Linear Algebra:

Vectors

1. Vector Spaces

- Scalars / Numbers:

  In defining vector spaces, we consider the *field* to be the set of reals: $\Re$. The elements of $\Re$ are called *scalars* or *numbers*.

- An n*-vector* x is an ordered set of n numbers $(x_1, x_2, ..., x_n)$.
  - 'Ordered' $\Rightarrow$ the ordering of the numbers matters.
  - The set of all n-vectors is called n*-space*, denoted by $\Re^n$.

- **Some Special Vectors:**
  - **Sum Vector:** The vector all of whose coordinates are 1.
  - **Null Vector:** The vector all of whose coordinates are 0.
  - **Unit Vectors:** The i-th unit vector is the vector whose i-th coordinate is 1, and whose other coordinates are 0.
    - We denote the i-th unit vector by e^i.

- **Vector Operations:**
  - Two n-vectors, $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$, are said to be *equal* (written $x = y$) if $x_i = y_i$, for $i = 1, 2, ..., n$.
  - **Addition:** If x and y are n-vectors, their sum $x + y$ is the n-vector whose i-th element is $x_i + y_i$, $i = 1, 2, ..., n$.
  - **Scalar Multiplication:** If x is an n-vector and λ is a number, the product λx is the vector whose i-th element is λx_i, $i = 1, 2, ..., n$.

- Given these two definitions, a number of properties follow immediately.

- **Properties of Vector Addition:**
  - Commutative Law: $x + y = y + x$.
  - Associative Law: $(x + y) + z = x + (y + z)$.
  - Law of Subtraction: For every x and y, there is z such that $x + z = y$.

- **Properties of Scalar Multiplication:**
  - Vector Distributive Law: $\lambda\,(x + y) = \lambda x + \lambda y$.
  - Scalar Distributive Law: $(\lambda + \mu)\,x = \lambda x + \mu x$.
  - Scalar Associative Law: $\lambda(\mu x) = (\lambda\mu)x$.
  - Identity Law: $1x = x$ (here 1 is the scalar 'one').

- **Vector Spaces:**

  The properties listed above may be taken as axioms for an abstract algebraic system. Such systems are called *vector spaces*.

  - The vector space that we will study consists of
    - the field $\Re$,
    - the n-space $\Re^n$,
    - the operations of addition and scalar multiplication.
  - We will, as a shorthand, refer to this vector space by $\Re^n$.

2. Linear Dependence of Vectors

- A set of vectors x^1, x^2, ..., x^m (here $x^i = \left(x_1^i, x_2^i, ..., x_n^i\right) \in \Re^n$ is a vector) is *linearly dependent* if there exist numbers $\lambda_1, \lambda_2, ..., \lambda_m$, *not all zero*, such that

$$\lambda_1 x^1 + \lambda_2 x^2 + ... + \lambda_m x^m = 0 \text{ (the zero vector in } \Re^n\text{).}$$

- A set of vectors is called *linearly independent* if the vectors are *not* linearly dependent.

#1. Are the following two vectors in $\Re^2$, $x^1 = (1, 2)$ and $x^2 = (-3, -6)$, linearly dependent?

#2. Are the following two vectors in $\Re^2$, $x^1 = (1, 2)$ and $x^2 = (0, 1)$, linearly dependent?

#3. Are the following three vectors in $\Re^2$, $x^1 = (1, 1)$, $x^2 = (2, 3)$ and $x^3 = (3, 7)$ linearly dependent?

#4. Show that the unit vectors in $\Re^2$, $e^1 = (1, 0)$ and $e^2 = (0, 1)$ are linearly independent.

- A vector $y \in \Re^n$ can be written as a *linear combination* of the set of vectors x^1, x^2, ..., x^m in $\Re^n$ if there are numbers $\lambda_1, \lambda_2, ..., \lambda_m$ such that

$$y = \lambda_1 x^1 + \lambda_2 x^2 + ... + \lambda_m x^m.$$

#5. Express $y = (0, 0)$ as a linear combination of $x^1 = (1, 2)$ and $x^2 = (1, 0)$.

#6. Express $y = (3, 2)$ as a linear combination of $x^1 = (1, 0)$ and $x^2 = (0, 1)$.

#7. Show that any vector in $\Re^n$ can be written as a linear combination of the unit vectors e^1, e^2, ..., e^n in $\Re^n$.

- **Theorem 1 (Fundamental Theorem on Vector Spaces):**

  If each of the $(m + 1)$ vectors $y^0, y^1, ..., y^m$ in $\Re^n$ can be expressed as a linear combination of the m vectors x^1, x^2, ..., x^m in $\Re^n$, then the vectors $y^0, y^1, ..., y^m$ are linearly dependent.

  – Proof: See Gale (1960).

