Math 271: Mathematical Methods
Semester I, 2023-24

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Final Exam (30 November 2023)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [25 marks]

   Let $A \subset \mathbb{R}^n$ be open, and $f : A \to \mathbb{R}$, $g^j : A \to \mathbb{R}$ be continuously differentiable functions on A, $j = 1, 2, ..., m$. Suppose x^* is a point of local maximum of f subject to the constraints $g^j(x) = 0$, $j = 1, 2, ..., m$. Form the Lagrangian

   $$L(x_1, ..., x_n, \lambda_0, \lambda_1, ..., \lambda_m) \equiv \lambda_0 f(x) - \lambda_1 g^1(x) - \lambda_2 g^2(x) - ... - \lambda_m g^m(x),$$

   with a multiplier λ_0 for the objective function.

   – Prove that there exist multipliers $\lambda^* = (\lambda_0^*, \lambda_1^*, ..., \lambda_m^*)$ such that

   (i) $\dfrac{\partial L}{\partial x_i}(x^*, \lambda^*) = 0, \ i = 1, 2, ..., n,$

   (ii) $g^j(x^*) = 0, \ j = 1, 2, ..., m,$

   (iii) $(\lambda_0^*, \lambda_1^*, ..., \lambda_m^*) \neq (0, 0, ..., 0)$, and

   (iv) $\lambda_0^* = 0$ or 1.

2. [35 marks]

   Let X be an open set in $\mathbb{R}^n$, and f, g^j $(j = 1, 2, ..., k)$ be continuously differentiable on X. Suppose that $x^* \in X$ is a point of constrained local maximum of f subject to k inequality constraints:

   $$g^1(x) \leq b_1, ..., g^k(x) \leq b_k.$$

   Without loss of generality, assume that the first k_0 constraints are binding at x^* and that the last $(k - k_0)$ constraints are not binding. Suppose that the following nondegenerate constraint qualification is satisfied at x^*:

The rank at x^* of the following Jacobian matrix of the binding constraints is k_0:

$$\begin{pmatrix} \frac{\partial g^1}{\partial x_1}(x^*) & \cdots & \frac{\partial g^1}{\partial x_n}(x^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g^{k_0}}{\partial x_1}(x^*) & \cdots & \frac{\partial g^{k_0}}{\partial x_n}(x^*) \end{pmatrix}.$$

(a) [4 marks]

Prove that there exists an open ball of radius $r > 0$ around x^*, $B_r(x^*)$, such that

$$g_j(x) < b_j, \text{ for all } x \in B_r(x^*), \text{ and for } j = k_0 + 1, ..., k.$$

- We will work in the open ball $B_r(x^*)$ for the rest of this question.

(b) [2 marks]

Prove that x^* maximizes f in $B_r(x^*)$ over the constraint set

$$g_1(x) = b_1, \ ..., \ g_{k_0}(x) = b_{k_0}.$$

(c) [6 marks]

Form the Lagrangian

$$L(x, \lambda) \equiv f(x) - \lambda_1 \left[g^1(x) - b_1\right] - ... - \lambda_k \left[g^k(x) - b_k\right].$$

Prove that there exist multipliers $(\lambda_1^*, ..., \lambda_k^*)$ such that

(i) $\frac{\partial L}{\partial x_1}(x^*, \lambda^*) = 0, ..., \frac{\partial L}{\partial x_n}(x^*, \lambda^*) = 0;$

(ii) $\lambda_1^*\left[g^1(x^*) - b_1\right] = 0, ..., \lambda_k^*\left[g^k(x^*) - b_k\right] = 0;$

(iii) $g^1(x^*) \leq b_1, ..., g^k(x^*) \leq b_k.$

(d) [9 marks]

Prove that there exists a C^1 function $x(t)$ defined for $t \in [0, \varepsilon)$, such that $x(0) = x^*$ and, for all $t \in [0, \varepsilon)$,

$$g_1(x(t)) = b_1 - t,$$
$$g_j(x(t)) = b_j, \text{ for } j = 2, ..., k_0.$$

(e) [13 marks]

Prove that $\lambda_1^* \geq 0$.

(f) [1 mark]

Prove that $\lambda_2^* \geq 0, ..., \lambda_k^* \geq 0$.

