Linear Algebra:

Simultaneous Linear Equations

1. System of Linear Equations

- Consider a system of m simultaneous linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= c_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= c_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= c_m, \end{aligned} \tag{*}$$

– In matrix-vector notation, we can write this system as

$$Ax = c, \text{ where}$$

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \; x_{n\times 1} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; c_{m\times 1} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}.$$

- The system of equations (*) is called *homogeneous* if $c = 0$, and *non-homogeneous* if $c \neq 0$.

- In analyzing a system of linear equations (*), the following questions naturally arise:

(i) Existence: Does there exist a solution to (*)?

(ii) Uniqueness: If there exists a solution to (*), is it unique?

(iii) Computation: If there exists a solution to (*), how can we find such a solution?

2. Existence of Solutions

- If the system of equations (*) is homogeneous, there is always a trivial solution, namely $x = 0$.

#1. Give an example to illustrate that if the system of equations is non-homogeneous, then, in general, a solution may not exist.

- In general, given the system of equations (*), we would like to know, given A and c, whether there is a solution to (*).

- Consider the system $Ax = c$.
  - The $m \times (n+1)$ matrix

$$A_c = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & c_m \end{pmatrix}$$

    is known as the *augmented matrix*.

– Note that the augmented matrix A_c can be interpreted as an ordered set of $n+1$ column vectors $\left(A^1, A^2, ..., A^n, c\right)$.

- **Theorem 1:**

Let A be an $m \times n$ matrix and c be a vector in $\Re^m$. Then the system of equations $Ax = c$ has a solution if and only if

$$\textit{rank } (A) = \textit{ rank } (A_c) .$$

– Proof: To be discussed in class.

– Hints:

1. Keep in mind that the augmented matrix A_c can be interpreted as an ordered set of $n+1$ column vectors $\left(A^1, A^2, ..., A^n, c\right)$.

2. Recognize that $Ax = c$ has a solution implies that c can be expressed as a linear combination of the column vectors of A, $\left(A^1, A^2, ..., A^n\right)$.

3. Uniqueness of Solutions

- **Theorem 2:**

  Let A be an $m \times n$ matrix and c be a vector in $\Re^m$. Then the system of equations $Ax = c$ has a unique solution if and only if

  $$\textit{rank } (A) = \textit{ rank } (A_c) = n.$$

  – Proof: To be discussed in class.

  – Hints:

  1. Step1: To show that if there exists a unique solution, then rank $(A) =$ rank $(A_c) = n$.

     - Given that there exists a unique solution, call it x^* (that is, $Ax^* = c$). Then, by Theorem 1, rank $(A) =$ rank (A_c) .

     - It remains to show that rank $(A) = n$.

     - If rank $(A) \neq n$, then it must be that rank $(A) < n$.

$\Rightarrow \left(A^1, A^2, ..., A^n\right)$ is a set of linearly dependent vectors.

$\Rightarrow$ There exists a vector $\lambda = (\lambda_1, \lambda_2, ...\lambda_n)$, $\lambda \neq 0$, such that $A\lambda = 0$.

- Now try to find out a contradiction to the fact that there exists a unique solution.

2. Step 2: To show that if rank (A) = rank $(A_c) = n$, then there exists a unique solution.

4. Calculation of Solutions

- Consider the case of n linear equations in n unknowns. Let A be an $n \times n$ matrix, and c be a vector in $\Re^n$. Consider the system of equations given by

$$Ax = c.$$

#2. Prove that if rank $(A) = n$, then rank $(A_c) = n$.

– In view of this and Theorem 2, we have to check only if rank $(A) = n$ to see whether a unique solution exists.

- rank $(A) = n$, $\Rightarrow A$ is non-singular, $\Rightarrow A$ is invertible,

$\Rightarrow$ Premultiplying $Ax = c$ by A^{-1} we get

$$A^{-1}Ax = A^{-1}c, \;\; \Rightarrow Ix = A^{-1}c, \;\; \Rightarrow x = A^{-1}c.$$

So $x = A^{-1}c$ is *the* solution.

