Classical Optimization Theory:

Unconstrained Optimization

1. Preliminaries

- **Framework:**
  - Consider a set $A \subset \Re^n$, and a function $f : A \to \Re$.
  - We are interested in identifying points in A at which the function f attains a (local or global) maximum and/or minimum.

- **Local Maximum:**

  Let $A \subset \Re^n$, and let f be a function from A to $\Re$. A point $x^* \in A$ is said to be a *point of local maximum* of f if there exists an open ball around x^*, $B_\epsilon(x^*)$, such that $f(x^*) \geq f(x)$ for all $x \in B_\epsilon(x^*) \cap A$.

  - That is, a point x^* is a local maximum if there are no nearby points at which f takes on a larger value.

- **Global Maximum:**

  Let $A \subset \Re^n$, and let f be a function from A to $\Re$. A point $x^* \in A$ is said to be a *point of global maximum* of f if $f(x^*) \geq f(x)$ for all $x \in A$.

- **Local** and **global minimum** can be defined similarly by just reverting the inequalities.

- Two issues:

(a) Existence of a solution: Weierstrass Theorem;

(b) Uniqueness of the solution: Strict Quasi-concavity.

- **Theorem 1 (Weierstrass Theorem: Existence of a Solution):**

  Let A be a compact subset of $\Re^n$ and $f : A \to \Re$ be continuous on A. Then there exists x_m and x_M in A such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in A$.

  – That is, x_M is a point of global maximum and x_m is a point of global minimum of f.

- **Theorem 2 (Uniqueness of the Solution):**

  Suppose A is a non-empty, compact and convex subset of $\Re^n$. Suppose $f : A \to \Re$ is a continuous and strictly quasi-concave (*respectively*, *quasi-convex*) *function on A. Then there exists $x^* \in A$ such that $f(x^*) > f(x)$* (*respectively*, $f(x^*) < f(x)$) *for all $x \in A$ where $x \neq x^*$.*

  – That is, x^* is the unique global maximum (respectively, minimum) of f.

  – Proof: To be discussed in class.

2. Necessary Conditions for Local Maximum & Minimum

- We will present two necessary conditions for local maximum and minimum:
  - one is a condition on the first-order partial derivatives of the relevant function (called the "first-order conditions"),
  - the other is a condition on the second-order partial derivatives of the relevant function (called the "second-order necessary conditions").

- **Theorem 3:**

  Let A be an open set in $\Re^n$, and let $f : A \to \Re$ be a continuously differentiable function on A. If $x^* \in A$ is a point of local maximum or minimum of f, then

$$\nabla f(x^*) = 0. \tag{1}$$

  - Proof: To be discussed in class.
  - Remark: The n equations given by (1) are called the *first-order conditions* for a local maximum or minimum.

- **Theorem 4:**

  Let A be an open set in $\Re^n$, and let $f : A \to \Re$ be a twice continuously differentiable function on A. If $x^* \in A$ is a point of local maximum (*respectively*, *minimum*) *of f, then*

$$H_f(x^*) \text{ is negative (respectively, positive) semi-definite.} \tag{2}$$

  – Proof: To be discussed in class. [Theorem 30.11, pages 840-841, of the textbook.]

  – Remark: Condition (2) is called the *second-order necessary condition* for a local maximum (respectively, minimum).

- *Necessary* conditions like (1) and (2) help us to rule out points where a local maximum *cannot* occur, thereby narrowing our search for points where a local maximum does occur. The following two examples illustrate this point.

#1. Let $f : \Re \to \Re$ be given by $f(x) = 1 - x^2$ for all $x \in \Re$. Does f have a local maximum at the point $x = 1$?

#2. Let $f : \Re \to \Re$ be given by $f(x) = 1 - 2x + x^2$ for all $x \in \Re$. Does f have a local maximum at the point $x = 1$?

3. Sufficient Conditions for Local Maximum and Minimum

- We present below a set of sufficient conditions for local maximum and minimum.
- **Theorem 5:**

  Let A be an open set in $\Re^n$, and let $f : A \to \Re$ be a twice continuously differentiable function on A.