3. [40 marks]

We consider the optimal tenancy contract problem between a landlord and a tenant cultivator. The landlord owns a plot of agricultural land, monetary wealth W_L, and no labour. The tenant owns no land, monetary wealth W_T, and 1 unit of labour. For the plot of land the production function is given by $Q = e + \theta$, where Q is output, e is effort put in by the cultivator, and θ stands for a random shock with zero mean and variance σ^2. Both the landlord (L) and the tenant (T) are risk-neutral so that the expected utility of agent i with an income y_i is given by $U(y_i) = E(y_i)$, the expected value of y_i, $i = L, T$. The cost of exerting effort is: $c(e) = \frac{1}{2}ce^2$, $c > 0$.

Effort, e, is neither observable nor monitorable. The tenant would need incentives to put in effort. We restrict ourselves to linear contracts, that is, the tenant's income, y_T, is a linear function of output:

$$y_T = s \cdot Q - R, \text{ where } 0 \leq s \leq 1.$$

Correspondingly, the landlord's income is:

$$y_L = Q - y_T = (1 - s) \cdot Q + R.$$

We summarize this contract as (s, R), where s is referred to as the 'share' component and R as the 'fixed-rent' component of the contract.

(a) [3 marks]

(i) Show that, under the contract, (s, R), the expected utilities of the tenant and the landlord are:

$$U_T(e, s, R) = s \cdot e - R - \frac{1}{2}ce^2,$$

and

$$U_L(e, s, R) = (1 - s) \cdot e + R.$$

(ii) The social surplus is defined as $S(e, s, R) = U_T(e, s, R) + U_L(e, s, R)$. The effort level that maximizes the social surplus is referred to as the 'efficient' or the 'first-best' effort level.

Find out the efficient effort level.

- Tenant's Participation Constraint: Let the market wage rate be $x > 0$. Since the tenant owns 1 unit of labour, his reservation (expected) utility is x. So the participation constraint of the tenant is: $U_T(e, s, R) \geq x$.

- Tenant's Incentive Compatibility Constraint: Since the landlord cannot observe e (and hence cannot specify e in the contract), he has to try to influence it through the choice of s and R. This adds the incentive compatibility constraint into the contracting problem:

$$e = \arg\max_{\{e\}} U_T(e, s, R) = \arg\max_{\{e\}} \left[s \cdot e - R - \frac{1}{2} c e^2 \right] = \frac{s}{c}.$$

- Tenant's Limited Liability Constraint: We add the constraint that the tenant has to pay the fixed component of the rent in advance and this amount is bounded above by his wealth W_T:

$$R \leq W_T.$$

- Optimal Contracting Problem: The optimal contracting problem for the landlord is to choose the contract (s, R) so as to maximize his expected utility, $U_L(e, s, R)$, subject to the tenant's participation constraint (PC), incentive compatibility constraint (ICC), and limited liability constraint (LLC).

(b) [5 marks]

Set up the optimal contracting problem as an optimization problem subject to only the tenant's participation constraint (PC) and the limited liability constraint (LLC) by substituting the incentive compatibility constraint (ICC) appropriately into $U_T(e, s, R)$ and $U_L(e, s, R)$.

(c) [5 marks]

Mention clearly which route (necessary or sufficient) you are taking to solve the optimal contracting problem and demonstrate carefully that all the required conditions of that route are satisfied.

(d) [27 marks: $22 + 2 + 3$]

Solve the optimal contracting problem showing your procedure clearly and answer the following questions. [**Instruction:** In order to reduce the number of cases, ignore the inequality constraints on s while setting up the Lagrangian. But verify that these constraints are satisfied in the optimal solution.]

(i) Show that there exist two thresholds for the tenant's wealth, $\underline{W_T}$ and $\overline{W_T}$, with $0 < \underline{W_T} < \overline{W_T}$, such that the nature of the solution to the contracting problem differs according as whether $0 \leq W_T < \underline{W_T}$, or $\underline{W_T} \leq W_T \leq \overline{W_T}$, or $\overline{W_T} < W_T$. (You have to derive $\underline{W_T}$ and $\overline{W_T}$ in terms of the parameters of the problem: c and x.)

For each of the three cases – (I) $0 \leq W_T < \underline{W_T}$, (II) $\underline{W_T} \leq W_T \leq \overline{W_T}$, and (III) $\overline{W_T} < W_T$ – derive the optimal contract (s, R) and the resulting effort level of the tenant (e) and the landlord's expected utility, U_L, in terms of the parameters of the problem.

(ii) Does the optimal contract achieve the 'efficient' effort level ever? Explain carefully.

(iii) The 'tenancy ladder hypothesis' says that wealthier tenants are preferred by the landlord and are given higher crop shares. Does the model support the tenancy ladder hypothesis? Explain carefully.