Modern Optimization Theory:

Concave Programming

1. Preliminaries

- We will present below the elements of "modern optimization theory" as formulated by Kuhn and Tucker, and a number of authors who have followed their general approach.
- Modern *constrained maximization theory* is concerned with the following problem:

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & g^j(x) \geq 0, \text{ for } j = 1, 2, ..., m \\ \text{and} & x \in X \end{array}\right\} \text{(P)}$$

where

- X is a non-empty subset of $\Re^n$, and
- f, g^j ($j = 1, 2, ..., m$) are functions from X to $\Re$.
- **Constraint Set:**

$$C = \left\{x \in X\text{: } g^j(x) \geq 0, \text{ for } j = 1, 2, ..., m\right\}.$$

– A point $\hat{x} \in X$ is a point of **constrained global maximum** if $\hat{x}$ solves the problem (P).

– A point $\hat{x} \in X$ is a point of **constrained local maximum** if there exists an open ball around $\hat{x}$, $B_\epsilon(\hat{x})$, such that $f(\hat{x}) \geq f(x)$ for all $x \in B_\epsilon(\hat{x}) \cap C$.

– A pair $\left(\hat{x}, \hat{\lambda}\right) \in (X \times \Re^m_+)$ is a **saddle point** if

$$\phi\left(x, \hat{\lambda}\right) \leq \phi\left(\hat{x}, \hat{\lambda}\right) \leq \phi(\hat{x}, \lambda) \text{ for all } x \in X \text{ and all } \lambda \in \Re^m_+,$$

where

$$\phi(x, \lambda) = f(x) + \lambda g(x) \text{ for all } (x, \lambda) \in (X \times \Re^m_+).$$

- $\left(\hat{x}, \hat{\lambda}\right)$ is simultaneously a point of maximum and minimum of $\phi(x, \lambda)$: maximum with respect to x, and minimum with respect to λ.

- The *constraint minimization problem* and the corresponding **constrained global minimum** and **constrained local minimum** can be defined analogously.

2. Constrained Global Maxima and Saddle Points

- A major part of modern optimization theory is concerned with establishing (under suitable conditions) an equivalence result between a point of constrained global maximum and saddle point.
  - We explore this theory in what follows.
- **Theorem 1:**
  If $\left(\hat{x}, \hat{\lambda}\right) \in (X \times \Re^m_+)$ *is a saddle point*, *then*

(i) $\hat{\lambda} g\left(\hat{x}\right) = 0$,

(ii) $g\left(\hat{x}\right) \geq 0$, *and*

(iii) $\hat{x}$ *is a point of constrained global maximum*.

  - Proof: To be discussed in class.
  - Hints:
    - For (i) and (ii) use the second inequality in the definition of a saddle point.
    - Then use (i), (ii) and the first inequality in the saddle point definition to prove (iii).

- A converse of Theorem 1 can be proved if
  - X is a convex set,
  - f, g^j ($j = 1, 2, ..., m$) are concave functions on X, and
  - a condition on the constraints, generally known as "Slater's condition" is satisfied.
    - Notice that none of these conditions are needed for the validity of Theorem 1.

- **Slater's Condition:**

  Given the problem (P), we will say that Slater's condition holds if *there exists* $\bar{x} \in X$ *such that* $g^j(\bar{x}) > 0$, *for* $j = 1, 2, ..., m$.

- **Theorem 2 (Kuhn-Tucker):**

  Suppose $\hat{x} \in X$ *is a point of constrained global maximum. If* X *is a convex set*, f, g^j ($j = 1, 2, ..., m$) *are concave functions on* X, *and Slater's condition holds*, *then there is* $\hat{\lambda} \in \Re^m_+$ *such that*

(i) $\hat{\lambda} g(\hat{x}) = 0$, *and*

(ii) $\left(\hat{x}, \hat{\lambda}\right)$ *is a saddle point*.

- **Examples:** The following examples demonstrate why the assumptions of Theorem 2 are needed for the conclusion to be valid.

