Math 271: Mathematical Methods Tridip Ray
Semester I, 2020-21 ISI, Delhi

Second Midterm Exam: Question 1 (28 February 2021)

- Maximum marks: **20**
- Time allotted (including uploading on Moodle): **45 minutes**

- In this question we will prove the following version of the Bolzano-Weierstrass Theorem:

  Any sequence contained in a compact interval of $\mathbb{R}$, $[a,b]$, *has a convergent subsequence whose limit lies in* $[a,b]$.

- Let $\{x_n\}_{n=1}^{\infty}$ be a sequence contained in $[a,b]$. Divide $[a,b]$ into two equal halves: $\left[a, \dfrac{a+b}{2}\right]$ and $\left[\dfrac{a+b}{2}, b\right]$.

(a) [3 marks]

Argue that infinitely many elements of the sequence $\{x_n\}_{n=1}^{\infty}$ must lie in one (or both) of these halves.

- Let I_1 denote a half which contains infinitely many members of the sequence. Now divide subinterval I_1 into two equal halves. Call the half which contains infinitely many elements of the sequence I_2. Continue dividing the interval into halves; each time choose a half which contains infinitely many elements of the sequence. The result of continuing this process indefinitely is a sequence of intervals $\{I_k\}_{k=1}^{\infty}$, with $I_{k+1} \subset I_k$. Construct a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ by choosing $x_{n_k} \in I_k$. Since each I_k contains infinitely many x_n's, we can ensure that $\{x_{n_k}\}_{k=1}^{\infty}$ really is an (infinite) subsequence of $\{x_n\}_{n=1}^{\infty}$.

(b) [14 marks]

Prove that the subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ has a limit.

(c) [3 marks]

Prove that this limit lies in $[a,b]$.

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Second Midterm Exam: Question 2 (28 February 2021)

- Maximum marks: **30**
- Time allotted (including uploading on Moodle): **70 minutes**

(a) [10 marks]

Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}$ be twice continuously differentiable. Consider the problem of maximizing $f(x; a)$ with respect to x, where $a \in \mathbb{R}$ is a *parameter* for the maximization problem.

Given any $a \in \mathbb{R}$, suppose $x^*(a)$ is a solution to the maximization problem. Provide (with a clear explanation) a *sufficient* condition on f for x^* to be a *continuously differentiable* function of a.

(b) [20 marks]

Recall the following characterization of twice differentiable quasi-concave functions.

Let $A \subset \mathbb{R}^n$ be an open convex set, and $f : A \to \mathbb{R}$ be twice continuously differentiable on A. The *bordered Hessian matrix* of f at $x \in A$ is denoted by $G_f(x)$ and is defined by the following $(n+1) \times (n+1)$ matrix:

$$G_f(x) = \begin{pmatrix} 0 & \nabla f(x) \\ & \\ \nabla f(x) & H_f(x) \end{pmatrix}.$$

We denote the $(k+1)$th leading principal minor of $G_f(x)$ by $|G_f(x; k)|$, where $k = 1, 2, ..., n$.

Theorem: Suppose $A \subset \mathbb{R}^n$ is an open convex set, and $f : A \to \mathbb{R}$ is twice continuously differentiable on A.

(i) *If f is quasi-concave on A, then $(-1)^k |G_f(x; k)| \geq 0$ for $x \in A$, and $k = 1, 2, ..., n$.*

(ii) *If* $(-1)^k |G_f(x;k)| > 0$ *for* $x \in A$*, and* $k = 1, 2, ..., n$*, then* f *is quasi-concave on* A*.*

- **Question:** Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be given by

$$f(x,y) = (x-1)^2 (y-1)^2, \ \ (x,y) \in \mathbb{R}^2_+.$$

In light of this example explain whether the 'strict inequalities' ('>0') in condition (ii) of the theorem can be replaced with 'weak inequalities' ('≥ 0').

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Second Midterm Exam: Question 3 (28 February 2021)

- Maximum marks: **20**
- Time allotted (including uploading on Moodle): **45 minutes**

