_n) = \max_{\{x_1, x_2, \dots, x_n\}} \{a_1 x_1 + a_2 x_2 + \dots + a_n x_n : g(x_1, x_2, \dots, x_n) \leq 0\}$ , where  $a_i$  is a scalar,  $i = 1, 2, \dots, n$ . Prove carefully whether  $V(\cdot)$  concave or convex in  $a_1, a_2, \dots, a_n$ .

(ii) Define  $w(a_1, a_2, \dots, a_n) = \min_{\{x_1, x_2, \dots, x_n\}} \{a_1 x_1 + a_2 x_2 + \dots + a_n x_n : g(x_1, x_2, \dots, x_n) \geq 0\}$ , where  $a_i$  is a scalar,  $i = 1, 2, \dots, n$ . Prove carefully whether  $w(\cdot)$  concave or convex in  $a_1, a_2, \dots, a_n$ .

(iii) For parts (i) and (ii), how does your answer depend on the nature of the function  $g(x_1, x_2, \dots, x_n)$ ?

(b) For  $(p_1, p_2, \dots, p_n, M) \gg 0$ , define

$$v(p_1, p_2, \dots, p_n, M) = \max_{\{x_1, x_2, \dots, x_n\}} \{u(x_1, x_2, \dots, x_n) : p_1 x_1 + p_2 x_2 + \dots + p_n x_n \leq M\},$$

where  $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$  is a continuous utility function. Prove that  $v(p_1, p_2, \dots, p_n, M)$  is quasiconvex in  $p_1, p_2, \dots, p_n, M$ .