Real Analysis:

Convex Analysis

1. Convex Sets

- **Line Segment:** If $x,\, y \in \Re^n$, the *line segment joining* x *and* y is given by the set of points

$$\{z \in \Re^n : z = \theta x + (1-\theta)\, y,\ \text{for some } 0 \le \theta \le 1\}\,.$$

- **Convex Set:** A set $S \subset \Re^n$ is a convex set if for every $x,\, y \in S$, the line segment joining x and y is contained in S.
  - For example, the set of points $\left\{(x,y) \in \Re^2\text{: } x^2 + y^2 \le 1\right\}$ is a convex set.
  - But the set of points $\left\{(x,y) \in \Re^2\text{: } x^2 + y^2 = 1\right\}$ is *not* a convex set.

#1. **Examples:** Suppose two sets S_1 and S_2 are convex in $\Re^n$.

(a) Prove that the intersection of S_1 and S_2, that is, the set

$$S = \{z \in \Re^n\text{: } z \in S_1 \text{ and } z \in S_2\}\,,$$

is a convex set in $\Re^n$.

(b) Prove that the sum of S_1 and S_2, that is, the set

$$S = \{z \in \Re^n \colon z = x + y, \text{ where } x \in S_1 \text{ and } y \in S_2\},$$

is a convex set in $\Re^n$.

(c) Prove that the Cartesian product of S_1 and S_2, that is, the set

$$S = \left\{z \in \Re^{2n} \colon z = (x, y), \text{ where } x \in S_1 \text{ and } y \in S_2\right\},$$

is a convex set in $\Re^n$.

(d) Is the union of S_1 and S_2, that is, the set

$$S = \{z \in \Re^n \colon z \in S_1 \text{ or } z \in S_2\},$$

a convex set in $\Re^n$?

- **Convex Combination:** A vector $y \in \Re^n$ is said to be a *convex combination* of the vectors $x^1, x^2, ..., x^m \in \Re^n$ if there exist m non-negative real numbers $\theta_1, \theta_2, ...\theta_m$ such that

$$\text{(i)} \sum_{i=1}^{m} \theta_i = 1, \text{ and (ii) } y = \sum_{i=1}^{m} \theta_i x^i.$$

  – A convex set $A \subset \Re^n$ can then be redefined as a set such that for every *two* vectors in the set A, all convex combinations of these two vectors are also in the set A.

  – It can be shown that in the above statement "two" can be replaced by "m" where m is any integer exceeding 1.

- **Proposition 1:**

  A set $S \subset \Re^n$ *is convex if and only if for any integer* $m > 1$ *and for any* m *vectors in* S, *every convex combination of these* m *vectors is in* S.

  – Proof: To be discussed in class.

  – Hints: Proof by induction.

    - Step 1: Show that the property is true for $m = 2$.

    - Step 2: Assume that the property is true for $m = k$; then show that it is also true for $m = k + 1$.

2. Continuous & Differentiable Functions on Convex Sets

- We now provide three very useful theorems on continuous and differential functions on convex sets.

- **Theorem 1 (Intermediate Value Theorem):**

  Suppose A *is a convex subset of* $\Re^n$, *and* $f : A \to \Re$ *is a continuous function on* A. *Suppose* x^1 *and* x^2 *are in* A, *and* $f\left(x^1\right) > f\left(x^2\right)$. *Then, given any* $c \in \Re$ *such that* $f\left(x^1\right) > c > f\left(x^2\right)$, *there is* $0 < \theta < 1$ *such that* $f\left(\theta x^1 + (1-\theta)\, x^2\right) = c$.

#2. Example: Suppose $X = [a, b]$ is a closed interval in $\Re$ (with $a < b$), and $f : X \to \Re$ is a continuous function. Use Weierstrass Theorem and the Intermediate Value Theorem to prove that $f(X)$ is itself a closed interval.

