Math 271: Mathematical Methods
Semester I, 2025-26

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Final Exam (10 November 2025)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [25 points: $5 + 10 + 10$]

   Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be C^1 functions given by

$$f(x) = \begin{cases} x^3, & x < 0 \\ 0, & 0 \leq x \leq 1 \\ (x-1)^2, & x > 1 \end{cases}$$

   and

$$g(x) = x, \text{ for all } x \in \mathbb{R}.$$

   Consider the problem of maximizing f over the inequality-constrained set $C = \{x \in \mathbb{R} \mid g(x) \geq 0\}$.

   (a) Argue that f and g are quasi-concave functions.

   (b) Demonstrate clearly that for any point $x^* \in [0, 1]$, there exists $\lambda^* \geq 0$ such that the pair (x^*, λ^*) satisfies the Kuhn-Tucker conditions.

   (c) Argue that no such point $x^* \in [0, 1]$ solves the problem of maximizing f over the constraint set C. What goes wrong? Provide a clear explanation.

2. [35 marks]

Consider the following optimization problem:

$$\left.\begin{array}{ll}\text{Maximize} & f(x) \\ \text{subject to} & g^j(x) = 0, \text{ for } j = 1, 2, ..., m \\ \text{and} & x \in A\end{array}\right\} \text{(P)}$$

where A is a non-empty open subset of $\mathbb{R}^n$, and f, g^j $(j = 1, 2, ..., m)$ are continuously differentiable functions from A to $\mathbb{R}$. We define the constraint set as

$$C = \left\{x \in A\text{: } g^j(x) = 0, \text{ for } j = 1, 2, ..., m\right\}.$$

We also define the Lagrangian, $L : A \times \mathbb{R}^m \to \mathbb{R}$, as

$$L(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g^j(x).$$

(a) [2 marks] Write down the appropriate *first-order conditions* for the optimization problem (P).

(b) [15 marks] Suppose a pair $(x^*, \lambda^*) \in (A \times \mathbb{R}^m)$ satisfies the first-order conditions. Prove that if C *is a convex set*, then

$$x \in C \text{ implies } (x - x^*) \cdot \nabla f(x^*) = 0.$$

- [Hint: Using the convexity of C, calculate, for each j, $(x - x^*) \cdot \nabla g^j(x^*)$.]

(c) [15 marks] Assume further that f is *quasi-concave* with $\nabla f(x^*) \neq 0$, where the pair $(x^*, \lambda^*) \in (A \times \mathbb{R}^m)$ satisfies the first-order conditions.

  (i) Choose $v \neq 0$ appropriately so that if $x \in C$, and C is a convex set, then $(x + tv - x^*) \cdot \nabla f(x^*) < 0$, for all $t > 0$.

  (ii) If $f(x) > f(x^*)$, then argue rigorously that there exists $t > 0$ small enough so that a contradiction arises.

(d) [3 marks] Observe that in the above parts we have proved a *sufficiency* theorem for the optimization problem (P).

- Provide a precise statement of this *sufficiency* theorem.

3. [40 points: $6 + 3 + 25 + 6$]

There are two goods x and y. A consumer consumes strictly positive amount of both goods and her preference for the two goods is given by the utility function

$$u(x, y) = \alpha \log x + (1 - \alpha) \log y, \quad 0 < \alpha < 1.$$

The consumer has endowments of $e_x > 0$ units of good x and $e_y > 0$ units of good y. She does not have any other income. The commodity market does not work perfectly. The price of good y is Re. 1 no matter whether an individual buys or sells good y. But for good x, the prices are Rs. $p_L > 0$ if an individual wishes to *sell* it and Rs. $p_H > 0$ if an individual wishes to *buy* it, with $p_H > p_L$.

(a) – Set up the consumer's utility maximization problem as an optimization problem with two inequality constraints. (Note that you are already told that the consumer consumes strictly positive amount of both goods.)

– Draw the consumer's budget constraint with consumption of x on x-axis and consumption of y on y-axis, and label the important points clearly.

(b) Mention clearly which route (necessary or sufficient) you are taking to solve the utility maximization problem and demonstrate carefully that all the required conditions of that route are satisfied.

(c) Solve the utility maximization problem showing your procedure clearly and answer the following questions.

– Show that there exist two thresholds of endowment of x, $\underline{e_x}$ and $\overline{e_x}$, with $0 < \underline{e_x} < \overline{e_x}$, such that the nature of the solution to the utility maximization problem differs according as whether $0 < e_x < \underline{e_x}$, or $\underline{e_x} \leq e_x \leq \overline{e_x}$, or $\overline{e_x} < e_x$. (You have to derive $\underline{e_x}$ and $\overline{e_x}$ in terms of the parameters of the problem.)

– For each of the three cases – (I) $0 < e_x < \underline{e_x}$, (II) $\underline{e_x} \leq e_x \leq \overline{e_x}$, and (III) $\overline{e_x} < e_x$ – derive the optimal choices of consumption of x and y in terms of the parameters of the problem mentioning clearly whether the consumer is a buyer or seller of x and y.

(d) Draw the *endowment-of-x expansion path*, that is, the combinations of choices of x and y when only e_x changes, with x on x-axis and y on y-axis, clearly illustrating your answers in part (c). (You must label all the important points clearly.)