Modern Optimization Theory:

Quasi-Concave Programming

1. Preliminaries

- The theory of concave programming, as developed by Kuhn and Tucker, is a very convenient and powerful tool in optimization theory.
  - For many economic applications, however, it is difficult to justify the assumption of *concavity* of the objective and constraint functions.
- In many such cases, one finds it easier to defend the assumption that these functions are *quasi-concave*.
  - It is with these economic applications in mind that Arrow and Enthoven developed the theory of quasi-concave programming.
- In what follows, we cover the elements of this theory by focusing on its two main results.
  - One result provides conditions under which a point satisfying the Kuhn-Tucker conditions is a point of constrained global maximum.
  - Another result provides conditions under which a point of constrained global maximum satisfies the Kuhn-Tucker conditions.

- **Modification in the Maximization Problem and the Kuhn-Tucker conditions:**

  – So far we were concerned with the following constrained maximization problem:

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & g^j(x) \geq 0, \text{ for } j = 1, 2, ..., m \\ \text{and} & x \in X \end{array}\right\} \text{(P)}$$

  where

  - X is a non-empty subset of $\Re^n$, and
  - f, g^j ($j = 1, 2, ..., m$) are functions from X to $\Re$.

  – And the corresponding Kuhn-Tucker conditions are:

- **The Kuhn-Tucker Conditions:**

  Let X be an open set in $\Re^n$, and f, g^j ($j = 1, 2, ..., m$) be continuously differentiable on X. A pair $\left(\hat{x}, \hat{\lambda}\right) \in (X \times \Re^m_+)$ satisfies Kuhn-Tucker conditions if

(a) $\frac{\partial f}{\partial x_i}(\hat{x}) + \sum_{i=1}^{m} \hat{\lambda}_j \frac{\partial g^j}{\partial x_i}(\hat{x}) = 0$, $i = 1, 2, ..., n$;

(b) $g^j(\hat{x}) \geq 0$, and $\hat{\lambda}_j g^j(\hat{x}) = 0$, $j = 1, 2, ..., m$.

– The maximization problem that we discuss under quasi-concave programming is somewhat more restrictive in theory than the one considered under concave programming, but not in practice for most applications.

– The optimization problem is:

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & g^j(x) \geq 0, \text{ for } j = 1, 2, ..., m \\ \text{and} & x \in \Re^n_+. \end{array}\right\} \text{(Q)}$$

where f, g^j ($j = 1, 2, ..., m$) are functions from $\Re^n$ to $\Re$.

– The (Modified) Kuhn-Tucker Conditions:

A pair $\left(\hat{x}, \hat{\lambda}\right) \in (\Re^n_+ \times \Re^m_+)$ satisfies the Kuhn-Tucker conditions if

(i) $\dfrac{\partial f}{\partial x_i}(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j \dfrac{\partial g^j}{\partial x_i}(\hat{x}) \leq 0,\ i = 1, 2, ..., n;$

(ii) $\hat{x}_i \left[\dfrac{\partial f}{\partial x_i}(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j \dfrac{\partial g^j}{\partial x_i}(\hat{x})\right] = 0,\ i = 1, 2, ..., n;$

(iii) $\hat{x}_i \geq 0,\ i = 1, 2, ..., n;$

(iv) $g^j(\hat{x}) \geq 0,\ j = 1, 2, ..., m;$

(v) $\hat{\lambda}_j g^j(\hat{x}) = 0,\ j = 1, 2, ..., m;$

(vi) $\hat{\lambda}_j \geq 0,\ j = 1, 2, ..., m.$

#1. Show that the (modified) Kuhn-Tucker conditions ((i) - (vi)) for problem (Q) are equivalent to the Kuhn-Tucker conditions ((a) - (b)) for problem (P).

- The constraint set for problem (Q) is defined as

$$C = \left\{x \in \Re^n_+ \colon g^j(x) \geq 0, \text{ for } j = 1, 2, ..., m\right\}.$$

2. The Sufficiency Theorem of Arrow-Enthoven

- **Lemma 1:**

  Suppose f, g^j ($j = 1, 2, ..., m$) are continuously differentiable functions on $\Re^n$. Suppose there is a pair $(\hat{x}, \hat{\lambda}) \in (\Re^n_+ \times \Re^m_+)$ such that $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions. If each g^j is quasi-concave, *then*

  $$x \in C \textit{ implies } (x - \hat{x}) \cdot \nabla f(\hat{x}) \leq 0.$$

  – Proof: To be discussed in class.