- **Theorem 2 (Mean Value Theorem):**

  Suppose A *is an open convex subset of* $\Re^n$, *and* $f : A \to \Re$ *is continuously differentiable on* A. *Suppose* x^1 *and* x^2 *are in* A. *Then there is* $0 \leq \theta \leq 1$ *such that*

$$f\left(x^2\right) - f\left(x^1\right) = \left(x^2 - x^1\right) \cdot \nabla f\left(\theta x^1 + (1-\theta)\, x^2\right).$$

#3. Example: We say that $f : A \to \Re$ (where $A \subset \Re$) is an *increasing function* if whenever $x^1 > x^2$, we have $f\left(x^1\right) > f\left(x^2\right)$.

(a) Suppose that $g : A \to \Re$, $A \subset \Re$, A is open and convex, and $g'(x) > 0$, for all $x \in A$. Prove, using the Mean Value Theorem, that g is an increasing function.

(b) Suppose g is an increasing function on A. Does it follow that $g'(x) > 0$, for all $x \in A$?

- **Theorem 3 (Taylor's Expansion upto Second Order):**

Suppose A is an open convex subset of $\Re^n$, and $f : A \to \Re$ is twice continuously differentiable on A. Suppose x^1 and x^2 are in A. Then there is $0 \leq \theta \leq 1$ such that

$$f\left(x^2\right) - f\left(x^1\right) = \left(x^2 - x^1\right) \cdot \nabla f\left(x^1\right) + \frac{1}{2}\left(x^2 - x^1\right) \cdot H_f\left(\theta x^1 + (1-\theta)\, x^2\right) \cdot \left(x^2 - x^1\right).$$

3. Concave Functions

- Let A be a convex set in $\Re^n$. Then $f : A \to \Re$ is a *concave function* (on A) if for all $x^1, x^2 \in A$, and for all $0 \le \theta \le 1$,

$$f\left(\theta x^1 + (1-\theta)\, x^2\right) \ge \theta f\left(x^1\right) + (1-\theta)\, f\left(x^2\right).$$

  – The function f is *strictly concave* on A if

$$f\left(\theta x^1 + (1-\theta)\, x^2\right) > \theta f\left(x^1\right) + (1-\theta)\, f\left(x^2\right)$$

  whenever $x^1, x^2 \in A$, $x^1 \ne x^2$, and $0 < \theta < 1$.

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  whenever $x^1, x^2 \in A$, $x^1 \ne x^2$, and $0 < \theta < 1$.

- **Relation between Concave Functions and Convex Sets:**

  The relation between concave function and convex sets is given by the following results.

- **Theorem 4:**

  Suppose A is a convex set in $\Re^n$ and $f : A \to \Re$. Then f is a concave function if and only if the set $\{(x, \alpha) \in A \times \Re : f(x) \geq \alpha\}$ is a convex set in $\Re^{n+1}$.

  – Proof: To be discussed in class.

- **Theorem 5:** *Suppose A is a convex set in $\Re^n$ and $f : A \to \Re$ is a concave function. Then, for every $\alpha \in \Re$, the set $S(\alpha) = \{x \in A : f(x) \geq \alpha\}$ is a convex set in $\Re^n$.*

  – Proof: To be discussed in class.

- A result on concave functions which parallels Proposition 1 on convex sets can now be noted. It is known as Jensen's Inequality, and is a very useful tool in convex analysis.

- **Proposition 2 (Jensen's Inequality):**

  Suppose A is a convex set in $\Re^n$ and $f : A \to \Re$ is a concave function. Then, *for any integer $m > 1$*,

$$f\left(\sum_{i=1}^{m} \theta_i x^i\right) \geq \sum_{i=1}^{m} \theta_i f\left(x^i\right)$$

  whenever $x^1, x^2, ..., x^m \in A$, $(\theta_1, \theta_2, ..., \theta_m) \in \Re_+^m$ and $\sum_{i=1}^{m} \theta_i = 1$.

  – Proof: To be discussed in class.

  – Hints: Proof by induction.