#1. Let $X = \Re_+$, $f : X \to \Re$ be given by $f(x) = x$, and $g : X \to \Re$ be given by $g(x) = -x^2$.

(a) What is the point of constrained global maximum $(\hat{x})$ for the problem (P) for this characterization of X, f and g?

(b) Can you find a $\hat{\lambda} \in \Re_+$ such that $\left(\hat{x}, \hat{\lambda}\right)$ is a saddle point? Explain clearly.

(c) What goes wrong? Explain clearly.

#2. Let $X = \Re_+$, $f : X \to \Re$ be given by $f(x) = x^2$, and $g : X \to \Re$ be given by $g(x) = 1 - x$.

(a) What is the point of constrained global maximum $(\hat{x})$ for the problem (P) for this characterization of X, f and g?

(b) Can you find a $\hat{\lambda} \in \Re_+$ such that $\left(\hat{x}, \hat{\lambda}\right)$ is a saddle point? Explain clearly.

(c) Is the Slater's condition satisfied? What goes wrong? Explain clearly.

3. The Kuhn-Tucker Conditions and Saddle Points

- **The Kuhn-Tucker Conditions:**

  Let X be an open set in $\Re^n$, and f, g^j ($j = 1, 2, ..., m$) be continuously differentiable on X. A pair $\left(\hat{x}, \hat{\lambda}\right) \in (X \times \Re^m_+)$ satisfies the Kuhn-Tucker conditions if

(i) $\frac{\partial f}{\partial x_i}(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j \frac{\partial g^j}{\partial x_i}(\hat{x}) = 0$, $i = 1, 2, ..., n$,

(ii) $g(\hat{x}) \geq 0$, and $\hat{\lambda} g(\hat{x}) = 0$.

– The condition $\hat{\lambda} g(\hat{x}) = 0$ is called the 'Complementary Slackness' condition. Note

$$\hat{\lambda} g(\hat{x}) = 0 \Rightarrow \hat{\lambda}_1 g^1(\hat{x}) + ... + \hat{\lambda}_m g^m(\hat{x}) = 0,$$

$\Rightarrow \hat{\lambda}_1 g^1(\hat{x}) = 0$, ..., $\hat{\lambda}_m g^m(\hat{x}) = 0$, since $\hat{\lambda}_j \geq 0$ as $\hat{\lambda} \in \Re^m_+$ and $g^j(\hat{x}) \geq 0$.

- So if $g^j(\hat{x}) > 0$, then $\hat{\lambda}_j = 0$. That is, if a constraint is not binding, then the corresponding multiplier is 0.
- But if $g^j(\hat{x}) = 0$, then $\hat{\lambda}_j$ can be either > 0 or equal to zero.

- A part of modern optimization theory is concerned with establishing the equivalence (under some suitable conditions) between a saddle point and a point where the Kuhn-Tucker conditions are satisfied.

- **Theorem 3:**

  Let X *be an open set in* $\Re^n$, *and* f, g^j ($j = 1, 2, ..., m$) *be continuously differentiable on* X. *Suppose a pair* $\left(\hat{x}, \hat{\lambda}\right) \in (X \times \Re_+^m)$ *satisfies the Kuhn-Tucker conditions. If* X *is convex and* f, g^j ($j = 1, 2, ..., m$) *are concave on* X, *then*

  (i) $\left(\hat{x}, \hat{\lambda}\right)$ *is a saddle point*, *and*

  (ii) $\hat{x}$ *is a point of constrained global maximum*.

  – Proof: To be discussed in class.

- **Theorem 4:**

  Let X *be an open set in* $\Re^n$, *and* f, g^j ($j = 1, 2, ..., m$) *be continuously differentiable on* X. *Suppose a pair* $\left(\hat{x}, \hat{\lambda}\right) \in (X \times \Re_+^m)$ *is a saddle point. Then* $\left(\hat{x}, \hat{\lambda}\right)$ *satisfies the Kuhn-Tucker conditions*.

  – Proof: To be discussed in class.

4. Sufficient Conditions for Constrained Global Maximum and Minimum

- Now we have all the ingredients to find out the sufficient conditions for a constrained global maximum or minimum involving the Kuhn-Tucker conditions.

