Math 271: Mathematical Methods — Tridip Ray

Semester I, 2023-24 — ISI, Delhi

Homework 6 (Class Test on 19 October)

1. **[Proof of Intermediate Value Theorem]**

   Consider the following version of the intermediate value theorem:

   Let f be a real-valued continuous function defined on an interval containing the real numbers a and b, with say $f(a) < f(b)$. Then, given any $y \in \Re$ such that $f(a) < y < f(b)$, there exists $x \in (a, b)$ such that $f(x) = y$.

   Proof: We use the divide and conquer method. We repeatedly bisect the interval, retaining the half that might contain a solution.

   – We consider a sequence of intervals $[a_1, b_1]$, $[a_2, b_2]$, ... where

     - $[a_1, b_1]$ is the original interval $[a, b]$,
     - $[a_2, b_2]$ is one of the halves $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$ – chosen so that $f(a_2) < y < f(b_2)$. Of course, if $f\left(\frac{a+b}{2}\right) = y$, then we are done. Otherwise either $f\left(\frac{a+b}{2}\right) < y$, in which case we choose $\left[\frac{a+b}{2}, b\right]$, or $f\left(\frac{a+b}{2}\right) > y$, in which case we choose $\left[a, \frac{a+b}{2}\right]$.
     - Iterating this process we obtain either the solution we seek or a pair of sequences $a_1, a_2, ...$ and $b_1, b_2, ...$ such that $f(a_n) < y < f(b_n)$, where $b_n - a_n = 2^{1-n}(b - a)$.

   – **Definition (Cauchy Sequence):** A sequence of real numbers $\{x_n\}$ is a Cauchy sequence if for any $\epsilon > 0$, there exists an integer N such that, for all $i, j \geq N$, $|x_i - x_j| < \epsilon$.

   – **Theorem:** *Any Cauchy sequence of real numbers converges.*

   [If you are interested, read section 29.1 (pages 803 – 806) of the textbook.]

   (a) Prove that both the sequences $a_1, a_2, ...$ and $b_1, b_2, ...$ constructed above are Cauchy sequences and converge to a common limit, say x.

   (b) Use part (a) and the continuity of f to prove the intermediate value theorem.

2. [**Alternative Proof of Intermediate Value Theorem**]

   (a) Prove the following proposition:

   Let f be a real-valued function defined on an interval S in $\Re$. Assume that f is continuous at a point c in S and that $f(c) \neq 0$. Then there is an open ball $B_\delta(c)$ such that $f(x)$ has the same sign as $f(c)$ in $B_\delta(c) \cap S$.

   (b) Prove the following theorem:

   Let f be a real-valued and continuous function on a compact interval $[\alpha, \beta]$ in $\Re$, and suppose that $f(\alpha)$ and $f(\beta)$ have opposite signs. Then there is at least one point γ in the open interval (α, β) such that $f(\gamma) = 0$.

   - **Hints:** For definiteness, assume $f(\alpha) > 0$ and $f(\beta) < 0$. Define the set

   $$A = \{x\colon\ x \in [\alpha, \beta] \text{ and } f(x) \geq 0\}.$$

   Then A is nonempty since $\alpha \in A$, and A is bounded above by β. Let $\gamma =$ supremum of A. Then $\alpha < \gamma < \beta$. Prove that $f(\gamma) = 0$. [Suppose not, and then use (a) to come up with a contradiction.]

   (c) Use (b) to prove the following version of the intermediate value theorem:

   Let f be a real-valued continuous function defined on an interval containing the real numbers a and b, with say $f(a) < f(b)$. Then, given any $y \in \Re$ such that $f(a) < y < f(b)$, there exists $x \in (a, b)$ such that $f(x) = y$.

3. Let $f : [a, b] \to [a, b]$ be a *continuous* function on $[a, b]$. Prove that there exists some $\beta \in [a, b]$ such that $f(\beta) = \beta$.

4. Let f be a *convex* function on the interval I and x_1, x_2 and x_3 be points of I which satisfy $x_1 < x_2 < x_3$.

   Prove that
   $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

5. If a and b are arbitrary positive real numbers, show that

   $$a^\theta b^{1-\theta} \leq \theta a + (1 - \theta) b, \text{ for every } 0 < \theta < 1.$$

6. Prove the following proposition (*Jensen's Inequality*):

   Suppose A is a convex set in $\Re^n$ and $f : A \rightarrow \Re$ is a concave function. Then, for any integer $m > 1$,

$$f\left(\sum_{i=1}^{m} \theta_i x^i\right) \geq \sum_{i=1}^{m} \theta_i f\left(x^i\right)$$

   whenever $x^1, x^2, ..., x^m \in A$, $(\theta_1, \theta_2, ..., \theta_m) \in \Re^m_+$ and $\sum_{i=1}^{m} \theta_i = 1$.