  – Hints:

  ∘ $(x - \hat{x}) \cdot \nabla f(\hat{x}) = (x - \hat{x}) \cdot [\nabla f(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j \nabla g^j(\hat{x})] - (x - \hat{x}) \cdot [\sum_{j=1}^{m} \hat{\lambda}_j \nabla g^j(\hat{x})]$.

  - Use the Kuhn-Tucker conditions to show that the R.H.S $\leq -(x - \hat{x}) \cdot [\sum_{j=1}^{m} \hat{\lambda}_j \nabla g^j(\hat{x})]$.

  ∘ If $\hat{\lambda}_j > 0$ for some j, use the Kuhn-Tucker conditions and the quasi-concavity of g^j to argue that $(x - \hat{x}) \cdot \hat{\lambda}_j \nabla g^j(\hat{x}) \geq 0$ for that j.

- **Relevant Index:**

  An index $k \in \{1, 2, ..., n\}$ is called a *relevant index* if there exists $x^* \in C$ such that $x_k^* > 0$.

- **Theorem 1:**

  Suppose f, g^j ($j = 1, 2, ..., m$) *are continuously differentiable quasi-concave functions on* $\Re^n$. *Suppose there is a pair* $(\hat{x}, \hat{\lambda}) \in (\Re_+^n \times \Re_+^m)$ *such that* $(\hat{x}, \hat{\lambda})$ *satisfies the Kuhn-Tucker conditions. Suppose further that at least one of the following conditions is satisfied*:

(a) $\dfrac{\partial f}{\partial x_i}(\hat{x}) < 0$ *for some* $i \in \{1, 2, ..., n\}$;

(b) $\dfrac{\partial f}{\partial x_i}(\hat{x}) > 0$ *for some* i *which is a relevant index*;

(c) f *is concave on* $\Re^n$.

Then $\hat{x}$ *solves problem* (Q).

– Proof: To be discussed in class.

- **Proof of Theorem 1(a):** There is some index k, such that $\dfrac{\partial f}{\partial x_k}(\hat{x}) < 0$.

  Let e^k be the k-th unit vector, and define $\bar{x} = \hat{x} + e^k$. Then $\bar{x} \in \Re^n_+$, and

  $$(\bar{x} - \hat{x}) \cdot \nabla f(\hat{x}) < 0. \tag{1}$$

  Let x be an arbitrary vector in C. We have to show that $f(x) \leq f(\hat{x})$.

  Define, for $0 < \theta < 1$, $x(\theta) = \theta\bar{x} + (1-\theta)x$, and $y(\theta) = \theta\bar{x} + (1-\theta)\hat{x}$.

  Then, using $\theta > 0$ and (1), we have

  $$(y(\theta) - \hat{x}) \cdot \nabla f(\hat{x}) = \theta(\bar{x} - \hat{x}) \cdot \nabla f(\hat{x}) < 0.$$

  Also, by Lemma 1,

  $$(x(\theta) - y(\theta)) \cdot \nabla f(\hat{x}) = (1-\theta)(x - \hat{x}) \cdot \nabla f(\hat{x}) \leq 0.$$

  Combining the above two inequalities we get $(x(\theta) - \hat{x}) \cdot \nabla f(\hat{x}) < 0$,

  and since f is quasi-concave, we have $f(x(\theta)) < f(\hat{x})$.

  Finally, letting $\theta \to 0$, we get $f(x) \leq f(\hat{x})$.

- **Proof of Theorem 1(b):**

  Suppose condition (b) holds. If condition (a) still holds, we are already done. So assume that condition (a) does not hold. That is, $\nabla f(\hat{x}) \geq 0$, and $\frac{\partial f}{\partial x_k}(\hat{x}) > 0$, for some index k which is a relevant index. Thus, there is $x^* \in C$ such that $x_k^* > 0$.