#3. State and prove rigorously a theorem that gives the *sufficient* conditions for a constrained global *maximum* involving the Kuhn-Tucker conditions.

#4. State and prove rigorously a theorem that gives the *sufficient* conditions for a constrained global *minimum* involving the Kuhn-Tucker conditions.

5. Constrained Local and Global Maxima

- It is clear that if $\hat{x}$ is a point of constrained global maximum, then $\hat{x}$ is also a point of constrained local maximum.
  - The circumstances under which the converse is true are given by the following theorem.

- **Theorem 5:**

  Let X be a convex set in $\Re^n$. Let f, g^j ($j = 1, 2, ..., m$) be concave functions on X. Suppose $\hat{x}$ is a point of constrained local maximum. Then $\hat{x}$ is also a point of constrained global maximum.

  - Proof: To be discussed in class.
  - Hints: Establish first that since X is a convex set and g^j ($j = 1, 2, ..., m$)'s are concave functions, the constraint set C is a convex set.

6. Necessary Conditions for Constrained Local Maximum and Minimum

- We now establish the useful result (corresponding to the classical Lagrange Theorem) that if $x^* \in X$ is a point of constrained local maximum then, under suitable conditions, there exists $\lambda^* \in \Re^k_+$ such that (x^*, λ^*) satisfies the Kuhn-Tucker conditions.

- **Theorem 6 (Constrained Local Maximum):**

  Let X be an open set in $\Re^n$, and f, g^j ($j = 1, 2, ..., k$) be continuously differentiable on X. Suppose that $x^* \in X$ is a point of constrained local maximum of f subject to k inequality constraints:

$$g^1(x) \leq b_1, ..., g^k(x) \leq b_k.$$

  Without loss of generality, *assume that the first k_0 constraints are binding at x^* and that the last $(k - k_0)$ constraints are not binding. Suppose that the following nondegenerate constraint qualification is satisfied at x^**:

The rank at x^* of the following Jacobian matrix of the binding constraints is k_0:

$$\begin{pmatrix} \frac{\partial g^1}{\partial x_1}(x^*) & \cdots & \frac{\partial g^1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g^{k_0}}{\partial x_1}(x^*) & \cdots & \frac{\partial g^{k_0}}{\partial x_n}(x^*) \end{pmatrix}.$$

Form the Lagrangian

$$L(x,\lambda) \equiv f(x) - \lambda_1 \left[g^1(x) - b_1\right] - ... - \lambda_k \left[g^k(x) - b_k\right].$$

Then, there exist multipliers $(\lambda_1^*, ..., \lambda_k^*)$ such that

(a) $\frac{\partial L}{\partial x_1}(x^*, \lambda^*) = 0, ..., \frac{\partial L}{\partial x_n}(x^*, \lambda^*) = 0;$

(b) $\lambda_1^* \left[g^1(x^*) - b_1\right] = 0, ..., \lambda_k^* \left[g^k(x^*) - b_k\right] = 0;$

(c) $\lambda_1^* \geq 0, ..., \lambda_k^* \geq 0;$

(d) $g^1(x^*) \leq b_1, ..., g^k(x^*) \leq b_k.$

– Proof: To be discussed in class (see Section 19.6, pages 480-482, of the textbook).

– Note that the conditions (a) – (d) are the Kuhn-Tucker conditions.

- **Example:**

  Consider the following problem:

$$\left.\begin{array}{ll} \text{Maximize} & x \\ \text{subject to} & (1-x)^3 \geq y, \\ & x \geq 0,\, y \geq 0. \end{array}\right\}$$

(a) Define carefully X, f, and the g^j's and b_j's for this problem.

(b) Draw carefully the constraint set for this problem and find out (x^*, y^*) such that (x^*, y^*) solves this problem.

(c) Are there λ_j^*'s (the number of λ_j^*'s should be in accordance with the number of g^j's) such that (x^*, y^*) and the λ_j^*'s satisfy the Kuhn-Tucker conditions? Explain carefully.