7. Let $f : \Re^2 \rightarrow \Re$ be a quasi-concave function. Let I be the closed interval $[0, 1]$. Let a, b be two arbitrary given vectors in $\Re^2$. Define a function $g : I \rightarrow \Re$ by $g(t) = f(ta + (1-t)b)$. Prove that g is a quasi-concave function on I.

8. Suppose A is a convex set in $\Re^n$ and $f : A \rightarrow \Re$. Prove that the following two statements are equivalent to each other:

   (a) $f(x^2) \geq f(x^1)$ implies $f(\theta x^1 + (1-\theta) x^2) \geq f(x^1)$ whenever $x^1, x^2 \in A$, and $0 \leq \theta \leq 1$;

   (b) For every $\alpha \in \Re$, the set $S(\alpha) = \{x \in A : f(x) \geq \alpha\}$ is a convex set in $\Re^n$.

9. Recall the following characterization of twice differentiable quasi-concave functions.

   Let $A \subset \mathbb{R}^n$ be an open convex set, and $f : A \rightarrow \mathbb{R}$ is twice continuously differentiable on A. The *bordered Hessian matrix* of f at $x \in A$ is denoted by $G_f(x)$ and is defined by the following $(n+1) \times (n+1)$ matrix:

$$G_f(x) = \begin{pmatrix} 0 & \nabla f(x) \\ & \\ \nabla f(x) & H_f(x) \end{pmatrix}.$$

   We denote the $(k+1)$th leading principal minor of $G_f(x)$ by $|G_f(x;k)|$, where $k = 1, 2, ..., n$.

   Theorem: Suppose $A \subset \Re^n$ is an open convex set, and $f : A \rightarrow \mathbb{R}$ is twice continuously differentiable on A.

   (i) *If f is quasi-concave on A, then $(-1)^k |G_f(x;k)| \geq 0$ for $x \in A$, and $k = 1, 2, ..., n$.*

   (ii) *If $(-1)^k |G_f(x;k)| > 0$ for $x \in A$, and $k = 1, 2, ..., n$, then f is quasi-concave on A.*

- **Question:** Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be given by

$$f(x,y) = (x-1)^2 (y-1)^2, \quad (x,y) \in \mathbb{R}^2_+.$$

In light of this example explain whether the 'strict inequalities' ('> 0') in condition (ii) of the theorem can be replaced with 'weak inequalities' ('≥ 0').

10. (a) Let $A \subset \mathbb{R}^n$ and $g : A \to \mathbb{R}$.

    (i) Define $V(a_1, a_2, ..., a_n) = \max_{\{x_1, x_2, ..., x_n\}} \{a_1 x_1 + a_2 x_2 + ... + a_n x_n : g(x_1, x_2, ..., x_n) \leq 0\}$, where a_i is a scalar, $i = 1, 2, ..., n$. Prove carefully whether $V(\cdot)$ *concave* or *convex* in $a_1, a_2, ..., a_n$.

    (ii) Define $w(a_1, a_2, ..., a_n) = \min_{\{x_1, x_2, ..., x_n\}} \{a_1 x_1 + a_2 x_2 + ... + a_n x_n : g(x_1, x_2, ..., x_n) \geq 0\}$, where a_i is a scalar, $i = 1, 2, ..., n$. Prove carefully whether $w(\cdot)$ *concave* or *convex* in $a_1, a_2, ..., a_n$.

    (iii) For parts (i) and (ii), how does your answer depend on the nature of the function $g(x_1, x_2, ..., x_n)$?

    (b) For $(p_1, p_2, ..., p_n, M) \gg 0$, define

$$v(p_1, p_2, ..., p_n, M) = \max_{\{x_1, x_2, ..., x_n\}} \{u(x_1, x_2, ..., x_n) : p_1 x_1 + p_2 x_2 + ... + p_n x_n \leq M\},$$

where $u : \mathbb{R}^n_+ \to \mathbb{R}$ is a continuous utility function. Prove that $v(p_1, p_2, ..., p_n, M)$ is *quasiconvex* in $p_1, p_2, ..., p_n, M$.