Math 271: Mathematical Methods — Tridip Ray
Semester I, 2022-23 — ISI, Delhi

Final Exam (04 December 2022)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [17 marks]

   Recall the following theorem.

   Theorem (Taylor's Expansion upto Second Order):

   Suppose A is an open convex subset of $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ is twice continuously differentiable on A. Suppose x^1 and x^2 are in A. Then there is $0 \leq \theta \leq 1$ such that

   $$f\left(x^2\right) - f\left(x^1\right) = \left(x^2 - x^1\right) \cdot \nabla f\left(x^1\right) + \frac{1}{2}\left(x^2 - x^1\right) \cdot H_f\left(\theta x^1 + (1-\theta)\, x^2\right) \cdot \left(x^2 - x^1\right).$$

   [$H_f(y)$ is the Hessian matrix of f evaluated at $y \in A$.]

   - Let A be an open convex subset of $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ be twice continuously differentiable on A. Using the above theorem prove carefully that if $x^* \in A$ is a point of *local maximum* of f, then $H_f(x^*)$ is *negative semi-definite*.

2. [18 marks]

   Consider the following theorem.

   Let $f : I \to \mathbb{R}$ be a C^3 function defined on an open interval I in $\mathbb{R}$. For any two points a and $a+h$ in I, there exists a point $a < c < a+h$ such that

   $$f(a+h) = f(a) + f'(a)\, h + \frac{1}{2!} f''(a)\, h^2 + \frac{1}{3!} f'''(c)\, h^3.$$

   - Let $f : I \to \mathbb{R}$ be a C^3 function defined on an open interval I in $\mathbb{R}$ and $x^* \in I$. Using the above theorem prove carefully that if $f'(x^*) = 0$ and $f''(x^*) < 0$, then x^* is a point of *strict local maximum* of f.

3. [25 marks: $13 + 12$]

- Consider the following optimization problem:

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & g^j(x) \geq 0, \text{ for } j = 1, 2, ..., m \\ \text{and} & x \in \mathbb{R}^n_+. \end{array}\right\} \text{(P)}$$

where f, g^j $(j = 1, 2, ..., m)$ are functions from $\mathbb{R}^n$ to $\mathbb{R}$. The constraint set for problem (P) is defined as $C = \left\{x \in \mathbb{R}^n_+ \colon g^j(x) \geq 0, \text{ for } j = 1, 2, ..., m\right\}$.

- Kuhn-Tucker Conditions:
A pair $\left(\hat{x}, \hat{\lambda}\right) \in \left(\mathbb{R}^n_+ \times \mathbb{R}^m_+\right)$ satisfies the Kuhn-Tucker conditions if

(i) $\frac{\partial f}{\partial x_i}(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j \frac{\partial g^j}{\partial x_i}(\hat{x}) \leq 0$, $\hat{x}_i \left[\frac{\partial f}{\partial x_i}(\hat{x}) + \sum_{j=1}^{m} \hat{\lambda}_j \frac{\partial g^j}{\partial x_i}(\hat{x})\right] = 0$, and $\hat{x}_i \geq 0$, $i = 1, 2, ..., n$;

(ii) $g^j(\hat{x}) \geq 0$, $\hat{\lambda}_j g^j(\hat{x}) = 0$, and $\hat{\lambda}_j \geq 0$, $j = 1, 2, ..., m$.

- **Lemma:**

Suppose f, g^j $(j = 1, 2, ..., m)$ are continuously differentiable functions on $\mathbb{R}^n$. Suppose there is a pair $(\hat{x}, \hat{\lambda}) \in \left(\mathbb{R}^n_+ \times \mathbb{R}^m_+\right)$ such that $(\hat{x}, \hat{\lambda})$ satisfies the Kuhn-Tucker conditions. If each g^j is quasi-concave, then

$$x \in C \textit{ implies } (x - \hat{x}) \cdot \nabla f(\hat{x}) \leq 0.$$

- For both the parts of this question assume that f, g^j $(j = 1, 2, ..., m)$ are continuously differentiable *quasi-concave* functions on $\mathbb{R}^n$ and the pair $(x^*, \lambda^*) \in \left(\mathbb{R}^n_+ \times \mathbb{R}^m_+\right)$ satisfies the *Kuhn-Tucker conditions.*

(a) [13 marks] In this part we will prove that *if* $\frac{\partial f}{\partial x_i}(x^*) < 0$ *for some* $i \in \{1, 2, ..., n\}$, *then* x^* *solves problem (P).*

  - Let x be an arbitrary vector in C. Let e^i be the i-th unit vector, and define $\bar{x} = x^* + e^i$. Also define, for $0 < \theta < 1$, $x(\theta) = \theta\bar{x} + (1 - \theta)x$, and $y(\theta) = \theta\bar{x} + (1 - \theta)x^*$.

