Linear Algebra:

Matrices

1. Matrix Algebra

- An $m \times n$ *matrix* is a rectangular array of numbers a_{ij}, $i = 1, 2, ..., m$; $j = 1, 2, ..., n$:

$$A_{m\times n} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

  – The vector $A_i = (a_{i1}, a_{i2}, ..., a_{in}) \in \Re^n$ is called the i-th *row vector* of $A_{m\times n}$.

  – The vector

$$A^j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

  is called the j-th *column vector* of $A_{m\times n}$.

  – The matrix $A_{m\times n}$ has m row vectors $A_1, A_2, ..., A_m$ and n column vectors $A^1, A^2, ..., A^n$.

– An $m \times n$ matrix can be interpreted as an ordered set of m row vectors $(A_1, A_2, ..., A_m)$, or as an ordered set of n column vectors $\left(A^1, A^2, ..., A^n\right)$.

$\Rightarrow$ The operations on matrices follow from the operations on vectors.

- **Matrix Operations:**

– Equality of Matrices:

Two $m \times n$ matrices A and B are *equal* (written $A = B$) if $a_{ij} = b_{ij}$, $i = 1, 2, ..., m$; $j = 1, 2, ..., n$.

– Addition of Matrices:

If A and B are $m \times n$ matrices, their sum, $A + B$, is an $m \times n$ matrix, $(a_{ij} + b_{ij})$.

– Multiplication of a Matrix by a Scalar:

If A is an $m \times n$ matrix, and λ is a scalar, their product, λA, is an $m \times n$ matrix (λa_{ij}).

– Matrix Multiplication:

Let A be an $m \times n$ matrix, and B an $n \times r$ matrix. Then B can be *premultiplied* by A, or A can be *postmultiplied* by B. The matrix product, AB, is an $m \times r$ matrix given by $\left(\sum_{k=1}^{n} a_{ik} b_{kj} \right), i = 1, 2, ..., m; j = 1, 2, ..., r.$

- **Properties of Matrix Operations:**

– Commutative Law for Addition: $A + B = B + A;$

– Associative Law for Addition: $(A + B) + C = A + (B + C);$

– Associative Law for Multiplication: $(AB)C = A(BC) = ABC;$

– Distributive Laws: $A(B + C) = AB + AC;$
$(B + C)A = BA + CA.$

- **Words of Caution:**

Some results which are true for real numbers are not necessarily true for matrices.

– AB is not necessarily equal to BA.
#1. Give an example.

– $AB = 0$ is possible with neither A nor B being the null matrix.
#2. Give an example.

– $CD = CE$ with C not a null matrix is possible without D and E being equal.
#3. Give an example.

- **Transpose of a Matrix:**

  If A is an $m \times n$ matrix, then the $n \times m$ matrix B defined by

$$b_{ij} = a_{ji},\ i = 1, 2, ..., n;\ j = 1, 2, ..., m$$

  is called the *transpose* of A, and is denoted by A^T.

- **Properties of Transpose:**

  – $\left(A^T\right)^T = A;$

  – $(A + B)^T = A^T + B^T;$

  – $(AB)^T = B^T A^T.$

- **Some Special Matrices:**
  - **Square Matrix:** An $m \times n$ matrix is a square matrix if $m = n$.
  - **Symmetric Matrix:** An $n \times n$ matrix is symmetric if $a_{ij} = a_{ji}$ for $i \neq j$.
  - **Diagonal Matrix:** An $n \times n$ matrix is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.
  - **Identity Matrix:** An $n \times n$ matrix is an identity matrix (denoted by I_n or I) if

$$\begin{aligned} a_{ii} &= 1 \text{ for } i = 1, 2, ..., n, \\ a_{ij} &= 0 \text{ for } i \neq j. \end{aligned}$$

  - **Null Matrix:** An $m \times n$ matrix is a null matrix (denoted by 0) if

$$a_{ij} = 0, \; i = 1, 2, ..., n; \; j = 1, 2, ..., m.$$

- **Properties of Identity Matrix and Null Matrices:**
  - $A + 0 = A = 0 + A$
  - $0A = 0 = A0$
  - $IA = A = AI$