- **Corollary 1:** *Any set of* $(n+1)$ *vectors in* $\Re^n$ *are linearly dependent*.
  - Proof: To be discussed in class.

- **Corollary 2:** *Any system of* n *homogenous linear equations in* $(n+1)$ *unknowns*,

$$\begin{aligned} a_{10}x_0 + a_{11}x_1 + ... + a_{1n}x_n &= 0, \\ a_{20}x_0 + a_{21}x_1 + ... + a_{2n}x_n &= 0, \\ &\vdots \\ a_{n0}x_0 + a_{n1}x_1 + ... + a_{nn}x_n &= 0, \end{aligned}$$

  has a non-zero solution.
  - Proof: To be discussed in class.

3. Rank and Basis

- **Rank:** Let S be a subset of the vector space $\Re^n$. The *rank* of S is the maximum number of linearly independent vectors which can be chosen from S.
  - Note that rank$(S) \leq n$ since $(n+1)$ vectors in S are linearly dependent by Corollary 1.

- **Basis:** Let S be a subset of the vector space $\Re^n$. If r is the rank of S, then any set of r linearly independent vectors in S is called *a basis* of S.

#8. Consider $S_1 = \{(1,0), (0,1)\}$ in $\Re^2$. What is the rank of S_1?

#9. Consider $S_2 = \{(0,0)\}$ in $\Re^2$. What is the rank of S_2?

#10. (a) Are the following three vectors in $\Re^3$, $x = (1,0,0)$, $y = (1,1,0)$ and $z = (1,1,1)$ linearly dependent?

(b) Consider $S_3 = \left\{(x_1, x_2, x_3) \in \Re^3 \colon x_1 = 1\right\}$. What is the rank of S_3? Find out a basis of S_3.

- **Corollary 3:** $\Re^n$ *has rank* n.
  - Proof: To be discussed in class.

- **Theorem 2: (Basis Theorem):**

  Suppose $x^1, x^2, ..., x^m$ *are linearly independent vectors in the set* S *(in* $\Re^n$).

  (i) *If every vector* y *in* S *can be expressed as a linear combination of* $x^1, x^2, ..., x^m$, *then* $(x^1, x^2, ..., x^m)$ *is a basis of* S.

  (ii) *If* $(x^1, x^2, ..., x^m)$ *is a basis of* S, *then every vector* y *in* S *can be expressed as a linear combination of* $x^1, x^2, ..., x^m$.

  - Proof: To be discussed in class.

4. Inner Product and Norm

- **Inner Product:** If x and y are two vectors in $\Re^n$, then their inner product is

$$xy = x_1 y_1 + x_2 y_2 + ... + x_n y_n.$$

  – Note that inner product is a scalar (not a vector).

  – $xe^i = x_i$, $i = 1, 2, ..., n$, where e^i is the i-th unit vector.

- **Properties of Inner Product:**

  – Commutative Law: $xy = yx$

  – Mixed Associative Law: $(\lambda x)\, y = \lambda\, (xy)$ [λ is a scalar]

  – Distributive Law: $(x + y)\, z = xz + yz$

  – $x^2 \equiv xx = 0$ if and only if $x = 0$.

- **Norm of a Vector:** The (Euclidean) norm of a vector x in $\Re^n$ is defined as

$$\|x\| = (xx)^{\frac{1}{2}} = \left(x_1^2 + x_2^2 + ... + x_n^2\right)^{\frac{1}{2}}.$$

- **Properties of Norm:**
  - $\|x\| = 0$ if and only if $x = 0$
  - $\|\lambda x\| = |\lambda| \,.\, \|x\|$
  - $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality for norms)
  - If $xy = 0$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ (Pythagoras Theorem).

- **Orthogonal Vectors:** Two vectors x and y are called *orthogonal* if their inner product is zero, that is, if $xy = 0$.

- **Orthonormal Vectors:** Two vectors x and y are called *orthonormal* if

  (a) x and y are orthogonal, and

  (b) $\|x\| = \|y\| = 1$.

References

- Must read the following sections from the textbook:

  Sections 10.1 – 10.4 (pages 199 – 214),

  Sections 11.1 – 11.3 (pages 237 – 249).

- Much of this material is standard in texts on linear algebra, like

1. Hadley, G., *Linear Algebra*, Massachusetts: Addison-Wesley, 1964 (chapter 2),
2. Hohn, Franz E., *Elementary Matrix Algebra*, New Delhi: Amerind, 1971 (chapters 4, 5).

- A good exposition can also be found in

3. Gale, David, *The Theory of Linear Economic Models*, New York: McGraw-Hill, 1960 (chapter 2).

   – You will find a proof of the fundamental theorem on vector spaces (by using mathematical induction) in Gale's book.