Preliminaries

1. Propositions: Contrapositives and Converses

- Given two propositions P and Q, the statement "If P, then Q" is interpreted as the statement that if the proposition P is true, then the statement Q is also true.
  - We denote this by $P \Rightarrow Q$.
    - ◦ We will also say that "P implies Q".
- We stress that $P \Rightarrow Q$ only says that if P is true, then Q is also true.
  - It has nothing to say about the case where P is *not* true; in this case, Q could be either true or false.
    - ◦ For example, if P is $x > 0$ and Q is $x^2 > 0$, then it is certainly true that $P \Rightarrow Q$, since the square of a positive number is positive.
    - ◦ However, Q can be true even if P is not true, since the square of a negative number is also positive.

- Given a statement of the form "if P, then Q," its **contrapositive** is the statement that "if Q is not true, then P is not true."
  - If we let "$\sim Q$" denote the statement that 'Q is not true' (call this "not Q"), then the contrapositive of the statement "$P \Rightarrow Q$" is the statement "$\sim Q \Rightarrow \sim P$".
- Example: The contrapositive of the statement "If x is positive, then x^3 is positive" is the statement "If x^3 is not positive, then x is not positive".
- A statement and its contrapositive are logically equivalent. That is, if the statement is true, then the contrapositive is also true, while if the statement is false, so is the contrapositive.
  - Suppose, first, that $P \Rightarrow Q$ is true.
    - Then, if Q is false, P must also be false: if P were true, then by $P \Rightarrow Q$, Q would have to be true, and a statement cannot be both true and false.
  - Now, suppose $P \Rightarrow Q$ is false.
    - The only way this can happen is if P were true and Q were false. But this is precisely the statement $\sim Q \Rightarrow \sim P$ is not true.

- The **converse** of the statement $P \Rightarrow Q$ is the statement that $Q \Rightarrow P$, that is, the statement that "if Q, then P".

- There is no logical relatonship between a statement and its converse.
  - As we have seen, if P is the proposition that $x > 0$ and Q is the proposition that $x^2 > 0$, then it is certainly true that $P \Rightarrow Q$, but the converse $Q \Rightarrow P$ is false: x could be negative and still satisfy $x^2 > 0$.

- If a statement and its converse both hold, we express this by saying that "P if and only if Q," and denote this by $P \Leftrightarrow Q$.
  - For example, if P is the proposition that $x > 0$ and Q is the proposition that $x^3 > 0$, then we have $P \Leftrightarrow Q$.

2. Quantifiers and Negation

- There are two kinds quantifiers:
  - the **universal** or “for all” quantifier is used to denote that a property Π holds *for every* element a in some set A;
  - the **existential** or “there exists” quantifier is used to denote that the property holds for *at least one* element a in the set A.
- The **negation** of proposition P is its denial, $\sim P$.
- If the proposition P involves a universal quantifier, then its negation involves an existential quantifier: to deny the truth of a universal statement requires us to find just one case where the statement fails.
  - Let A be some set and let $\Pi(a)$ be some property defined for elements $a \in A$. Suppose P is the proposition of the form

$$\text{For all } a \in A, \text{ property } \Pi(a) \text{ holds.}$$

– Then, P is false if there is just a single element $a \in A$ for which the property $\Pi(a)$ does not hold. Thus, the negation of P is the proposition

There exists $a \in A,$ such that property $\Pi(a)$ does not hold.

- The negation of an existential quantifier involves a universal quantifier: to deny that there is at least one case where the proposition holds requires us to show that the proposition fails in every case.

  – That is, if Q is a proposition of the form

  There exists $b \in B,$ such that property $\Pi'(b)$ holds,

  its negation is the proposition

  For all $b \in B,$ property $\Pi'(b)$ does not hold.

- Example: Consider the following.
  - Given a real number x, let $\Pi(x)$ be the property that $x^2 > 0$.
  - Let P be the proposition that "Property $\Pi(x)$ holds for every real number x." In the language of quantifiers, we would express P as

$$\text{For every } x \in \mathbb{R},\ x^2 > 0.$$

  - P is negated if there is at least one real number whose square is not strictly positive. So the negation $\sim P$ is the statement

$$\text{There exists } x \in \mathbb{R} \text{ such that } x^2 \not> 0.$$

- When multiple quantifiers are involved in a statement, the situation gets a little more complicated.
  - If all the quantifiers in a given proposition are of the ame type (that is, they are all universal, or are all existential) the order of the quantifiers is immaterial.
    - For instance, the statement

$$\text{For all } x \in \mathbb{R}, \text{ for all } y \in \mathbb{R}, \ (x+y)^2 = x^2 + 2xy + y^2,$$

is the same as the statement

$$\text{For all } y \in \mathbb{R}, \text{ for all } x \in \mathbb{R}, \ (x+y)^2 = x^2 + 2xy + y^2.$$

– However, the order of the quantifiers becomes significant if quantifiers of different types are involved.