(a) If $x^* \in A$ and

$$\nabla f\left(x^*\right) = 0, \text{ and} \tag{3a}$$

$$H_f\left(x^*\right) \text{ is negative definite,} \tag{4a}$$

then x^* *is a point of local maximum of* f.

(b) If $x^* \in A$ and

$$\nabla f\left(x^*\right) = 0, \text{ and} \tag{3b}$$

$$H_f\left(x^*\right) \text{ is positive definite,} \tag{4b}$$

then x^* *is a point of local minimum of* f.

(c) If $x^* \in A$ and

$$\nabla f\left(x^*\right) = 0, \text{ and} \tag{3c}$$
$$H_f\left(x^*\right) \textit{ is indefinite,} \tag{4c}$$

then x^* *is neither a point of local maximum nor a point of local minimum of* f.

– Proof: To be discussed in class. [Theorem 30.10, pages 836-840, of the textbook.]

– Remark: Condition (4a) (respectively, (4b)) is called the *second-order sufficient condition* for a local maximum (respectively, minimum).

- It should be noted that in the statement of Theorem 5 condition (4) *cannot* be weakened to condition (2).

#3. Provide an example to establish this point.

- It should also be observed that in the statement of Theorem 4 condition (2) *cannot* be strengthened to condition (4).

#4. Provide an example to establish this point.

- The outcome of this discussion is the following:
  - The second-order *necessary* conditions for a local maximum and minimum are different from (*weaker* than) the second-order *sufficient* conditions.
  - This simply reflects the fact that, in general, the first and second derivatives of a function at a point do not capture all aspects relevant to the occurrence of a local maximum or minimum of the function at that point.

- The sufficient conditions of Theorem 5 enable us to find points of local maximum or minimum of a function as the following example shows.

#5. Let $f : \Re^2 \to \Re$ be given by $f(x_1, x_2) = 2x_1x_2 - 2x_1^2 - x_2^2$ for all $(x_1, x_2) \in \Re^2$. Find out all the points of local maximum of f.

4. Sufficient Conditions for Global Maximum & Minimum

- **Theorem 6:**

  Let A be an open convex set in $\Re^n$, and let $f : A \to \Re$ be a continuously differentiable function on A. If $x^* \in A$ satisfies $\nabla f(x^*) = 0$, and f is a concave (*respectively*, *convex*) *function on A, then x^* is a point of global maximum* (*respectively*, *minimum*) *of f.*

  – Proof: To be discussed in class.

- **Theorem 7:**

  Let A be an open convex set in $\Re^n$, and let $f : A \to \Re$ be a twice continuously differentiable function on A.

(a) If $x^* \in A$ and

$$\nabla f(x^*) = 0, \text{ and}$$
$$H_f(x) \textit{ is negative semi-definite for all } x \in A,$$

then x^* *is a point of global maximum of f.*

(b) If $x^* \in A$ and

$$\nabla f\left(x^*\right) = 0, \text{ and}$$
$$H_f\left(x\right) \textit{ is positive semi-definite for all } x \in A,$$

then x^* *is a point of global minimum of* f.

– Proof: To be discussed in class.

- It is worth noting that Theorem 6 or 7 might be applicable in cases where Theorem 5 is not applicable.

  – Example: Consider $f : \Re \rightarrow \Re$ be given by $f\left(x\right) = -x^4$.

    - Note that $f'\left(0\right) = 0$ and $f''\left(x\right) = -12x^2 \leq 0$ *for all* $x \in \Re$.

    $\Rightarrow$ Applying Theorem 6 or 7 we can conclude that $x = 0$ is a point of global maximum, and hence of local maximum.

    - But the conclusion that $x = 0$ is a point of local maximum cannot be derived from Theorem 5, since $f''\left(0\right) = 0$.

References

- Must read the following chapterand section from the textbook:
  - Chapter 17 (pages 396 – 410): Unconstrained Optimization,
  - Section 30.4 (pages 836 – 841): Second Order Optimization Conditions.

- This material is based on

1. Bartle, R., *The Elements of Real Analysis*, (chapter 7),
2. Apostol, T., *Mathematical Analysis: A Modern Approach to Advanced Calculus*, (chapter 6).
3. Takayama, A., *Mathematical Economics*, (chapter 1).