– In terms of calculating this solution, it remains to learn how to calculate A^{-1}, the inverse of a non-singular matrix.

- This leads us naturally into the study of *determinants*.

5. Determinants

- Let A be an $n \times n$ matrix. We can associate with A a number, denoted by $|A|$, called the determinant of A.
- The determinant of the $n \times n$ matrix is defined recursively as follows:

(1) For a 1×1 matrix, which is a number, we define the determinant to be the number itself.

(2) For any $m \times m$ matrix A ($m \geq 2$), the *cofactor* A_{ij} of the element a_{ij} is $(-1)^{i+j}$ times the determinant of the submatrix obtained from A by deleting row i and column j. The *determinant* of the $m \times m$ matrix is then given by

$$|A| = \sum_{j=1}^{m} a_{1j} A_{1j}.$$

– Thus using (2) and knowing (1), the determinant of a 2×2 matrix is

$$a_{11}a_{22} - a_{12}a_{21}.$$

– This information can then be used in (2) again to obtain the determinant of a 3×3 matrix:

$$\begin{aligned} & a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ = \ & a_{11}\left(a_{22}a_{33} - a_{32}a_{23}\right) - a_{12}\left(a_{21}a_{33} - a_{31}a_{23}\right) + a_{13}\left(a_{21}a_{32} - a_{31}a_{22}\right). \end{aligned}$$

– This procedure can be continued to obtain the determinant of any $n \times n$ matrix.

– It is implicit in the definition of $|A|$ that the expansion is done by the first row. However, it can be shown that for every $i = 1, 2, ..., n$,

$$|A| = \sum_{j=1}^{n} a_{ij}A_{ij}$$

so that expansion by any row will give the same result.

– Expansion by any column will also give the same result, that is, for every $j = 1, 2, ..., n$,

$$|A| = \sum_{i=1}^{n} a_{ij}A_{ij}.$$

• Properties of Determinants:

(i) $|A| = \left|A^T\right|$.

(ii) The multiplication of any one row by a scalar k will change the determinant k-fold.

(iii) The interchange of any two rows will alter the sign, but not the numerical value, of the determinant.

(iv) If one row is a multiple of another row, the determinant is zero.

(v) The addition of a multiple of any row to another row will leave the determinant unaltered.

(vi) The expansion of a determinant by “alien” cofactors yields a value of zero. That is,

$$\sum_{j=1}^{n} a_{ij} A_{kj} = 0, \text{ if } i \neq k.$$

[Here the expansion is by the i-th row, using cofactors of the k-th row].

(vii) $|AB| = |A|\,|B|$.

– Properties (ii) – (v) hold if the word “row” is replaced uniformly by “column” in each statement.

– The proofs will be discussed in class.

6. Matrix Inversion

- **Theorem 3:**

  Let A be an $n \times n$ matrix. Then A is invertible if and only if $|A| \neq 0.$ Furthermore, in case A is invertible, $\left|A^{-1}\right| = |A|^{-1}$.

  – Proof: To be discussed in class.

  – Hints:

  1. Step1: To show that if A is invertible then $|A| \neq 0.$ Use Property (vii).
  2. Step 2: To show that if $|A| \neq 0$ then A is invertible.

     - It is equivalent to show that A is nonsingular.

     - Suppose not. Then the column vectors of $A,$ $\left(A^1, A^2, ..., A^n\right)$, are linearly dependent.

     $\Rightarrow$ One column vector can be expressed as a linear combination of the other column vectors.

     - Now use Property (v) to show that a contradiction arises.

- It follows that for an $n \times n$ matrix A, the following statements are equivalent:
  - A is invertible;
  - A is non-singular;
  - $|A| \neq 0$;
  - rank $(A) = n$;
  - Column vectors of A are linearly independent.
- **Cofactor Matrix:** For an $n \times n$ matrix A, we define the cofactor matrix of A to be the $n \times n$ matrix given by

$$C = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}.$$

- **Adjoint Matrix:** The transpose of C is called the adjoint of A, and denoted by *adj* A, that is, $adj\ A = C^T$.

