Math 271: Mathematical Methods — Tridip Ray
Semester I, 2021-22 — ISI, Delhi

Final Exam: Question 1 (19 January 2022)

- Maximum marks: **20**
- Time allotted (including uploading on Moodle): **45 minutes**

- Prove the following theorem.

  Let f, h^1, ..., $h^m : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be C^1 functions. Consider the problem of maximizing $f(x_1, ..., x_n; a)$ subject to the constraints

  $$h^1(x_1, ..., x_n; a) = 0, \; ..., \; h^m(x_1, ..., x_n; a) = 0.$$

  Let $x^*(a) = (x_1^*(a), \; ..., \; x_n^*(a))$ solves the problem with corresponding Lagrange multipliers $\mu_1^*(a), ..., \mu_m^*(a)$. Suppose further that $x_1^*, ..., x_n^*$ and $\mu_1^*, ..., \mu_m^*$ are C^1 functions of a and that the nondegenerate constraint qualification (NDCQ) holds. Then

  $$\frac{d}{da} f(x^*(a); a) = \frac{\partial}{\partial a} L(x^*(a), \mu^*(a); a)$$

  where $L(x, \mu; a) = f(x; a) - \mu_1 h^1(x; a) - \; ... \; - \mu_m h^m(x; a)$, the Lagrangian for this problem.

Tridip Ray
ISI, Delhi

Final Exam: Question 2 (19 January 2022)

- Maximum marks: **30**
- Time allotted (including uploading on Moodle): **60 minutes**

- Consider the following constrained maximization problem:

$$\underset{\{x\in\mathbb{R}^n\}}{\text{Maximize}}\ f(x) = x^T A x \quad \text{subject to} \quad x^T x = 1,$$

where A is a given *symmetric* $n \times n$ matrix.

(a) [5 marks] Mention clearly which route (necessary or sufficient) you are taking to solve this problem and demonstrate carefully that all the required conditions of that route are satisfied.

- Define the Lagrangian for this problem as $L(x, \lambda) = x^T A x + \lambda \left(1 - x^T x\right)$.

(b) [15 marks] Suppose $x^* \in \mathbb{R}^n$ is a solution to the problem. Show that x^* is a normalized *eigenvector* of A and $\lambda = f(x^*)$.

(c) [10 marks] Explain how you can obtain $f(x^*)$. Illustrate your method when

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

with $a > 0$ and $c > 0$.

Math 271: Mathematical Methods | Tridip Ray
Semester I, 2021-22 | ISI, Delhi

Final Exam: Question 3 (19 January 2022)

- Maximum marks: **25**
- Time allotted (including uploading on Moodle): **50 minutes**

- Recall the following theorem.

  Theorem (Taylor's Expansion upto Second Order):

  Suppose A is an open convex subset of $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ is twice continuously differentiable on A. Suppose x^1 and x^2 are in A. Then there is $0 \leq \theta \leq 1$ such that

  $$f\left(x^2\right) - f\left(x^1\right) = \left(x^2 - x^1\right) \cdot \nabla f\left(x^1\right) + \frac{1}{2}\left(x^2 - x^1\right) \cdot H_f\left(\theta x^1 + (1-\theta)\, x^2\right) \cdot \left(x^2 - x^1\right).$$

- Suppose D is an open convex set in $\mathbb{R}^n$, and $g : D \to \mathbb{R}$ is twice continuously differentiable and *quasiconcave* on D. Suppose there exists $x^* \in D$ satisfying

  (i) $\nabla g\left(x^*\right) = 0,$ and
  (ii) $H_g\left(x^*\right)$ is *negative definite*.

(a) [15 marks] Use the above theorem to prove that g has a *strict* local maximum at x^*.

(b) [10 marks] Prove further that x^* is a point of *global maximum* of g on D.

Math 271: Mathematical Methods
Semester I, 2021-22

Tridip Ray
ISI, Delhi

Final Exam: Question 4 (19 January 2022)

- Maximum marks: **25**
- Time allotted (including uploading on Moodle): **50 minutes**

- In this question we will prove the following *sufficiency theorem* under quasiconcave programming.

  Suppose f, g^j $(j = 1, 2, ..., m)$ *are continuously differentiable quasiconcave functions on* $\mathbb{R}^n$. *Suppose there is a pair* $(\hat{x}, \hat{\lambda}) \in \left(\mathbb{R}^n_+ \times \mathbb{R}^m_+\right)$ *such that* $(\hat{x}, \hat{\lambda})$ *satisfies the Kuhn-Tucker conditions. Suppose further that* $\nabla f(\hat{x}) \neq 0$. *Then* $\hat{x}$ *solves the following problem:*

  $$\text{Maximize } f(x), \qquad \text{subject to} \qquad x \in C \equiv \left\{x \in \mathbb{R}^n_+ \colon g^j(x) \geq 0, \text{ for } j = 1, 2, ..., m\right\}.$$

- We will develop the proof in the following three steps.

(a) [4 marks] **Step I:** Given the premises of the theorem, in the lectures we have proved that

$$x \in C \textit{ implies } (x - \hat{x}) \cdot \nabla f(\hat{x}) \leq 0.$$

  – Which premises of the theorem we do *not* need to get this result?

(b) [5 marks] **Step II:** Prove that if $x \in C$ and $(x - \hat{x}) \cdot \nabla f(\hat{x}) < 0$, then $\hat{x}$ solves the problem.

(c) [16 marks: $5 + 6 + 5$] **Step III:** In this step we will prove that if $x \in C$ and $(x - \hat{x}) \cdot \nabla f(\hat{x}) = 0$, then $\hat{x}$ solves the problem.

  We will prove this by contradiction. That is, we will show that if $(x - \hat{x}) \cdot \nabla f(\hat{x}) = 0$ and $f(x) > f(\hat{x})$, then, given the premises of the theorem, a contradiction will arise.

  (i) Given the premises of the theorem, choose $v \neq 0$ appropriately so that if $(x - \hat{x}) \cdot \nabla f(\hat{x}) = 0$, then $(x + tv - \hat{x}) \cdot \nabla f(\hat{x}) < 0$, for all $t > 0$.

(ii) If $f(x) > f(\hat{x})$, then argue that there exists $t > 0$ small enough so that $f(x + tv) > f(\hat{x})$.

(iii) Explain the contradiction clearly.