Math 271: Mathematical Methods
Semester I, 2024-25

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First Midterm Exam (08 September 2024)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [45 marks: $10 + 20 + 15$]

(a) Prove that the eigenvalues of an upper-triangular matrix or a lower-triangular matrix are precisely its diagonal entries.

(b) **Theorem** (Diagonalization): *Let A be a $k \times k$ matrix. Let $r_1, r_2, ..., r_k$ be eigenvalues of A, and $v^1, v^2, ..., v^k$ the corresponding eigenvectors.*

Form the matrix P whose columns are these k eigenvectors. If P is invertible, then $P^{-1}AP = L$, where L is the diagonal matrix with the eigenvalues of A on its diagonal.

Conversely, if for any invertible $k \times k$ matrix B, $B^{-1}AB$ is a *diagonal matrix D, then the columns of B must be eigenvectors of A (that is, $B = P$) and the diagonal entries of D must be eigenvalues of A (that is, $D = L$).*

(i) Prove the above theorem.

(ii) A matrix is *diagonalizable* if it can be diagonalized by the method described in the above theorem. Suppose A is a $n \times n$ matrix, not necessarily symmetric, with n *distinct* and real eigenvalues. Is A diagonalizable? Explain clearly.

(c) Let A be a 2×2 matrix with two equal eigenvalues. Prove that A is diagonalizable if and only if A is already a diagonal matrix.

2. [55 marks: $33 + 22$]

Recall the definitions of *minor* and *cofactor*.

Minor of order k: For any $m \times n$ matrix A, consider the k-th order submatrix M obtained by deleting all but some k rows and k columns of A. Then $|M|$ is called a k-th order minor of A.

Cofactor: The cofactor A_{ij} of the element a_{ij} of any square matrix A is $(-1)^{i+j}$ times the determinant of the submatrix obtained from A by deleting row i and column j.

In this question we will prove the following theorem.

Theorem: *The rank of an $m \times n$ matrix A is k if and only if every minor in A of order $k+1$ vanishes, while there is at least one minor of order k which does not vanish.*

In part (a) we will prove that if *rank* $(A) = k$ then every minor in A of order $k+1$ vanishes, while there is at least one minor of order k that does not vanish. In part (b) we will prove the converse.

(a) [33 marks: $10 + 18 + 5$]

Suppose *rank* $(A) = k$.

(i) [10 marks] Argue that the columns of any submatrix of order $k+1$ can be expressed as linear combinations of k columns from A. Prove that all minors of order $k+1$ vanish.

– Now we will show that there is at least one minor of order k that does not vanish. Assume that the opposite holds, that is, that all determinants of order k vanish.

(ii) [18 marks] Select k columns from A which are linearly independent. Without loss of generality, we can consider them to be the first k columns. Assume that all determinants of order k formed from the first k columns vanish.

- [2 marks] Argue that $\sum_{i=1}^{k} a_{ki} A_{ki} = 0$.
- [5 marks] Consider the $k \times k$ submatrices formed using the first $k-1$ rows and any other row j, $j = k+1, k+2, ..., m$.

  Explain clearly that $\sum_{i=1}^{k} a_{ji} A_{ki} = 0$, $j = k+1, k+2, ..., m$.
- [2 marks] Argue that $\sum_{i=1}^{k} a_{ji} A_{ki} = 0$, $j = 1, 2, ..., k-1$.
- [6 marks] Using the last few steps argue that the first k columns of A are *linearly dependent* if A_{k1}, A_{k2}, A_{kk} are not all zero.
- In fact, if there were any minor of order $k-1$ in the first k columns which did not vanish, we could rearrange the rows and columns so that at least one A_{ki}, $i = 1, 2, ..., k$, would be different from zero, as in the last step.
- [3 marks] Summarize, from the steps in this part, what you can conclude about the determinants of order $k-1$ formed from the first k columns of A

starting with the assumption that all determinants of order k formed from the first k columns vanish.

– If all determinants of order $k-1$ formed from the first k columns of A were zero, we could repeat the procedure with $k-1$ columns and show that the $k-1$ columns are linearly dependent if the determinants of order $k-2$ do not vanish.

(iii) Conceivably, we will in this way arrive at a point where all determinants of order 2 vanish. Now we have reduced the problem to showing that any set of two columns is linearly dependent.

  ◦ [5 marks] Argue that this linear dependence would follow.

– Thus, using the above procedure, we have contradicted our assumption that the k columns were linearly independent. Hence, there must be at least one nonvanishing determinant of order k in any set of k linearly independent columns.

(b) [22 marks: $11+11$]

Now let us assume that all determinants of order $k+1$ vanish and there is one determinant of order k which does not vanish.

(i) [11 marks] Argue that any $k+1$ columns whose determinants of order $k+1$ all vanish are linearly dependent. Hence $rank\ (A) \leq k$.

(ii) [11 marks] Now let us consider the columns associated with any determinant of order k which does not vanish.

Prove that these k columns cannot be linearly dependent. Hence $rank\ (A) = k$.