Math 271: Mathematical Methods | Tridip Ray
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Homework 7 (Class Test on 04 November)

0. **Practice Problems from the Textbook**:

   Exercises **17.1, 17.3, 18.2, 18.3, 18.5** and **19.14** (only for **18.2, 18.3, 18.5**).

1. Let $f : \Re \to \Re$ be a continuous function on $\Re$ satisfying the property

$$f(x) < f(0) \text{ for all } x > 1.$$

   Now consider the following constrained maximization problem:

$$\left.\begin{array}{l} \text{Maximize } f(x) \\ \text{subject to } x \geq 0. \end{array}\right\}$$

   Does this problem have a solution? Explain carefully.

2. **[The Method of Least Squares]**

   - Suppose we are given n points (x_i, y_i), $i = 1, 2, ..., n$ in $\Re^2$. Let $f : \Re \to \Re$ be given by $f(x) = ax + b$, for all $x \in \Re$. We wish to find a function f (that is, we want to choose $a \in \Re$ and $b \in \Re$) such that the quantity $\sum_{i=1}^{n} [f(x_i) - y_i]^2$ is minimized.

   - We can set up the problem as an *unconstrained maximization problem* as follows. Define $F : \Re^2 \to \Re$ by

$$F(a, b) = -\sum_{i=1}^{n} [ax_i + b - y_i]^2 .$$

     The maximization problem is:

$$\underset{\{a,b\}}{\text{Maximize}} \, F(a, b) .$$

   (a) Write down the first-order conditions of the maximization problem.

   (b) Argue clearly that if (a^*, b^*) satisfies the first-order conditions, then (a^*, b^*) is a point of *global maximum* of F.

   (c) Find out (a^*, b^*) which is the point of *global maximum* of F.

3. Recall the following theorem.

   Theorem (Taylor's Expansion upto Second Order):

   Suppose A is an open convex subset of $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ is twice continuously differentiable on A. Suppose x^1 and x^2 are in A. Then there is $0 \leq \theta \leq 1$ such that

$$f\left(x^2\right) - f\left(x^1\right) = \left(x^2 - x^1\right) \cdot \nabla f\left(x^1\right) + \frac{1}{2}\left(x^2 - x^1\right) \cdot H_f\left(\theta x^1 + (1-\theta)x^2\right) \cdot \left(x^2 - x^1\right).$$

   Suppose D is an open convex set in $\mathbb{R}^n$, and $g : D \to \mathbb{R}$ is twice continuously differentiable and *quasiconcave* on D. Suppose there exists $x^* \in D$ satisfying

$$\text{(i) } \nabla g\left(x^*\right) = 0, \text{ and}$$
$$\text{(ii) } H_g\left(x^*\right) \text{ is } \textit{negative definite.}$$

   (a) Use the above theorem to prove that g has a *strict* local maximum at x^*.

   (b) Prove further that x^* is a point of *global maximum* of g on D.

4. Consider the following constrained maximization problem:

$$\left.\begin{array}{ll} \text{Maximize} & \prod_{i=1}^{n} x_i \\ \text{subject to} & \sum_{i=1}^{n} x_i = n, \\ \text{and} & x_i \geq 0,\ i = 1, 2, ...n. \end{array}\right\} \text{(P)}$$

   Using Weierstrass theorem we have proved in class that there exists $x^* \in C$ $= \left\{x \in \Re^n_+ : \sum_{i=1}^{n} x_i = n\right\}$ such that x^* solves (P).

   (a) Apply Lagrange theorem to find x^*.

   (b) **The Arithmetic Mean-Geometric Mean Inequality:** Let $a_1, a_2, ..., a_n$ be positive real numbers. Their AM is $\frac{\sum_{i=1}^{n} a_i}{n}$, and GM is $\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$.

      – Use the conclusion of part (a) to prove that AM $\geq$ GM.

5. Let $c = (c_1, ..., c_n)$ be a non-zero vector in $\Re^n$. Consider the following constrained maximization problem:

$$\left.\begin{array}{ll} \text{Maximize} & \sum_{i=1}^{n} c_i x_i \\ \text{subject to} & \sum_{i=1}^{n} x_i^2 = 1, \\ & (x_1, ..., x_n) \in \Re^n. \end{array}\right\} \text{(Q)}$$

(a) Show, by using Weierstrass theorem, that there exists $(x_1^*, ..., x_n^*) \in \Re^n$ which solves (Q).

(b) Use Lagrange theorem to show that

$$\sum_{i=1}^{n} c_i x_i^* = \|c\| .$$[1]

(c) Let p and q be arbitrary non-zero vectors in $\Re^n$. Using the result in (b) show that

$$|pq| \leq \|p\| \|q\| . \text{ (Cauchy-Schwarz Inequality)}$$

[1] Recall that $\|y\| = \sqrt{y_1^2 + y_2^2 + ... + y_n^2}$.

6. In the lectures we have also discussed (without a proof) the following Theorem. In this question we will prove this Theorem.

   - Let $A \subset \mathbb{R}^n$ be an open convex set, and $f : A \to \mathbb{R}$ is twice continuously differentiable on A. The *bordered Hessian matrix* of f at $x \in A$ is denoted by $G_f(x)$ and is defined by the following $(n+1) \times (n+1)$ matrix:

$$G_f(x) = \begin{pmatrix} 0 & \nabla f(x) \\ \\ \nabla f(x) & H_f(x) \end{pmatrix}.$$

     We denote the $(k+1)$th leading principal minor of $G_f(x)$ by $|G_f(x;k)|$, where $k = 1, 2, ..., n$.

   - **Theorem:** *Suppose $A \subset \mathbb{R}^n$ is an open convex set, and $f : A \to \mathbb{R}$ is twice continuously differentiable on A. If $(-1)^k |G_f(x;k)| > 0$ for $x \in A$, and $k = 1, 2, ..., n$, then f is quasi-concave on A.*

- We will prove this Theorem through the following three steps. Fix an arbitrary point $x^* \in A$, and define g by $g(x) = \nabla f(x^*) \cdot (x^* - x)$, for $x \in \mathbb{R}^n$.

(a) **Step I:** Given the premise of the Theorem, prove that x^* is a *local maximum* of f on the constraint set $C'[x^*] = \{x \in A \mid g(x) = 0\}$ by showing (i) that the constraint qualification is met at x^*, (ii) that there exists $\lambda^* \in \mathbb{R}$ such that (x^*, λ^*) meets the first-order conditions of the Lagrange Theorem, and (iii) that (x^*, λ^*) also meets second-order sufficient conditions for a local maximum.

- **Step II:** It can also be shown that (*you do not have to show this*) x^* is actually a *global maximum* of f on the constraint set $C[x^*] = \{x \in A \mid g(x) \geq 0\}$.

(b) **Step III:** Prove the following proposition:

If a C^1 function $F : A \to \mathbb{R}$ satisfies the following property for all $z \in A$:

$$\nabla F(z) \cdot (z - w) \geq 0 \text{ implies } F(w) \leq F(z), \text{ for all } w \in A,$$

then F is quasi-concave on A.

Using this proposition, complete the proof that given the premise of the Theorem, f is quasi-concave on A.