(d) What goes wrong? Explain carefully.

- **Theorem 7 (Mixed Constraints):**

  Let X be an open set in $\Re^n$, and f, g^j ($j = 1, 2, ..., k$) and h^i ($i = 1, 2, ..., m$) be continuously differentiable on X. Suppose that $x^* \in X$ is a point of constrained local maximum of f subject to k inequality constraints and m equality constraints:

$$\begin{aligned} g^1(x) &\leq b_1, ..., g^k(x) \leq b_k, \\ h^1(x) &= c_1, ..., h^m(x) = c_m. \end{aligned}$$

  Without loss of generality, *assume that the first k_0 inequality constraints are binding at x^* and that the last $(k - k_0)$ constraints are not binding. Suppose that the following nondegenerate constraint qualification is satisfied at x^**:

The rank at x^* of the Jacobian matrix of the equality constraints and the binding inequality constraints

$$\begin{pmatrix} \frac{\partial g^1}{\partial x_1}(x^*) & \cdots & \frac{\partial g^1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g^{k_0}}{\partial x_1}(x^*) & \cdots & \frac{\partial g^{k_0}}{\partial x_n}(x^*) \\ \frac{\partial h^1}{\partial x_1}(x^*) & \cdots & \frac{\partial h^1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial h^m}{\partial x_1}(x^*) & \cdots & \frac{\partial h^m}{\partial x_n}(x^*) \end{pmatrix}$$

is $(k_0 + m)$.

Form the Lagrangian

$$\begin{aligned} L(x,\lambda,\mu) \;\equiv\; & f(x) - \lambda_1\left[g^1(x) - b_1\right] - ... - \lambda_k\left[g^k(x) - b_k\right] \\ & -\mu_1\left[h^1(x) - c_1\right] - ... - \mu_m\left[h^m(x) - c_m\right]. \end{aligned}$$

Then, there exist multipliers $(\lambda_1^*, ..., \lambda_k^*, \mu_1^*, ..., \mu_m^*)$ *such that*

(a) $\dfrac{\partial L}{\partial x_1}(x^*, \lambda^*, \mu^*) = 0, ..., \dfrac{\partial L}{\partial x_n}(x^*, \lambda^*, \mu^*) = 0;$

(b) $\lambda_1^*\left[g^1(x^*) - b_1\right] = 0, ..., \lambda_k^*\left[g^k(x^*) - b_k\right] = 0;$

(c) $h^1(x^*) = c_1, ..., h^m(x^*) = c_m;$

(d) $\lambda_1^* \geq 0, ..., \lambda_k^* \geq 0;$

(e) $g^1(x^*) \leq b_1, ..., g^k(x^*) \leq b_k.$

- **Theorem 8 (Constrained Local Minimum):**

  Let X *be an open set in* $\Re^n$, *and* f, g^j ($j = 1, 2, ..., k$) and h^i ($i = 1, 2, ..., m$) *be continuously differentiable on* X. *Suppose that* $x^* \in X$ *is a point of constrained local minimum of* f *subject to* k *inequality constraints and* m *equality constraints*:

$$\begin{aligned} g^1(x) &\geq b_1, ..., g^k(x) \geq b_k, \\ h^1(x) &= c_1, ..., h^m(x) = c_m. \end{aligned}$$

  Without loss of generality, *assume that the first* k_0 *inequality constraints are binding at* x^* *and that the last* $(k - k_0)$ *constraints are not binding. Suppose that the following nondegenerate constraint qualification is satisfied at* x^*:

The rank at x^* of the Jacobian matrix of the equality constraints and the binding inequality constraints

$$\begin{pmatrix} \frac{\partial g^1}{\partial x_1}(x^*) & \cdots & \frac{\partial g^1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g^{k_0}}{\partial x_1}(x^*) & \cdots & \frac{\partial g^{k_0}}{\partial x_n}(x^*) \\ \frac{\partial h^1}{\partial x_1}(x^*) & \cdots & \frac{\partial h^1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial h^m}{\partial x_1}(x^*) & \cdots & \frac{\partial h^m}{\partial x_n}(x^*) \end{pmatrix}$$

is $(k_0 + m)$.