    - Step 1: Show that the property is true for $m = 2$.

    - Step 2: Assume that the property is true for $m = k$; then show that it is also true for $m = k + 1$.

- **Continuity of Concave Functions:**

  In general, if A is a convex set in $\Re^n$ and $f : A \to \Re$ is concave on A, then f need not be continuous on A.

#4. Give an example to illustrate the above statement.

  – But if f is an *open* convex set in $\Re^n$ and $f : A \to \Re$ is concave on A, then one can show that f is continuous on A.

- **Theorem 6:**

  Suppose A is an open convex set in $\Re^n$ and $f : A \to \Re$ is a concave function on A. Then f is a continuous function on A.

  – Proof: Homework.

- **Differentiable Concave Functions:**

  In general, if A is an open convex set in $\Re^n$ and $f : A \to \Re$ is concave on A, then f need not be differentiable on A.

#5. Give an example to illustrate the above statement.

  – If f is continuously differentiable on A, then a convenient characterization for f to be concave on A can be given in terms of the gradient vector of f.

- **Theorem 7:**

  Suppose $A \subset \Re^n$ is an open convex set, and $f : A \to \Re$ is continuously differentiable on A. Then f is concave on A if and only if for all x^1 and x^2 in A

$$f\left(x^2\right) - f\left(x^1\right) \leq \nabla f\left(x^1\right) \cdot \left(x^2 - x^1\right).$$

  – Proof: To be discussed in class.

- **Corollary 1:** *Suppose $A \subset \Re^n$ is an open convex set*, *and $f : A \to \Re$ is continuously differentiable on A. Then f is concave on A if and only if for all x^1 and x^2 in A*

$$\left(\nabla f\left(x^2\right) - \nabla f\left(x^1\right)\right) \cdot \left(x^2 - x^1\right) \leq 0.$$

- It is interesting to note that a characterization of *strictly* concave functions can be given by replacing the weak inequalities in Theorem 7 and Corollary 1 with strict inequalities (for x^1, x^2 in A with $x^1 \neq x^2$).

- **Theorem 8:**

  Suppose $A \subset \Re^n$ is an open convex set, *and $f : A \to \Re$ is continuously differentiable on A. Then f is strictly concave on A if and only if for all x^1*, *x^2 in A with $x^1 \neq x^2$,*

$$f\left(x^2\right) - f\left(x^1\right) < \nabla f\left(x^1\right) \cdot \left(x^2 - x^1\right).$$

- **Corollary 2:**

  Suppose $A \subset \Re^n$ is an open convex set, *and $f : A \to \Re$ is continuously differentiable on A. Then f is strictly concave on A if and only if for all x^1*, *x^2 in A with $x^1 \neq x^2$,*

$$\left(\nabla f\left(x^2\right) - \nabla f\left(x^1\right)\right) \cdot \left(x^2 - x^1\right) < 0.$$

- **Twice Differentiable Concave Functions:**

  If $A \subset \Re^n$ is an open convex set, and $f : A \to \Re$ is twice continuously differentiable on A, then we can find a convenient characterization for f to be concave on A in terms of the negative semi-definiteness of the Hessian matrix of f.

- **Theorem 9:**

  Suppose $A \subset \Re^n$ is an open convex set, and $f : A \to \Re$ is twice continuously differentiable on A. Then f is concave on A if and only if the Hessian matrix $H_f(x)$ is negative semi-definite for all x in A.

- **Theorem 10:**

  Suppose $A \subset \Re^n$ is an open convex set, and $f : A \to \Re$ is twice continuously differentiable on A. If the Hessian matrix $H_f(x)$ is negative definite for all x in A, then f is strictly concave on A.

#6. Give a counter-example to establish that the converse of Theorem 10 does not hold.

4. Quasi-Concave Functions

- Let $A \subset \Re^n$ be a convex set, and f a real-valued function on A. Then f is *quasi-concave* on A if

$$f\left(x^2\right) \geq f\left(x^1\right) \text{ implies } f\left(\theta x^1 + (1-\theta)\, x^2\right) \geq f\left(x^1\right)$$

whenever $x^1, x^2 \in A$, and $0 \leq \theta \leq 1$.