  Using Lemma 1,

$$\hat{x} \cdot \nabla f(\hat{x}) \geq x^* \cdot \nabla f(\hat{x}) > 0. \tag{2}$$

  Let x be an arbitrary vector in C. Then, for $0 < \theta < 1$,

$$(\theta x) \cdot \nabla f(\hat{x}) \leq (\theta \hat{x}) \cdot \nabla f(\hat{x}) < \hat{x} \cdot \nabla f(\hat{x}).$$

  Then quasi-concavity of f implies $f(\theta x) < f(\hat{x})$.

  Finally, letting $\theta \to 1$, we get $f(x) \leq f(\hat{x})$.

- **Corollary 1:**

  Suppose f, g^j ($j = 1, 2, ..., m$) *are continuously differentiable quasi-concave functions on* $\Re^n$*. Suppose there is a pair* $(\hat{x}, \hat{\lambda}) \in (\Re^n_+ \times \Re^m_+)$ *such that* $(\hat{x}, \hat{\lambda})$ *satisfies the Kuhn-Tucker conditions. Suppose there is* $x^* \gg 0$*, such that* $g^j(x^*) \geq 0$*, for* $j = 1, 2, ..., m$*, and* $\nabla f(\hat{x}) \neq 0$*. Then* $\hat{x}$ *solves problem* (Q).

  – Proof: To be discussed in class.

- **Slater's Condition:**

  We say that Slater's condition holds if there exists $\bar{x} \in C$ such that $g^j(\bar{x}) > 0$, for $j = 1, 2, ..., m$.

- **Corollary 2:**

  Suppose f, g^j ($j = 1, 2, ..., m$) *are continuously differentiable quasi-concave functions on* $\Re^n$*. Suppose there is a pair* $(\hat{x}, \hat{\lambda}) \in (\Re^n_+ \times \Re^m_+)$ *such that* $(\hat{x}, \hat{\lambda})$ *satisfies the Kuhn-Tucker conditions. Suppose Slater's condition is satisfied*, *and* $\nabla f(\hat{x}) \neq 0$*. Then* $\hat{x}$ *solves problem* (Q).

  – Proof: To be discussed in class.

3. The Necessity Theorem of Arrow-Enthoven

- **Theorem 2:**

  Suppose f, g^j ($j = 1, 2, ..., m$) *are continuously differentiable functions on* $\Re^n$. *Suppose* g^j ($j = 1, 2, ..., m$) *are quasi-concave on* $\Re^n$, *and there is* $x^* \in C$ *such that* $g^j(x^*) > 0$, *for* $j = 1, 2, ..., m$. *If* $\hat{x} \in \Re^n_+$ *solves problem* (Q), *and for each* $j = 1, 2, ..., m$, $\nabla g^j(\hat{x}) \neq 0$, *then there is* $\hat{\lambda} \in \Re^m_+$ *such that* $(\hat{x}, \hat{\lambda})$ *satisfies the Kuhn-Tucker conditions*.

#2. **Example 1:**

Suppose $a, b, c > 0$. Consider the following minimization problem:

$$\left.\begin{array}{ll} \text{Minimize} & ax_1 + bx_2 \\ \text{subject to} & x_1 x_2 \geq c, \\ \text{and} & (x_1, x_2) \in \Re^2_+. \end{array}\right\}$$

Find out the solution(s) to the minimization problem.

– (Mention clearly which theorem you are using, and demonstrate carefully that all the required conditions of that theorem are satisfied.)

#3. **Example 2:**

Consider the following maximization problem:

$$\left.\begin{array}{ll} \text{Maximize} & x_1 + x_1x_2 + x_2 \\ \text{subject to} & 1 - \alpha x_1 - \beta x_2 \geq 0,\ \alpha > 0,\ \beta > 0 \\ \text{and} & (x_1, x_2) \in \Re^2_+. \end{array}\right\}$$

(a) Take the *sufficiency route* to find the solution(s) to this problem. Explain your steps clearly.

(b) Take the *necessary route* to find the solution(s) to this problem. Explain your steps clearly.

References

- Must read the following sections from the textbook:
  - Section 21.5 (pages 532 – 537): Concave Programming.

- This material is primarily based on

1. Arrow, K.J. and Enthoven, A.C., (1961), “Quasi-concave Programming,” *Econometrica*, pages 779-800.