Modern Optimization Theory:

Topics in Optimization Theory

1. Meaning of the Multiplier

- The multipliers play an important role in economic analysis.
  - The multipliers measure the sensitivity of the optimal value of the objective function to changes in the right-hand sides of the constraints.
    - As a result, they provide a natural measure of value for scarce resources in maximization problems in economics.

- **Theorem 1 (One Equality Constraint):**

  Let f and h be C^1 functions on $\Re^2$. For any fixed value of the parameter a, let $(x^*(a), y^*(a))$ maximizes f subject to the constraint $h(x, y) = a$, and let the corresponding Lagrange multiplier be $\mu^*(a)$. Suppose that x^*, y^* and μ^* are C^1 functions of a and that NDCQ holds at $(x^*(a), y^*(a), \mu^*(a))$. Then

$$\mu^*(a) = \frac{d}{da} f(x^*(a), y^*(a)).$$

  - Proof: To be discussed in class.

- **Theorem 2 (Several Equality Constraints):**

  Let f, h^1, ..., h^m *be* C^1 *functions on* $\Re^n$. *Let* $a = (a_1, ..., a_m)$ *be an* m*-tuple of exogenous parameters, and consider the problem of maximizing* $f(x_1, ..., x_n)$ *subject to the constraints*

$$h^1(x_1, ..., x_n) = a_1, \ ..., \ h^m(x_1, ..., x_n) = a_m.$$

  Let $x_1^*(a)$, ..., $x_n^*(a)$ *solves the problem with corresponding Lagrange multipliers* $\mu_1^*(a)$, ..., $\mu_m^*(a)$. *Suppose further that* x_1^*, ..., x_n^* *and* μ_1^*, ..., μ_m^* *are* C^1 *functions of* a *and that NDCQ holds. Then*

$$\mu_j^*(a) = \frac{\partial}{\partial a_j} f(x_1^*(a), \ ..., \ x_n^*(a)).$$

  – Proof: To be discussed in class.

- **Theorem 3 (Inequality Constraints):**

  Let f, g^1, ..., g^m *be* C^1 *functions on* $\Re^n$. *Let* $a = (a_1, ..., a_m)$ *be an* m*-tuple of exogenous parameters, and consider the problem of maximizing* $f(x_1, ..., x_n)$ *subject to the constraints*

$$g^1(x_1, ..., x_n) \leq a_1,\ ...,\ g^m(x_1, ..., x_n) \leq a_m.$$

  Let $x_1^*(a)$, ..., $x_n^*(a)$ *solves the problem with corresponding Lagrange multipliers* $\lambda_1^*(a)$, ..., $\lambda_m^*(a)$. *Suppose further that* x_1^*, ..., x_n^* *and* λ_1^*, ..., λ_m^* *are* C^1 *functions of* a *and that NDCQ holds. Then*

$$\lambda_j^*(a) = \frac{\partial}{\partial a_j} f\left(x_1^*(a),\ ...,\ x_n^*(a)\right).$$

  – Proof: To be discussed in class.

2. Envelope Theorems

- Theorems 1, 2 and 3 are special cases of a class of theorems which describe how the optimal value of the objective function in a parameterized optimization problem changes as one of the parameters change.
  - Such theorems are called *Envelope Theorems*.

- **Theorem 4 (Unconstrained Problems):**

  Let $f(x;\ a)$ *be a* C^1 *function of* $x \in \Re^n$ *and the scalar* a*. For each choice of the parameter* a, *consider the unconstrained maximization problem*:

$$\text{Maximize } f(x;\ a) \text{ \textit{with respect to} } x.$$

  Let $x^*(a)$ be a solution to this problem. *Suppose that* $x^*(a)$ *is a* C^1 *function of* a*. Then*

$$\frac{d}{da} f(x^*(a);\ a) = \frac{\partial}{\partial a} f(x^*(a);\ a).$$

  - Proof: To be discussed in class.

- **Theorem 5 (Constrained Problems):**

  Let $f, h^1, ..., h^m : \Re^n \times \Re \to \Re$ *be* C^1 *functions. Consider the problem of maximizing* $f(x_1, ..., x_n; a)$ *subject to the constraints*

$$h^1(x_1, ..., x_n; a) = 0, \ ..., \ h^m(x_1, ..., x_n; a) = 0.$$

  Let $x^*(a) = (x_1^*(a), ..., x_n^*(a))$ *solves the problem with corresponding Lagrange multipliers* $\mu_1^*(a), ..., \mu_m^*(a)$. *Suppose further that* $x_1^*, ..., x_n^*$ *and* $\mu_1^*, ..., \mu_m^*$ *are* C^1 *functions of* a *and that NDCQ holds. Then*

$$\frac{d}{da} f(x^*(a); a) = \frac{\partial}{\partial a} L(x^*(a), \mu^*(a); a)$$

  where $L(x, \mu; a) = f(x; a) - \mu_1 h^1(x; a) - \ ... \ - \mu_m h^m(x; a)$, *the Lagrangian for this problem.*

  – Proof: To be discussed in class.