Form the Lagrangian

$$L(x,\lambda,\mu) \equiv f(x) - \lambda_1\left[g^1(x) - b_1\right] - ... - \lambda_k\left[g^k(x) - b_k\right]$$
$$-\mu_1\left[h^1(x) - c_1\right] - ... - \mu_m\left[h^m(x) - c_m\right].$$

Then, there exist multipliers $(\lambda_1^*, ..., \lambda_k^*, \mu_1^*, ..., \mu_m^*)$ *such that*

(a) $\frac{\partial L}{\partial x_1}(x^*, \lambda^*, \mu^*) = 0, ..., \frac{\partial L}{\partial x_n}(x^*, \lambda^*, \mu^*) = 0;$

(b) $\lambda_1^*\left[g^1(x^*) - b_1\right] = 0, ..., \lambda_k^*\left[g^k(x^*) - b_k\right] = 0;$

(c) $h^1(x^*) = c_1, ..., h^m(x^*) = c_m;$

(d) $\lambda_1^* \geq 0, ..., \lambda_k^* \geq 0;$

(e) $g^1(x^*) \geq b_1, ..., g^k(x^*) \geq b_k.$

7. Sufficient Conditions for Constrained Local Maximum and Minimum

- We use techniques similar to the necessary conditions.
  - Given a solution (x^*, λ^*, μ^*) of the Kuhn-Tucker conditions (the first-order conditions), divide the inequality constraints into binding constraints and non-binding constraints at x^*.
    - On the one hand, we treat the binding inequality constraints like equality constraints;
    - on the other hand, the multipliers for the non-binding constraints must be zero and these constraints drop out of the Lagrangian.

- **Theorem 9:**

Let X be an open set in $\Re^n$, and f, g^j ($j = 1, 2, ..., k$) and h^i ($i = 1, 2, ..., m$) *be twice continuously differentiable on X. Consider the problem of maximizing f on the constraint set:*

$$C_{g,h} \equiv \left\{ x \in X: \begin{array}{l} g^j(x) \leq b_j, \text{ for } j = 1, 2, ..., k, \\ h^i(x) = c_i, \text{ for } i = 1, 2, ..., m. \end{array} \right\}$$

Form the Lagrangian

$$\begin{aligned} L(x, \lambda, \mu) \equiv\ & f(x) - \lambda_1 \left[g^1(x) - b_1\right] - ... - \lambda_k \left[g^k(x) - b_k\right] \\ & - \mu_1 \left[h^1(x) - c_1\right] - ... - \mu_m \left[h^m(x) - c_m\right]. \end{aligned}$$

(a) *Suppose that there exist multipliers* $(\lambda_1^*, ..., \lambda_k^*, \mu_1^*, ..., \mu_m^*)$ *such that*

$$\frac{\partial L}{\partial x_1}(x^*, \lambda^*, \mu^*) = 0, ..., \frac{\partial L}{\partial x_n}(x^*, \lambda^*, \mu^*) = 0;$$

$$\lambda_1^* \geq 0, ..., \lambda_k^* \geq 0;$$

$$\lambda_1^* \left[g^1(x^*) - b_1\right] = 0, ..., \lambda_k^* \left[g^k(x^*) - b_k\right] = 0;$$

$$h^1(x^*) = c_1, ..., h^m(x^*) = c_m.$$

(b) *Without loss of generality*, *assume that the first* k_0 *inequality constraints are binding at* x^* *and that the last* $(k - k_0)$ *constraints are not binding. Write* $\left(g^1, ..., g^{k_0}\right)$ *as* g_{k_0}, $\left(h^1, ..., h^m\right)$ *as* h, *the Jacobian derivative of* g_{k_0} *at* x^* *as* $Dg_{k_0}(x^*)$, *and the Jacobian derivative of* h *at* x^* *as* $Dh(x^*)$.

