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First Midterm Exam (30 August 2025)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [15 marks]

   Suppose A is an $m \times n$ matrix.

   (a) Show that the $m \times m$ matrix AA^T is symmetric (A^T denotes the transpose of a matrix).

   (b) If the determinant of AA^T is non-zero, are the row vectors of A linearly dependent? Explain clearly.

2. [30 marks: $18 + 12$]

   (a) Consider a set of *basis vectors* for $\mathbb{R}^n$, $\{x^1, x^2, ..., x^n\}$, and any other *non-null* vector y in $\mathbb{R}^n$. Consider also the expression of y as a linear combination of the set of basis vectors: $y = \sum_{i=1}^{n} \alpha_i x^i$. In the above expression of y, suppose, without loss of generality, that $\alpha_1 \neq 0$. Consider the set of vectors $\{y, x^2, ..., x^{n-1}, x^n\}$, that is, x^1 is removed from the set $\{x^1, x^2, ..., x^n\}$, and it is replaced by the vector y.

   Prove that the set of vectors $\{y, x^2, ..., x^{n-1}, x^n\}$ forms a basis for $\mathbb{R}^n$.

   (b) Prove that any $m < n$ linearly independent vectors from $\mathbb{R}^n$ form part of a basis for $\mathbb{R}^n$, that is, if $x^1, x^2, ..., x^m$ from $\mathbb{R}^n$ are linearly independent, then $n - m$ additional vectors in $\mathbb{R}^n$ can be found such that $x^1, x^2, ..., x^m$ and the other $n - m$ additional vectors in $\mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

3. [25 marks]

   Let A be a *symmetric* matrix.

   (a) Prove that A is *positive definite* if and only if there exists a *nonsingular* matrix B so that $A = B^T B$.

   (b) State and prove the *negative definite* version of the above theorem.

4. [30 marks]

   Suppose A is an $m \times n$ matrix with $m \neq n$. The $n \times m$ matrix B is a *right inverse* for A if $AB = I$, where I is the $m \times m$ identity matrix.

   Prove that if A has one *right inverse*, then it has infinitely many right inverses.

   [Hint: Results on simultaneous linear equations might help.]