  ○ The statement

$$\text{For every } x > 0, \text{ there exists } y > 0 \text{ such that } y^2 = x$$

  is most definitely not the same as the statement that

$$\text{There exists } y > 0 \text{ such that for every } x > 0, \ y^2 = x.$$

  ○ In fact, while the first statement is true (it asserts essentially that every positive real number has a positive square root), the second is false (it claims that a single fixed real number is the square root of every positive number).

- The importance of the order of quantifiers makes it necessary to exercise caution in forming the negation of statements with multiple quantifiers, since the negation will also involve the use of multiple quantifiers.
  - Let $\Pi(a, b)$ denote a property defined on elements a and b in sets A and B, respectively. Consider the statement P

    For every $a \in A,$ there exists $b \in B$ such that $\Pi(a, b)$ holds.

  - The statement P will be falsified if there is even one $a \in A$ for which the property $\Pi(a, b)$ fails to hold, no matter what we take for the value of $b \in B$.
  - Thus, the negation of P is the statement $\sim P$ defined by

    There exists $a \in A$ such that for every $b \in B,$ $\Pi(a, b)$ fails.

  - We reiterate the importance of the order of quantifiers in forming this negation. The negation of P is *not* the statement

    For every $b \in B,$ there exists $a \in A$ such that $\Pi(a, b)$ fails.

3. Necessary vs. Sufficient Conditions

- $A \Rightarrow B$:
  - A implies B;
  - if A then B;
  - A only if B;
  - A is **sufficient** for B;
  - B is **necessary** for A.
- Example 1:
  - A: The natural number n is divisible by 6.
  - B: The natural number n is divisible by 3.
- Example 2:
  - A: A person is a father.
  - B: A person is a male.

- Example 3:
  - Necessary and sufficient conditions for *local maximum* and *minimum*.
- “$A \Rightarrow B$” $\equiv$ “not $B \Rightarrow$ not A”.
- $A \Leftrightarrow B$:
  - A if and only if B;
  - A is **necessary and sufficient** for B.

4. Types of Mathematical Proof

- A mathematical theorem can be formulated as an implication $A \Rightarrow B$ where
  - A represents a proposition called the *hypothesis* or the *premise*, and
  - B represents a proposition called the *conclusion*.
- One can prove such an implication in different ways.

1. **Direct Proof:**
   - Assume A;
   - Then try to show B.
2. **Indirect Proof:**

2.1 **Proof of Contrapositive:**

- Assume “not B”;
- Then try to show “not A”.

2.2 **Proof by Contradiction:**

– Assume "$A \Rightarrow$ not B";

– Then show that this leads to a *contradiction*.

- Example:

  Consider the propositions

$$A\text{: } 3x - x^2 \geq 0, \text{ and}$$

$$B\text{: } x \geq 0.$$

  Let us prove the implication $A \Rightarrow B$ by all the three types of mathematical proofs.

  – Direct Proof:

  - Assume A, that is, suppose that $3x - x^2 \geq 0$.

  $\Rightarrow 3x \geq x^2 \geq 0.$

  $\Rightarrow x \geq 0$, which is B.

– Proof of Contrapositive:

- Assume "not B", that is, suppose that $x < 0$.

$\Rightarrow 3x < 0$, and hence $3x - x^2 < 0$, which is "not A".

– Proof by Contradiction:

- Assume "$A \Rightarrow$ not B", which corresponds to the following proposition:

∘ There exists an x such that

$$3x - x^2 \geq 0, \text{ and } x < 0.$$

$\Rightarrow x^2 \leq 3x < 0$, that is, $x^2 < 0$, a contradiction.

3. **Proof by Induction:**

This type of proof can be used for propositions $A(n)$, where $n = k$, $k + 1$, ... , and k is a natural number.

Two steps:

– *Initial step*: Prove that the proposition is true for an initial natural number $n = k$, that is, $A(k)$ is true.

– *Inductive step*: Prove the implication $A(n) \Rightarrow A(n{+}1)$ for an arbitrary $n \in \{k, k+1, ...\}$.

- Example: Prove by induction that $A(n)$: $2^n > n$, where n is a natural number.

– Initial step: $A(1)$: $2^1 = 2 > 1$, that is, $A(1)$ is true.

– Inductive step: Need to show that

$$2^n > n \Rightarrow 2^{n+1} > n + 1.$$

- Multiplying both sides of $A(n)$ by 2 yields $2^{n+1} > 2n$.
- For $n \geq 1$, we have $2n \geq n + 1$.
- Combining the above two inequalities gives $2^{n+1} > n + 1$.