Math 271: Mathematical Methods | Tridip Ray
Semester I, 2023-24 | ISI, Delhi

## Homework 8

0. **Practice Problems from the Textbook:**

   Exercises **18.11, 18.12, 19.1, 19.4, 19.5, 19.12, 19.13**.

1. Let $a$, $b$ be arbitrary positive real numbers. Let $f : \Re \to \Re$ be a concave twice continuously differentiable function with $f' > 0$ and $f'' < 0$ on $\Re$. Define $\alpha = f'\left(\frac{1}{a}\right)$, $\beta = f'\left(\frac{1}{b}\right)$, and $\theta = f'(0)$.

   Consider the following maximization problem:

$$\left.\begin{array}{ll} \text{Maximize} & f(x) + f(y) \\ \text{subject to} & 1 - ax - by \geq 0, \\ \text{and} & (x, y) \in \Re^2_+. \end{array}\right\} \text{(P)}$$

   – In answering all the questions below, mention clearly which route you are using, and demonstrate carefully that all the required conditions of that route are satisfied.

   (a) Suppose $\frac{\alpha}{\theta} \geq \frac{a}{b}$. Obtain a solution to problem (Q), and draw an appropriate diagram to illustrate your solution.

   (b) Suppose $\frac{\theta}{\beta} \leq \frac{a}{b}$. Obtain a solution to problem (Q), and draw an appropriate diagram to illustrate your solution.

   (c) Suppose $\frac{\alpha}{\theta} < \frac{a}{b} < \frac{\theta}{\beta}$. If $(x^*, y^*)$ solves (Q), can we conclude that $x^* > 0$ and $y^* > 0$? Explain clearly.

2. A consumer's utility function for sugar (S) and bread (B) is given by

$$u = (x_S)^{\alpha} (x_B)^{1-\alpha}, \ 0 < \alpha < 1.$$

A rationing scheme is in place so that the consumer needs both money and ration coupons to purchase bread and sugar. The consumer has Rs. 1,000 and the per unit prices of sugar (S) and bread (B) are $P_S =$ Rs. 10 and $P_B =$ Rs. 5, respectively. He also has 1000 ration coupons and must give 5 per unit of sugar and 10 per unit of bread.

(a) Set up the consumer's utility maximization problem and draw the constraint set clearly.

(b) Can either $x_S = 0$ or $x_B = 0$ be a solution to the utility maximization problem? Explain clearly.

(c) Suppose $0 < \alpha \leq \frac{1}{3}$. Obtain a solution to the utility maximization problem, and draw an appropriate diagram to illustrate your solution.

(d) Suppose $\frac{1}{3} < \alpha < \frac{2}{3}$. Obtain a solution to the utility maximization problem, and draw an appropriate diagram to illustrate your solution.

(e) Suppose $\frac{2}{3} \leq \alpha < 1$. Obtain a solution to the utility maximization problem, and draw an appropriate diagram to illustrate your solution.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x, y) = \alpha x + \beta y + xy,$$

where $\alpha > 0$ and $\beta > 0$.

(a) Show that $f$ is a quasi-concave function on $\mathbb{R}^2_+$.

(b) Consider the following constrained maximization problem:

$$\left.\begin{array}{ll} \text{Maximize} & \alpha x + \beta y + xy \\ \text{subject to} & x + 2y \leq 4, \\ & 2x + y \leq 4, \\ \text{and} & (x, y) \in \mathbb{R}^2_+. \end{array}\right\} \quad \text{(R)}$$

- While solving problem (R), mention clearly which route (necessary or sufficient) you are taking and demonstrate carefully that all the required conditions of that route are satisfied.

- Solve problem (R), showing your procedure clearly, under each of the following restrictions on the parameter values of $\alpha$ and $\beta$:

  (i) $\dfrac{\beta}{2} - 2 < \alpha < \dfrac{\beta}{2} - \dfrac{2}{3}$;

  (ii) $\dfrac{\beta}{2} - \dfrac{2}{3} \leq \alpha \leq 2\beta + \dfrac{4}{3}$;

  (iii) $2\beta + \dfrac{4}{3} < \alpha < 2\beta + 4$.