  – The function f is *strictly quasi-concave* on A if

$$f\left(x^2\right) \geq f\left(x^1\right) \text{ implies } f\left(\theta x^1 + (1-\theta)\, x^2\right) > f\left(x^1\right)$$

whenever $x^1, x^2 \in A$, $x^1 \neq x^2$, and $0 < \theta < 1$.

- While the conditions stated in Theorem 5 did not characterize concave functions, it does characterize quasi-concave functions.

- **Theorem 11:**

  Suppose A is a convex set in $\Re^n$ and $f : A \to \Re$. Then f is a quasi-concave function on A if and only if for every $\alpha \in \Re$, the set $S(\alpha) = \{x \in A : f(x) \geq \alpha\}$ is a convex set in $\Re^n$.

- **Differentiable Quasi-Concave Functions:**

  A characterization of differentiable quasi-concave function can be given which parallels the characterization of differentiable concave functions given in Theorem 8.

- **Theorem 12:**

  Suppose $A \subset \Re^n$ is an open convex set, and $f : A \to \Re$ is continuously differentiable on A. Then f is quasi-concave on A if and only if for all x^1 and x^2 in A

$$f\left(x^2\right) \geq f\left(x^1\right) \text{ \textit{implies} } \left(x^2 - x^1\right) \cdot \nabla f\left(x^1\right) \geq 0.$$

  – Proof: To be discussed in class.

- **Twice Differentiable Quasi-Concave Functions:**

  An interesting characterization of twice differentiable quasi-concave functions can be given in terms of the "bordered" Hessian matrix associated with the functions.

- Let $A \subset \Re^n$ is an open convex set, and $f : A \to \Re$ is twice continuously differentiable on A. The *bordered Hessian matrix* of f at $x \in A$ is denoted by $G_f(x)$ and is defined by the following $(n+1) \times (n+1)$ matrix:

$$G_f(x) = \begin{pmatrix} 0 & \nabla f(x) \\ \\ \nabla f(x) & H_f(x) \end{pmatrix}.$$

  – We denote the $(k+1)$th leading principal minor of $G_f(x)$ by $|G_f(x;k)|$, where $k = 1, 2, ..., n.$

- **Theorem 13:**

  Suppose $A \subset \Re^n$ is an open convex set, and $f : A \to \Re$ is twice continuously differentiable on A.

  (i) *If f is quasi-concave on A, then $(-1)^k |G_f(x;k)| \geq 0$ for $x \in A$, and $k = 1, 2, ..., n$.*

  (ii) *If $(-1)^k |G_f(x;k)| > 0$ for $x \in A$, and $k = 1, 2, ..., n$, then f is strictly quasi-concave on A.*

References

- Must read the following chapters and sections from the textbook:
  - Section 30.1 (pages 822 – 827): Weierstrass Theorem and Mean Value Theorems,
  - Section 30.2 (pages 827 – 832): Taylor Polynomials on $\Re^1$,
  - Section 30.3 (pages 832 – 836): Taylor Polynomials on $\Re^n$,
  - Section 21.1 (pages 505 – 516): Concave and Convex Functions,
  - Section 21.2 (pages 516 – 520): Properties of Concave Functions,
  - Section 21.3 (pages 522 – 527): Quasi-concave and Quasi-convex Functions.
- This material is based on

1. Nikaido, H., *Convex Structures and Economic Theory*, (chapter 1),
2. Takayama, A., *Mathematical Economics*, (chapters 0, 1),
3. Apostol, T., *Mathematical Analysis*, (chapters 4, 6),
4. Bartle, R., *The Elements of Real Analysis*, (chapter 7),
5. Mangasarian, O. L., *Non-Linear Programming*, (chapters 2, 3, 4, 6, 9).