Math 271: Mathematical Methods
Semester I, 2024-25

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Final Exam (22 November 2024)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [40 marks]

Consider the following optimization problem:

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & g^j(x) = 0, \text{ for } j = 1, 2, ..., m \\ \text{and} & x \in A \end{array}\right\} \text{(P)}$$

where A is a non-empty open subset of $\mathbb{R}^n$, and f, g^j $(j = 1, 2, ..., m)$ are continuously differentiable functions from A to $\mathbb{R}$. We define the constraint set as

$$C = \left\{x \in A\colon\ g^j(x) = 0,\ \text{for } j = 1, 2, ..., m\right\}.$$

We also define the Lagrangian, $L : A \times \mathbb{R}^m \to \mathbb{R}$, as

$$L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g^j(x).$$

(a) [3 marks] Write down the appropriate *first-order conditions* for the optimization problem (P).

(b) [10 marks] Suppose a pair $(x^*, \lambda^*) \in \left(A \times \mathbb{R}^m_+\right)$ satisfies the first-order conditions. Prove that if *each* g^j *is quasi-concave*, then

$$x \in C \text{ implies } (x - x^*) \cdot \nabla f(x^*) \leq 0.$$

[Note that $\lambda^*_j \geq 0$, for each j.]

(c) [17 marks: $5 + 7 + 5$] Assume further that f is *quasi-concave* with $\nabla f(x^*) \neq 0$. In this part we will prove that if $x \in C$ and $(x - x^*) \cdot \nabla f(x^*) = 0$, then x^* solves problem (P).

We will prove this by contradiction. That is, we will show that if $x \in C$, $(x - x^*) \cdot \nabla f(x^*) = 0$ and $f(x) > f(x^*)$, then, given the assumptions, a contradiction will arise.

(i) Choose $v \neq 0$ appropriately so that if $(x - x^*) \cdot \nabla f(x^*) = 0$, then $(x + tv - x^*) \cdot \nabla f(x^*) < 0$, for all $t > 0$.

(ii) If $f(x) > f(x^*)$, then argue rigorously that there exists $t > 0$ small enough so that $f(x + tv) > f(x^*)$.

(iii) Explain the contradiction clearly.

(d) [10 marks: $5+5$] We are very close to prove a *sufficiency* theorem for the optimization problem (P).

(i) Provide a precise statement of this *sufficiency* theorem.

(ii) Complete the proof of this theorem.

2. [60 marks]

We study credit contracts between borrowers and lenders. There is a group of borrower-entrepreneurs whose projects benefit from access to working capital provided by lenders. Let w denote the level of *illiquid* wealth a borrower-entrepreneur is endowed with (for example, a house or a piece of land). Each borrower-entrepreneur supplies effort $e \in [0, 1]$. Cost of effort is $\frac{e^2}{2}$. In addition to supplying effort, the borrower-entrepreneur can use working capital x to enhance productivity. Working capital x is a discrete variable that takes only two values: 0 and 1. When $x = 1$, output is $A(1 + \Delta)$ with probability e and 0 with probability $(1 - e)$. When $x = 0$, output is A with probability e and 0 with probability $(1 - e)$.

The working capital is supplied only by the lenders; no borrower-entrepreneur owns any working capital. The cost of a unit of working capital is ρ.

We assume that effort (e) is not observable and hence not contractible. The borrower-entrepreneur can use his wealth w as collateral. But there is *limited liability* in the sense that he can pay only up to $A(1 + \Delta) + w$ when output is high, and w when output is low.

A *debt contract* is a tuple (R, C) where R is the payment that the borrower has to make when the project is successful, and C is the payment to be made when the project is unsuccessful.

