Tridip Ray
ISI, Delhi

First Midterm Exam (11 September 2022)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [20 marks]

   An $n \times n$ matrix A is given, with $|A| \neq 0$. Prove that *for any* arbitrary $n \times n$ matrix B, there exist matrices C and D, with $rank\,(C) = rank\,(D)$, such that $AC = B = DA$.

2. [25 marks]

   In this problem you will prove, using the method of induction, the following theorem:

   *Let λ_1, λ_2, ..., λ_k be k **distinct** eigenvalues of the $n \times n$ matrix A. Let x^1, x^2, ..., x^k be the corresponding eigenvectors. Then, x^1, x^2, ..., x^k are linearly independent vectors.* [Note that A may not be a symmetric matrix.]

   (a) Initial step: Prove that the theorem is true for $k = 2$.

   (b) Inductive step: Define the inductive step carefully and then prove it to complete the proof of the theorem.

3. [55 marks]

- **Elementary Row Operations:** Given any matrix, we can perform the following three elementary row operations:

  - interchange two rows of a matrix,
  - change a row by adding to it a multiple of another row,
  - multiply each element in a row by the same nonzero number.

- **Row Echelon Form:** A row of a matrix is said to have k **leading zeros** if the first k elements of the row are all zeros and the $(k+1)$th element of the row is *not* zero. With this terminology, a matrix is in **row echelon form** if each row has more leading zeros than the row preceding it.

– For example, the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is in row echelon form, while the matrix $\begin{pmatrix} 0 & 7 \\ 9 & 0 \\ 0 & 2 \end{pmatrix}$ is not in row echelon form.

- **Result:** It can be shown that elementary row operations can be used, possibly repeated finitely, to reduce any matrix to a row echelon form.

- **Spanning Set:** Let $x_1,\ x_2,\ ...,\ x_k$ be a fixed set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $x_1,\ x_2,\ ...,\ x_k$,

$$\mathcal{L}\left[x_1, x_2, ..., x_k\right] \equiv \left\{c_1 x_1 + c_2 x_2 + ... + c_k x_k : c_i \in \mathbb{R},\ \forall i = 1, 2, ...k\right\},$$

is called the set **generated** or **spanned** by $x_1,\ x_2,\ ...,\ x_k$.

(a) [13 marks]

Let $x_1,\ x_2,\ ...,\ x_k$ be a collection of vectors in $\mathbb{R}^n$. For some $j > 1$, let

$$w = b_1 x_1 + b_j x_j, \text{ with } b_1, b_j \in \mathbb{R}, \text{ and } b_1 \neq 0.$$

Prove that $\mathcal{L}\left[x_1, x_2, ..., x_k\right] = \mathcal{L}\left[w, x_2, ..., x_k\right]$.

- **Row Space:** For an $m \times n$ matrix A, let $A_1,\ A_2,\ ...,A_m \in \mathbb{R}^n$ denote the m rows of A. The **row space** of A, denoted by $Row\left(A\right)$, is defined by

$$Row\left(A\right) = \mathcal{L}\left[A_1, A_2, ..., A_m\right].$$

(b) [13 marks]

(i) Prove that the elementary row operations on A leave the row space, $Row\left(A\right)$, unchanged.

(ii) Prove that any row echelon form of A, denoted by A^R, has the same row space as A.

(c) [15 marks]

Prove the following result.

Let x_1, x_2, ..., x_k be *nonzero* vectors in $\mathbb{R}^n$ such that each x_{i+1} has more leading zeros than x_i. Then the vectors x_1, x_2, ..., x_k are linearly independent.

(d) [14 marks]

Prove the following theorem.

Let A^R be any row echelon form of A. Then the rank of A is the number of nonzero rows of A^R.