2. Rank of a Matrix

- Let A be an $m \times n$ matrix.
  - Consider the set of row vectors: $\{A_1, A_2, ..., A_m\}$.
    Row rank of $A =$ rank $\{A_1, A_2, ..., A_m\}$.
  - Consider the set of column vectors: $\left\{A^1, A^2, ..., A^n\right\}$.
    Column rank of $A =$ rank $\left\{A^1, A^2, ..., A^n\right\}$.

- **Theorem 1 (Rank Theorem):**

  For any $m \times n$ matrix A, row rank of $A =$ column rank of A.
  - Proof: See Gale (1960).
  - In view of the Rank Theorem, simply refer to the rank of A (denoted by *rank* (A)).

- **Non-Singular Matrix:**
  - If A is an $n \times n$ matrix, then A is called *non-singular* if *rank* $(A) = n$.
  - A is called *singular* if *rank* $(A) < n$.

3. Inverse of a Matrix

- Let A be an $n \times n$ matrix. If B is an $n \times n$ matrix such that

$$AB = I, \text{ and } BA = I,$$

then A is *invertible* and B is called the *inverse* of A (denoted by A^{-1}).

- **Properties of Inverse:**
  - $\left(A^{-1}\right)^{-1} = A;$
  - $(AB)^{-1} = B^{-1}A^{-1};$
  - $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$

- **Relationship between Invertible and Non-Singular Matrices:**

  – *If A is an $n \times n$ matrix which is invertible*, *then A is non-singular*.

  - **Proof:** To be discussed in class.

  - **Hints:**

    1. Method of contradiction: Assume that A is invertible, but singular, and then show that there will be a contradiction.

    2. Use the implication of the definition of 'singularity' of A working through the linear dependence/independence of the column/row vectors of A.

  – *If A is an $n \times n$ matrix which is non-singular*, *then A is invertible*.

  - **Proof:** To be discussed in class.

  - **Hints:**

    Step 1: A is non-singular $\Rightarrow$ rank $(A) = n \Rightarrow$ column vectors of A, $\left(A^1, A^2, ..., A^n\right)$, are linearly independent.

$\Rightarrow \left(A^1, A^2, ..., A^n\right)$ is a basis of $\Re^n$ so that any vector in $\Re^n$ can be expressed as a linear combination of $\left(A^1, A^2, ..., A^n\right)$ (by the Basis Theorem).

– Observe that the column vectors of I (the identity matrix) are the unit vectors in $\Re^n$.

– Think about how you can construct a matrix B with column vectors $\left(B^1, B^2, ..., B^n\right)$ such that $AB = I$.

Step 2: Same as Step 1 for the row vectors of A, $(A_1, A_2, ..., A_n)$.

– Think about how you can construct a matrix C with row vectors $(C_1, C_2, ..., C_n)$ such that $CA = I$.

Step 3: Summarizing, you have constructed matrices B and C such that $AB = I$ and $CA = I$. Now show that $B = C$.

References

- Must read the following sections from the textbook:

  Sections 8.1 – 8.2 (pages 153 – 162).

- This material on matrix algebra can be found in standard texts like

1. Hohn, Franz E., *Elementary Matrix Algebra*, New Delhi: Amerind, 1971 (chapters 1, 6),
2. Hadley, G., *Linear Algebra*, Massachusetts: Addison-Wesley, 1964 (chapter 3).

- A good discussion on the rank of a matrix is in

3. Gale, David, *The Theory of Linear Economic Models*, New York: McGraw-Hill, 1960 (chapter 2).
   – A proof of the “rank theorem” can be found here.

- You will also find a good exposition of matrix algebra in

4. Dorfman, Robert, Paul A. Samuelson and Robert M. Solow, *Linear Programming and Economic Analysis*, New York: McGraw-Hill, 1958 (Appendix B).