Math 271: Mathematical Methods | Tridip Ray
Semester I, 2020-21 | ISI, Delhi

Final Exam: Question 1 (01 April 2021)

- Maximum marks: **15**
- Time allotted (including uploading on Moodle): **35 minutes**

- Consider the following definition of differentiability. Suppose that f is defined on an open interval I containing the point ξ. Then f is said to be *differentiable* at the point ξ if and only if the limit

$$\lim_{x \to \xi} \frac{f(x) - f(\xi)}{x - \xi}$$

exists. If the limit exists, it is called the *derivative* of f at the point ξ and denoted by $f'(\xi)$.

- **Question:** Use the above definition of differentiability to prove the following theorem.

Suppose that f is differentiable on (a, b) and that $\xi \in (a, b)$. If f has a local maximum or local minimum at ξ, then $f'(\xi) = 0$.

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Final Exam: Question 2 (01 April 2021)

- Maximum marks: **35**
- Time allotted (including uploading on Moodle): **80 minutes**

- Suppose $w_1, w_2, q > 0$. Consider the following constrained minimization problem:

$$\left.\begin{array}{ll} \underset{\{x_1,x_2\}}{\text{Minimize}} & w_1 x_1 + w_2 x_2 \\ \text{subject to} & x_1 + x_1 x_2 + x_2 \geq q, \\ \text{and} & (x_1, x_2) \in \mathbb{R}^2_+. \end{array}\right\} \quad \text{(Q)}$$

(a) [7 marks] Mention clearly which route (necessary or sufficient) you are taking to solve problem (Q) and demonstrate carefully that all the required conditions of that route are satisfied.

(b) [25 marks] Solve the constrained minimization problem (Q) showing your procedure clearly and answer the following questions.

(i) Suppose $\frac{w_1}{w_2} \leq \frac{1}{1+q}$. Obtain the solution(s) to the problem, and draw an appropriate diagram to illustrate your solution.

(ii) Suppose $\frac{w_1}{w_2} \geq 1+q$. Obtain the solution(s) to the problem, and draw an appropriate diagram to illustrate your solution.

(iii) Suppose $\frac{1}{1+q} < \frac{w_1}{w_2} < 1+q$. Obtain the solution(s) to the problem, and draw an appropriate diagram to illustrate your solution.

(c) [3 marks] For each specification of $(w_1, w_2, q) \in \Re^3_{++}$, define the *value function*,

$$V(w_1, w_2, q) = \left.\begin{array}{ll} \underset{\{x_1,x_2\}}{\text{Minimize}} & w_1 x_1 + w_2 x_2 \\ \text{subject to} & x_1 + x_1 x_2 + x_2 \geq q, \\ \text{and} & (x_1, x_2) \in \mathbb{R}^2_+. \end{array}\right\}.$$

For each case in part (b), compute $\frac{\partial}{\partial q} V(w_1, w_2, q)$ and compare it with the Kuhn-Tucker multiplier associated with the constraint "$x_1 + x_1 x_2 + x_2 \geq q$" obtained in that case.

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Final Exam: Question 3 (01 April 2021)

- Maximum marks: **28**
- Time allotted (including uploading on Moodle): **50 minutes**

- Consider the following optimization problem:

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & g^j(x) = 0, \text{ for } j = 1, 2, ..., m \\ \text{and} & x \in A \end{array}\right\} \text{(P)}$$

where A is a non-empty open subset of $\mathbb{R}^n$, and f, g^j $(j = 1, 2, ..., m)$ are continuously differentiable functions from A to $\mathbb{R}$. We define the constraint set as

$$C = \left\{x \in A\text{: } g^j(x) = 0, \text{ for } j = 1, 2, ..., m\right\}.$$

We also define the Lagrangian, $L : A \times \mathbb{R}^m \to \mathbb{R}$, as

$$L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g^j(x).$$

(a) [3 marks] Write down the appropriate *first-order conditions* for the optimization problem.

(b) [10 marks] Suppose a pair $(x^*, \lambda^*) \in \left(A \times \mathbb{R}^m_+\right)$ satisfies the first-order conditions. Prove that if *each* g^j *is quasi-concave*, then

$$x \in C \text{ implies } (x - x^*) \cdot \nabla f(x^*) \leq 0.$$

[Note that $\lambda_j \geq 0$, for each j.]

(c) [15 marks] Suppose a pair $(x^*, \lambda^*) \in (A \times \mathbb{R}^m)$ satisfies the first-order conditions. Prove that if C *is a convex set*, then

$$x \in C \text{ implies } (x - x^*) \cdot \nabla f(x^*) \leq 0.$$

[Observe that the non-negativity of the multipliers is not required here.]

- [Hint: Calculate, for each j, $(x - x^*) \cdot \nabla g^j(x^*)$.]

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Final Exam: Question 4 (01 April 2021)

- Maximum marks: **22**
- Time allotted (including uploading on Moodle): **45 minutes**

- Consider the following optimization problem:

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & g^j(x) = 0, \text{ for } j = 1, 2, ..., m \\ \text{and} & x \in A \end{array}\right\} \text{(P)}$$

where A is a non-empty open subset of $\mathbb{R}^n$, and f, g^j $(j = 1, 2, ..., m)$ are continuously differentiable functions from A to $\mathbb{R}$. We define the constraint set as

$$C = \left\{x \in A\text{: } g^j(x) = 0, \text{ for } j = 1, 2, ..., m\right\}.$$

We also define the Lagrangian, $L : A \times \mathbb{R}^m \to \mathbb{R}$, as

$$L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g^j(x).$$

- In part (b) of Question 3 you have proved the following. Suppose a pair $(x^*, \lambda^*) \in \left(A \times \mathbb{R}^m_+\right)$ satisfies the first-order conditions. If *each* g^j *is quasi-concave*, then

$$x \in C \text{ implies } (x - x^*) \cdot \nabla f(x^*) \leq 0.$$

- In part (c) of Question 3 you have proved the following. Suppose a pair $(x^*, \lambda^*) \in (A \times \mathbb{R}^m)$ satisfies the first-order conditions. If C *is a convex set*, then

$$x \in C \text{ implies } (x - x^*) \cdot \nabla f(x^*) \leq 0.$$

- **Question:** Prove the following two theorems.

- **Theorem 1:** *If* f, g^j $(j = 1, 2, ..., m)$ *are quasi-concave functions, and if* (x^*, λ^*) *satisfy the first-order conditions,* $\nabla f(x^*) \neq 0$, $\lambda_j^* \geq 0$ *for each* j, *and* $x^* \in C$, *then* x^* *solves the optimization problem* (P).
- **Theorem 2:** *If* f *is a quasi-concave function,* C is *a convex set, and if* (x^*, λ^*) *satisfy the first-order conditions,* $\nabla f(x^*) \neq 0$, *and* $x^* \in C$, *then* x^* *solves the optimization problem* (P).
- **Hints:**
  - With reference to your answers to Question 3, think about a common proposition that will prove both the theorems.
  - In order to prove that common proposition, you need to rule out the case

$$(x - x^*) \cdot \nabla f(x^*) = 0 \qquad \text{and } f(x) > f(x^*).$$

Consider perturbing x to x' given by $x' = x + tv$, where $t > 0$ is a scalar and v is the non-zero vector $-\nabla f(x^*)$.