Math 271: Mathematical Methods
Semester I, 2020-21

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First Midterm Exam: Question 1 (17 January 2021)

- Maximum marks: **20**
- Time allotted (including uploading on Moodle): **50 minutes**

- Suppose A is an $m \times n$ matrix where $m < n$. The $n \times m$ matrix B is a *right inverse* for A if $AB = I$, where I is the $m \times m$ identity matrix.

  Prove that A has a right inverse if and only if $rank\ (A) = m$.

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First Midterm Exam: Question 2 (17 January 2021)

- Maximum marks: **30**
- Time allotted (including uploading on Moodle): **75 minutes**

- Suppose A is an $n \times n$ real symmetric matrix.

  (a) [4 marks] Prove that the sum of all the eigenvalues of A is equal to the *trace* of A.

  – [Hint: For any two matrices C and D, *trace* $(CD) =$ *trace* (DC).]

  (b) [3 marks] Prove that the product of all the eigenvalues of A is equal to the *determinant* of A.

  (c) [6 marks] Prove that the rank of A is equal to the number of nonzero eigenvalues.

  (d) [3 marks] Prove that the eigenvalues of A^2 (defined as $A^2 \equiv AA$), are the squares of the eigenvalues of A, but the eigenvectors of both matrices are the same.

  (e) [14 marks] A matrix A is called an *idempotent* matrix if $A = A^2 = A^3 = ...$, that is, multiplying A by itself, however many times, simply reproduces the original matrix.

  (i) Prove that each eigenvalue of an idempotent matrix is either 0 or 1.

  (ii) Prove that the rank of an idempotent matrix is equal to the sum of its diagonal elements.

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First Midterm Exam: Question 3 (17 January 2021)

- Maximum marks: **15**
- Time allotted (including uploading on Moodle): **40 minutes**

- Suppose that the set of r vectors, $\{x^1, x^2, ..., x^r\}$, are linearly independent in $\mathbb{R}^n$. Prove that there exists a vector y in $\mathbb{R}^n$ such that

$$x^i y = \alpha_i, \qquad i = 1, 2, ..., r,$$

for any numbers $\alpha_1, \alpha_2, ..., \alpha_r$.

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First Midterm Exam: Question 4 (17 January 2021)

- Maximum marks: **35**
- Time allotted (including uploading on Moodle): **90 minutes**

- Consider a system of m simultaneous linear equations in n unknowns, $Ax = c$, where

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \; x_{n\times 1} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; c_{m\times 1} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}.$$

(a) [5 marks] Prove that the system of equations must have either no solution, one solution, or infinitely many solutions.

(b) [15 marks] Prove that the system of equations will have at most one solution for every choice of right-hand side $(c_1, c_2, ..., c_m)$ if and only if

$$\text{rank } (A) = \text{ number of columns of } A.$$

(c) [5 marks] Prove that the system of equations has one and only one solution for every choice of right-hand side $(c_1, c_2, ..., c_m)$ if and only if

$$\text{number of rows of } A = \text{ number of columns of } A = \text{ rank } (A)\,.$$

- A general linear model will have m equations in n variables:

$$\begin{array}{ccc} a_{11}x_1 + a_{12}x_2 + & \cdots & + a_{1n}x_n = c_1, \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + & \cdots & + a_{mn}x_n = c_m. \end{array} \tag{1}$$

The variables whose values are determined by the system of equations (1) are called *endogenous variables*. On the other hand, the variables whose values are determined outside of system (1) are called *exogenous variables*. The division of the n variables into endogenous and exogenous variables will be successful only if, after choosing values for the exogenous variables and plugging them into system (1), one can then uniquely solve the system for the endogenous variables.

(d) [10 marks] Let $x_1, x_2, ..., x_k$ and $x_{k+1}, x_{k+2}, ..., x_n$ be a partition of the n variables in (1) into endogenous and exogenous variables, respectively. Provide, with clear explanations, the necessary and sufficient conditions so that there is, for each choice of values $x^0_{k+1}, x^0_{k+2}, ..., x^0_n$ for the exogenous variables, a *unique* set of values $x^0_1, x^0_2, ..., x^0_k$ for the endogenous variables which solves (1).