- *Limits from the left*: Suppose that f is defined on an interval (a, b). We say that $f(x)$ tends to a limit l as x tends to b from the left and write $f(x) \to l$ as $x \to b-$, or, alternatively, $\lim_{x \to b-} f(x) = l$ if the following criterion is satisfied.
  - Given any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - l| < \epsilon$ provided that $b - \delta < x < b$.
- *Limits from the right*: Suppose that f is defined on an interval (a, b). We say that $f(x)$ tends to a limit l as x tends to a from the right and write $f(x) \to l$ as $x \to a+$, or, alternatively, $\lim_{x \to a+} f(x) = l$ if the following criterion is satisfied.
  - Given any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - l| < \epsilon$ provided that $a < x < a + \delta$.
- *$f(x) \to l$ as $x \to \xi$*: Suppose that f is defined on an interval (a, b) and $\xi \in (a, b)$. We say that $f(x)$ tends to a limit l as x tends to ξ and write $f(x) \to l$ as $x \to \xi$, or, alternatively, $\lim_{x \to \xi} f(x) = l$ if the following criterion is satisfied.
  - Given any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - l| < \epsilon$ provided that $0 < |x - \xi| < \delta$.
- **Proposition:** *Let f be defined on an interval (a, b) and $\xi \in (a, b)$. Then $f(x) \to l$ as $x \to \xi$ if and only if $f(x) \to l$ as $x \to \xi-$ and $f(x) \to l$ as $x \to \xi+$.*
- Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$. We say that f is *increasing* on A if and only if, for each $x \in A$ and $y \in A$ with $x < y$, it is true that $f(x) \leq f(y)$. Similar definition holds for a *decreasing* function.

- **Question:** Prove the following theorem.

  (i) Let $f : (a, b) \to \mathbb{R}$ be increasing and bounded above with the smallest upper bound L. Then $f(x) \to L$ as $x \to b-$.

  (ii) Let $f : (a, b) \to \mathbb{R}$ be increasing and bounded below with the largest lower bound l. Then $f(x) \to l$ as $x \to a+$.

Math 271: Mathematical Methods — Tridip Ray
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Second Midterm Exam: Question 4 (28 February 2021)

- Maximum marks: **30**
- Time allotted (including uploading on Moodle): **75 minutes**

- *Limits from the left*: Suppose that f is defined on an interval (a, b). We say that $f(x)$ tends to a limit l as x tends to b from the left and write $f(x) \to l$ as $x \to b-$, or, alternatively, $\lim_{x \to b-} f(x) = l$ if the following criterion is satisfied.
  - Given any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - l| < \epsilon$ provided that $b - \delta < x < b$.

- *Limits from the right*: Suppose that f is defined on an interval (a, b). We say that $f(x)$ tends to a limit l as x tends to a from the right and write $f(x) \to l$ as $x \to a+$, or, alternatively, $\lim_{x \to a+} f(x) = l$ if the following criterion is satisfied.
  - Given any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - l| < \epsilon$ provided that $a < x < a + \delta$.

- *$f(x) \to l$ as $x \to \xi$*: Suppose that f is defined on an interval (a, b) and $\xi \in (a, b)$. We say that $f(x)$ tends to a limit l as x tends to ξ and write $f(x) \to l$ as $x \to \xi$, or, alternatively, $\lim_{x \to \xi} f(x) = l$ if the following criterion is satisfied.
  - Given any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - l| < \epsilon$ provided that $0 < |x - \xi| < \delta$.

- **Proposition:** *Let f be defined on an interval (a, b) and $\xi \in (a, b)$. Then $f(x) \to l$ as $x \to \xi$ if and only if $f(x) \to l$ as $x \to \xi-$ and $f(x) \to l$ as $x \to \xi+$.*

- Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$. We say that f is *increasing* on A if and only if, for each $x \in A$ and $y \in A$ with $x < y$, it is true that $f(x) \leq f(y)$. Similar definition holds for a *decreasing* function.

- **Theorem:**

  (i) Let $f:(a,b)\to\mathbb{R}$ be increasing and bounded above with the smallest upper bound L. Then $f(x)\to L$ as $x\to b-$.

  (ii) Let $f:(a,b)\to\mathbb{R}$ be increasing and bounded below with the largest lower bound l. Then $f(x)\to l$ as $x\to a+$.

- In a homework problem you have established that if f is a *convex* function on the interval I and x_1, x_2 and x_3 are points of I which satisfy $x_1<x_2<x_3$, then

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\frac{f(x_3)-f(x_1)}{x_3-x_1}.$$

(a) [6 marks]

Suppose f is a *convex* function on the open interval I and $x\in I$. Argue that the function

$$F(h)=\frac{f(x+h)-f(x)}{h}$$

is *increasing* in some interval $(0,\delta)$.

(b) [14 marks]

Suppose f is a *convex* function on the open interval I. Prove that the limits

$$\lim_{h\to 0+}\frac{f(x+h)-f(x)}{h},\qquad \lim_{h\to 0-}\frac{f(x+h)-f(x)}{h}$$

both exist for each $x\in I$.

(c) [10 marks]

Suppose f is a *convex* function on the open interval I. Prove that f is *continuous* on I.

- Hints: Note that, for $h\neq 0$,

$$f(x+h)-f(x)=\left[\frac{f(x+h)-f(x)}{h}\right]\cdot h.$$