  (i) Show that $(y(\theta) - x^*) \cdot \nabla f(x^*) < 0$, and $(x(\theta) - y(\theta)) \cdot \nabla f(x^*) \leq 0$.

  (ii) Show that $f(x(\theta)) < f(x^*)$, and complete the proof for this part.

(b) [12 marks] An index $k \in \{1, 2, ..., n\}$ is called a *relevant index* if there exists $\tilde{x} \in C$ such that $\tilde{x}_k > 0$. In this part we will prove that *if* $\frac{\partial f}{\partial x_i}(x^*) > 0$ *for some* i *which is a relevant index, then* x^* *solves problem (P).*

(i) Argue that, given part (a) above, we need to consider only the case where $\nabla f(x^*) \geq 0$, and $\frac{\partial f}{\partial x_k}(x^*) > 0$, for some index k which is a relevant index.

(ii) Let $\tilde{x} \in C$ be such that $\tilde{x}_k > 0$. Show that $x^* \cdot \nabla f(x^*) > 0$.

(iii) Let x be an arbitrary vector in C. Show that, for $0 < \theta < 1$, $(\theta x) \cdot \nabla f(x^*) < x^* \cdot \nabla f(x^*)$, and complete the proof for this part.

4. [40 marks: $5 + 3 + 1 + 25 + 6$]

A person lives for two periods: period 1 (youth) and period 2 (working age). In her youth (period 1) she inherits an wealth $W \geq 0$ from her parents and acquires education for which she has to spend an amount $E > 0$, $E \leq W$. Her education gets her a job that pays her an income $Y > 0$ during her working period (period 2). Her preference between period 1 consumption (c_1) and period 2 consumption (c_2) is given by the utility function

$$u(c_1, c_2) = \log c_1 + \beta \log c_2, \ \beta > 0.$$

She can do intertemporal substitution of consumption through borrowing and saving. But the credit market does not work perfectly and she faces two different interest rates for borrowing and lending: the borrowing interest rate, $r_B > 0$, is strictly greater than the lending interest rate, $r_L > 0$, that is, $r_B > r_L$.

(a) [5 marks] Set up the utility maximization problem. Draw the budget constraint with c_1 on x-axis and c_2 on y-axis, and label the important points clearly.

(b) [3 marks] Mention clearly which route (necessary or sufficient) you are taking to solve the utility maximization problem and demonstrate carefully that all the required conditions of that route are satisfied.

(c) [1 mark] Can either $c_1 = 0$ or $c_2 = 0$ be a solution to the utility maximization problem? Explain clearly.

(d) [25 marks] Solve this utility maximization problem showing your procedure clearly and answer the following questions.

(i) Show that there exist two wealth thresholds, $\underline{W}$ and $\overline{W}$, with $0 < \underline{W} < \overline{W}$, such that the nature of the solution to the utility maximization problem differs according as whether $0 \leq W < \underline{W}$, or $\underline{W} \leq W \leq \overline{W}$, or $\overline{W} < W$. (You have to derive $\underline{W}$ and $\overline{W}$ in terms of the parameters of the problem: E, Y, β, r_L and r_B.)

(ii) For each of the three cases – (I) $0 \leq W < \underline{W}$, (II) $\underline{W} \leq W \leq \overline{W}$, and (III) $\overline{W} < W$ – derive the optimal choices of c_1, c_2, *borrowing* and *saving* in terms of the parameters of the problem.

(e) [6 marks] Draw the *wealth expansion path*, that is, the combinations of choices of c_1 and c_2 when only W changes, with c_1 on x-axis and c_2 on y-axis, clearly illustrating your answers in part (d). (You must label all the important points clearly.)