We make the following two assumptions:

$$A(1 + \Delta) < 1, \qquad \text{(Assumption 1)}$$

and

$$\frac{A^2 (1+\Delta)^2}{4} - \rho > 0. \qquad \text{(Assumption 2)}$$

(a) [3 marks]

Explain clearly that under a debt contract (R, C), the expected payoffs of the borrower (π^B) and the lender (π^L) are

$$\pi^B (e, R, C) = e \cdot [A (1+\Delta) - R] - (1-e) \cdot C - \frac{e^2}{2},$$

and

$$\pi^L (e, R, C) = e \cdot R + (1-e) \cdot C - \rho.$$

(b) **The First-best** [3 marks]

The expected social surplus is defined as $S(e, R, C) = \pi^B (e, R, C) + \pi^L (e, R, C)$. The effort level that maximizes the expected social surplus is referred to as the 'efficient' or the 'first-best' effort level.

(i) Derive, with a clear explanation, the 'efficient' or the 'first-best' effort level.

(ii) Derive the expression for expected social surplus at this efficient effort level.

(c) **The Constraints** [9 marks: $4+2+3$]

(i) *Borrower's Participation Constraint*: The outside option of the borrower-entrepreneur is not to borrow the working capital from the lender, that is, to choose $x = 0$.

Derive, with a clear explanation, that the *borrower's participation constraint* is

$$\pi^B \geq \frac{A^2}{2}.$$

(ii) *Borrower's Incentive Compatibility Constraint*: Since the lender cannot observe e (and hence cannot specify e in the contract), he has to try to influence it through the choice of R and C. This adds the incentive compatibility constraint into the contracting problem:

$$e = \arg\max_{\{e\}} \ \pi^B (e, R, C).$$

Derive, with a clear explanation, the *borrower's incentive compatible* effort level under a debt contract (R, C).

(iii) *Borrower's Limited Liability Constraint*: As mentioned above, there is *limited liability* on part of the borrower in the sense that he can pay only up to $A(1+\Delta)+w$ when the project is successful, and w when the project is unsuccessful. This imposes the following two *limited liability constraints* for a debt contract (R, C): $R \leq A(1+\Delta)+w$, and $C \leq w$.

Argue clearly that only the following *limited liability constraint* will be effective:

$$C \leq w,$$

that is, if this constraint holds then the other limited liability constraint follows.

(d) **The Debt Contracting Problem** [7 marks]

The optimal debt contracting problem for the lender is to choose the debt contract (R, C) so as to maximize his expected payoff, $\pi^L(e, R, C)$, subject to the borrower's participation constraint (PC), incentive compatibility constraint (ICC), and limited liability constraint (LLC).

- Set up the optimal debt contracting problem as an optimization problem subject to only the borrower's participation constraint (PC) and the limited liability constraint (LLC) by substituting the incentive compatibility constraint (ICC) appropriately into $\pi^L(e, R, C)$ and $\pi^B(e, R, C)$.

(e) [5 marks]

Mention clearly which route (necessary or sufficient) you are taking to solve the optimal debt contracting problem and demonstrate carefully that all the required conditions of that route are satisfied.

(f) **Structure of the Debt Contract** [28 marks]

Solve the optimal debt contracting problem showing your procedure clearly and answer the following questions.

- Show that there exist two thresholds for the borrower's wealth, $\underline{w}$ and $\overline{w}$, with $0 < \underline{w} < \overline{w}$, such that the nature of the solution to the contracting problem differs according as whether $0 \leq w < \underline{w}$, or $\underline{w} \leq w \leq \overline{w}$, or $\overline{w} < w$. (You have to derive $\underline{w}$ and $\overline{w}$ in terms of the parameters of the problem.)

– For each of the three cases – (I) $0 \leq w < \underline{w}$, (II) $\underline{w} \leq w \leq \overline{w}$, and (III) $\overline{w} < w$ – derive the optimal contract (R, C), the resulting effort level of the borrower (e), and the expected payoffs of the borrower (π^B) and the lender (π^L) in terms of the parameters of the problem.

(g) **Effects of Wealth** [5 marks]

(i) Demonstrate that as long as the borrower does not have enough wealth to guarantee the full value of the loan, the effort choice will be less than the first-best.

(ii) Identify, with a clear explanation, a range of borrower's wealth where marginal improvement in wealth does not affect the effort choice but has a redistributive effect with lenders gaining relative to borrowers.