4. Consider the following constrained maximization problem:

$$\left.\begin{array}{ll}\text{Maximize} & (x_1-3)^6+(x_2-4)^6 \\ \text{subject to} & x_1^2+x_2^2 \le 25, \\ & x_1+x_2 \ge 7, \\ & (x_1,x_2) \in \Re^2_+.\end{array}\right\} \text{(S)}$$

(a) Does the constraint set in problem (S) satisfy Slater's condition? Explain clearly.

(b) Use Weierstrass theorem to show that there exists a solution to problem (S).

(c) Obtain a solution to problem (S), showing your procedure clearly.

5. Suppose $a, b, c > 0$. Consider the following minimization problem:

$$\left.\begin{array}{ll}\text{Minimize} & ax_1+bx_2 \\ \text{subject to} & x_1x_2 \ge c, \\ \text{and} & (x_1,x_2) \in \Re^2_+.\end{array}\right\}$$

(a) Find out the solution(s) to the minimization problem. (Mention clearly which theorem you are using, and demonstrate carefully that all the required conditions of that theorem are satisfied.)

(b) Also find out the Kuhn-Tucker multiplier, $\lambda^*$, associated with the constraint "$x_1x_2 \ge c$".

(c) For each specification of $(a,b,c) \in \Re^3_{++}$, define the *value function*,

$$V(a,b,c) = \left.\begin{array}{ll}\text{Minimize} & ax_1+bx_2 \\ \text{subject to} & x_1x_2 \ge c, \\ \text{and} & (x_1,x_2) \in \Re^2_+\end{array}\right\}.$$

Compute $\frac{\partial}{\partial c}V(a,b,c)$, and compare it with $\lambda^*$ obtained in part (b).

6. A person lives for two periods: period 1 (youth) and period 2 (working age). In her youth (period 1) she inherits an wealth $W \geq 0$ from her parents and acquires education for which she has to spend an amount $E > 0$, $E \leq W$. Her education gets her a job that pays her an income $Y > 0$ during her working period (period 2). Her preference between period 1 consumption ($c_1$) and period 2 consumption ($c_2$) is given by the utility function

$$u(c_1, c_2) = \log c_1 + \beta \log c_2, \ \ \beta > 0.$$

She can do intertemporal substitution of consumption through borrowing and saving. But the credit market does not work perfectly and she faces two different interest rates for borrowing and lending: the borrowing interest rate, $r_B > 0$, is strictly greater than the lending interest rate, $r_L > 0$, that is, $r_B > r_L$.

(a) Set up the utility maximization problem. Draw the budget constraint with $c_1$ on $x$-axis and $c_2$ on $y$-axis, and label the important points clearly.

(b) Mention clearly which route (necessary or sufficient) you are taking to solve the utility maximization problem and demonstrate carefully that all the required conditions of that route are satisfied.

(c) Can either $c_1 = 0$ or $c_2 = 0$ be a solution to the utility maximization problem? Explain clearly.

(d) Solve this utility maximization problem showing your procedure clearly and answer the following questions.

(i) Show that there exist two wealth thresholds, $\underline{W}$ and $\overline{W}$, with $0 < \underline{W} < \overline{W}$, such that the nature of the solution to the utility maximization problem differs according as whether $0 \leq W < \underline{W}$, or $\underline{W} \leq W \leq \overline{W}$, or $\overline{W} < W$. (You have to derive $\underline{W}$ and $\overline{W}$ in terms of the parameters of the problem: $E$, $Y$, $\beta$, $r_L$ and $r_B$.)

(ii) For each of the three cases – (I) $0 \leq W < \underline{W}$, (II) $\underline{W} \leq W \leq \overline{W}$, and (III) $\overline{W} < W$ – derive the optimal choices of $c_1, c_2$, *borrowing* and *saving* in terms of the parameters of the problem.

(e) Draw the *wealth expansion path*, that is, the combinations of choices of $c_1$ and $c_2$ when only $W$ changes, with $c_1$ on $x$-axis and $c_2$ on $y$-axis, clearly illustrating your answers in part (d). (You must label all the important points clearly.)