3. Smooth Dependence on the Parameters

- Theorems 1 – 5 have two basic hypotheses:
  - the smooth dependence of the maximizers on the parameters of the problem, and
  - the nondegenerate constraint qualification (NDCQ).
- Now we will look at these two hypotheses a little more carefully and show how to phrase them in terms of the problem's objective and constraint functions.
- **Parameterized Unconstrained Problems:**

  Consider the problem:

$$\text{Maximize } f(x;\ a) \textit{ with respect to } x.$$

  - Since we are assuming that a maximizer $x^*(a)$ exists, then $x^*(a)$ is a solution to the first-order conditions:

$$\frac{\partial}{\partial x_1} f(x_1, ..., x_n; a) = 0, ..., \ \frac{\partial}{\partial x_n} f(x_1, ..., x_n; a) = 0.$$

– By the Implicit Function Theorem, we can solve these n equations for the n unknowns $x_1, ..., x_n$ as C^1 functions of the exogenous variable a provided that the Jacobian of the functions $\frac{\partial f}{\partial x_1}$, ..., $\frac{\partial f}{\partial x_n}$, with respect to the endogenous variables $x_1, ..., x_n$, is *nonsingular* at $(x^*(a); a)$.

– But the Jacobian is simply the Hessian of f at $(x^*(a); a)$:

$$H_f(x^*(a); a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 x_1} & \cdots & \frac{\partial^2 f}{\partial x_n x_1} \\ \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_n x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 x_n} & \frac{\partial^2 f}{\partial x_2 x_n} & \cdots & \frac{\partial^2 f}{\partial x_1^2} \end{pmatrix}.$$

– Hence, we can replace the hypothesis that $x^*(a)$ is a C^1 function of a, by the hypothesis that the Hessian matrix of f is *nonsingular* at $(x^*(a); a)$.

- **Parameterized Constrained Problems:**

Consider the problem:

$$\text{Maximize} \quad f(x_1, ..., x_n; a)$$

$$\text{subject to} \quad h^j(x) = 0, \text{ for } j = 1, 2, ..., m.$$

– Assuming that NDCQ

$$rank \begin{pmatrix} \frac{\partial h^1}{\partial x_1}(x^*(a); a) & \cdots & \frac{\partial h^1}{\partial x_n}(x^*(a); a) \\ \vdots & \ddots & \vdots \\ \frac{\partial h^m}{\partial x_1}(x^*(a); a) & \cdots & \frac{\partial h^m}{\partial x_n}(x^*(a); a) \end{pmatrix} = m$$

holds at $x^*(a)$, we write the Lagrangian for this problem as

$$L(x, \mu; a) = f(x; a) - \mu_1 h^1(x; a) - ... - \mu_m h^m(x; a).$$

– The constrained maximizer $x^*(a)$ must satisfy the first order conditions

$$\frac{\partial L}{\partial \mu_j}(x,\mu;a) = 0,\ j=1,2,...,m, \text{ and}$$
$$\frac{\partial L}{\partial x_i}(x,\mu;a) = 0,\ i=1,2,...,n,$$

a system of $(n+m)$ equations in $(n+m)$ unknowns $\mu_1,...,\mu_m, x_1,...,x_n$.

– Once again, we call on the Implicit Function Theorem for conditions that will guarantee that $x^*(a)$ and $\mu^*(a)$ are C^1 functions of a: the Jacobian of the functions $\frac{\partial L}{\partial \mu_1}$, ..., $\frac{\partial L}{\partial \mu_m}$, $\frac{\partial L}{\partial x_1}$, ..., $\frac{\partial L}{\partial x_n}$, with respect to the endogenous variables $\mu_1,...,\mu_m$, $x_1,...,x_n$, is *nonsingular* at $(x^*(a), \mu^*(a); a)$.

– This Jacobian is simply the Hessian of the Lagrangian:

$$H_L = \begin{pmatrix} 0 & \cdots & 0 & -\frac{\partial h^1}{\partial x_1} & \cdots & -\frac{\partial h^1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{\partial h^m}{\partial x_1} & \cdots & -\frac{\partial h^m}{\partial x_n} \\ -\frac{\partial h^1}{\partial x_1} & \cdots & -\frac{\partial h^m}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_n x_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial h^1}{\partial x_n} & \cdots & -\frac{\partial h^m}{\partial x_n} & \frac{\partial^2 L}{\partial x_1 x_n} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{pmatrix}$$

evaluated at $(x^* (a), \mu^* (a); a)$.

– So the required condition is that H_L is *nonsingular* at $(x^* (a), \mu^* (a); a)$.

– Note the important role played by the NDCQ.

- If NDCQ does not hold, then H_L is *singular* at $(x^* (a), \mu^* (a); a)$.

– In summary, we can replace the conditions in Theorem 5 that "x_1^*, ..., x_n^* and μ_1^*, ..., μ_m^* are C^1 functions of a and that NDCQ holds" by the condition that the Hessian of the Lagrangian, H_L, is *nonsingular* at $(x^*(a), \mu^*(a); a)$.

- **Theorem 6:**

Let $f, h^1, ..., h^m : \Re^n \times \Re \to \Re$ be C^1 functions. Consider the problem of maximizing $f(x_1, ..., x_n; a)$ subject to the constraints

$$h^1(x_1, ..., x_n; a) = 0, \ ..., \ h^m(x_1, ..., x_n; a) = 0.$$

Let $x^*(a) = (x_1^*(a), ..., x_n^*(a))$ solves the problem with corresponding Lagrange multipliers $\mu_1^*(a), ..., \mu_m^*(a)$. If the Hessian matrix of the Lagrangian, H_L, is nonsingular at $(x^*(a), \mu^*(a); a)$, then

(a) $x^*(a)$ *and* $\mu^*(a)$ *are* C^1 *functions of* a, *and*

(b) *the NDCQ holds at* $(x^*(a), \mu^*(a); a)$.

References

- Must read the following sections from the textbook:
  - Section 19.1 (pages 448 – 453): The Meaning of the Multiplier;
  - Section 19.2 (pages 453 – 457): Envelope Theorems;
  - Section 19.4 (pages 469 – 472): Smooth Dependence on the Parameters.