- By the rules of matrix multiplication,

$$
\begin{aligned}
AC^T &= \begin{pmatrix} \sum_{j=1}^{n} a_{1j}A_{1j} & \sum_{j=1}^{n} a_{1j}A_{2j} & \cdots & \sum_{j=1}^{n} a_{1j}A_{nj} \\ \sum_{j=1}^{n} a_{2j}A_{1j} & \sum_{j=1}^{n} a_{2j}A_{2j} & \cdots & \sum_{j=1}^{n} a_{2j}A_{nj} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} a_{nj}A_{1j} & \sum_{j=1}^{n} a_{nj}A_{2j} & \cdots & \sum_{j=1}^{n} a_{nj}A_{nj} \end{pmatrix} \\
&= \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |A| \end{pmatrix} \\
&= |A|\, I.
\end{aligned}
$$

– Note that this calculation is valid for any $n \times n$ matrix, invertible or otherwise.

- **Inverse of a Matrix:**

  – If A is invertible, there is A^{-1} such that $AA^{-1} = A^{-1}A = I$.

  – Consider the relation $AC^T = |A|\, I$. Premultiplying by A^{-1} we get

$$C^T = |A|\, A^{-1}.$$

  – Since A is invertible, we have $|A| \neq 0$. Then we can divide by $|A|$ and get

$$A^{-1} = \frac{C^T}{|A|} = \frac{adj\ A}{|A|}.$$

  - This gives a formula for computing the inverse of an invertible matrix A in terms of the determinant and cofactors of A.

7. Cramer's Rule

- Recall that we wanted to calculate *the unique solution* of a system of n equations in n unknowns given by

$$Ax = c$$

where A is an $n \times n$ matrix and c is a vector in $\Re^n$.

– We found that *the unique solution* is given by

$$x = A^{-1}c.$$

– Using the formula for A^{-1} derived above we conclude that

$$x = A^{-1}c = \frac{adj\ A}{|A|}c.$$

- Let us evaluate x_i using the above relationship:

$$\begin{aligned} x_i &= e_i x = e_i \frac{adj\ A}{|A|} c \\ &= \frac{(A_{1i}\ A_{2i}\ ...\ A_{ni})\, c}{|A|} \\ &= \frac{(c_1 A_{1i} + c_2 A_{2i} + ... + c_n A_{ni})}{|A|} \\ &= \frac{1}{|A|} \begin{vmatrix} a_{11} & \cdots & a_{1,\,i-1} & c_1 & a_{1,\,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,\,i-1} & c_2 & a_{2,\,i+1} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,\,i-1} & c_n & a_{n,\,i+1} & \cdots & a_{nn} \end{vmatrix}. \end{aligned}$$

- This gives us an easy way to compute the solution of x_i:
  - Replace the i-th column of A by the vector c and find the determinant of this matrix.
  - Dividing this number by the determinant of A yields the solution of x_i.
  - This rule is known as *Cramer's Rule*.

References

- Must read the following sections from the textbook:

  Section 9.1 (pages 188 – 194) [You can skip Theorems 9.1 and 9.2.],

  Section 9.2 (pages 194 – 197).

- This material is based on

1. Hadley, G., *Linear Algebra*, Massachusetts: Addison-Wesley, 1964 (chapters 3, 5),
2. Hohn, Franz E., *Elementary Matrix Algebra*, New Delhi: Amerind, 1971 (chapters 2, 6, 7),
3. Gale, David, *The Theory of Linear Economic Models*, New York: McGraw-Hill, 1960 (chapter 2).

- Some of this material is also covered in

4. Dorfman, Robert, Paul A. Samuelson and Robert M. Solow, *Linear Programming and Economic Analysis*, New York: McGraw-Hill, 1958 (Appendix B).