Suppose that the Hessian of L *with respect to* x *at* (x^*, λ^*, μ^*) *is negative definite on the linear constraint set*

$$\{v\text{: } Dg_{k_0}(x^*) \cdot v = 0 \text{ and } Dh(x^*) \cdot v = 0\},$$

that is,

$$\begin{aligned} v &\neq 0,\ Dg_{k_0}(x^*) \cdot v = 0 \text{ and } Dh(x^*) \cdot v = 0 \\ &\Rightarrow v^T \cdot H_L(x^*, \lambda^*, \mu^*) \cdot v < 0. \end{aligned}$$

Then x^* *is a point of constrained local maximum of* f *on the constraint set* $C_{g,h}$.

- To check condition (b), form the **bordered Hessian**

$$\overline{H} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & | & \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & | & \frac{\partial g^{k_0}}{\partial x_1} & \cdots & \frac{\partial g^{k_0}}{\partial x_n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & | & \frac{\partial h^1}{\partial x_1} & \cdots & \frac{\partial h^1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & | & \frac{\partial h^m}{\partial x_1} & \cdots & \frac{\partial h^m}{\partial x_n} \\ - & - & - & - & - & - & | & - & - & - \\ \frac{\partial g^1}{\partial x_1} & \cdots & \frac{\partial g^{k_0}}{\partial x_1} & \frac{\partial h^1}{\partial x_1} & \cdots & \frac{\partial h^m}{\partial x_1} & | & \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_n x_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & | & \vdots & \ddots & \vdots \\ \frac{\partial g^1}{\partial x_n} & \cdots & \frac{\partial g^{k_0}}{\partial x_n} & \frac{\partial h^1}{\partial x_n} & \cdots & \frac{\partial h^m}{\partial x_n} & | & \frac{\partial^2 L}{\partial x_1 x_n} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{pmatrix}$$

Check the signs of the last $(n-(k_0+m))$ leading principal minors of $\overline{H}$, starting with the determinant of $\overline{H}$ itself.

– If $\left|\overline{H}\right|$ has the same sign as $(-1)^n$ and if these last $(n-(k_0+m))$ leading principal minors alternate in sign, then condition (b) holds.

- We need to make the following changes in the wording of Theorem 9 for an inequality-constrained **minimization** problem:

(i) write the inequality constraints as $g^j(x) \geq b_j$ in the presentation of the constraint set $C_{g,h}$,

(ii) change "negative definite" and "< 0" in condition (b) to "positive definite" and "> 0".

– The bordered Hessian check requires that the last $(n-(k_0+m))$ leading principal minors of $\overline{H}$ all have the same sign as $(-1)^{k_0+m}$.

- **Example 1:**

  Consider the following constrained maximization problem:

$$\left.\begin{array}{ll} \text{Maximize} & \prod_{i=1}^{n} x_i \\ \text{subject to} & \sum_{i=1}^{n} x_i \leq n, \\ \text{and} & x_i \geq 0,\ i = 1, 2, ...n. \end{array}\right\} \text{(P)}$$

  Find out the solution to (P) by showing your steps clearly.

- **Example 2:**

  Consider the following constrained maximization problem:

$$\left.\begin{array}{ll} \text{Maximize} & x^2 + x + 4y^2 \\ \text{subject to} & 2x + 2y \leq 1, \\ \text{and} & x \geq 0, y \geq 0. \end{array}\right\} \text{(Q)}$$

  Find out the solution to (Q) by showing your steps clearly.

References

- Must read the following sections from the textbook:
  - Section 18.3, 18.4, 18.5, 18.6 (pages 424 – 447): Inequality Constraints, Mixed Constraints, Constrained Minimization Problems, and Kuhn-Tucker Formulation;
  - Section 19.3 (pages 466 – 469): Second-Order Conditions (Inequality Constraints),
  - Section 19.6 (pages 480 – 482): Proofs of First Order Conditions.
- This material is based on

1. Mangasarian, O. L., *Non-Linear Programming*, (chapters 5, 7),
2. Takayama, A., *Mathematical Economics*, (chapter 1),
3. Nikaido, H., *Convex Structures and Economic Theory*, (chapter 1).