Math 271: Mathematical Methods Tridip Ray
Semester I, 2021-22 ISI, Delhi

Second Midterm Exam: Question 1 (26 December 2021)

- Maximum marks: **25**
- Time allotted (including uploading on Moodle): **50 minutes**

- Let $A \subset \mathbb{R}^n$, $f\colon\ A \to \mathbb{R}^m$, and $x_0 \in A$. Prove that the following two definitions of *continuity* are equivalent to each other.
  - **Definition 1:** The function f is *continuous* at x_0 if whenever $\{x_n\}$ is sequence in A which converges to x_0, then the sequence $\{f(x_n)\}$ in $\mathbb{R}^m$ converges to $f(x_0)$.
  - **Definition 2:** The function f is *continuous* at x_0 if given any $\epsilon > 0$, there is a number $\delta > 0$, such that if $x \in A$, and $0 < \|x - x_0\| < \delta$, then $\|f(x) - f(x_0)\| < \epsilon$.

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Second Midterm Exam: Question 2 (26 December 2021)

- Maximum marks: **25**
- Time allotted (including uploading on Moodle): **50 minutes**

(a) [12 marks] Let $A \subset \mathbb{R}^n$ be a convex set and $f\colon A \to \mathbb{R}$. Prove that if f is *quasiconcave*, then it *cannot* have a *strict interior minimum* in A.

(b) [13 marks] Consider the following constrained maximization problem:

$$\left.\begin{array}{l} \underset{\{x\geq 0, y\geq 0\}}{\text{Maximize}} \;\; x^{\frac{1}{4}} y^{\frac{1}{4}} \\ \text{subject to } x + y \leq 2. \end{array}\right\} \qquad \text{(P)}$$

Without applying any theorem of optimization prove that $(x^*, y^*) = (1, 1)$ solves problem (P).

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Second Midterm Exam: Question 3 (26 December 2021)

- Maximum marks: **20**
- Time allotted (including uploading on Moodle): **45 minutes**

- Suppose that $u : X \to \mathbb{R}$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ (an "at-least-as-good-as" relation) defined on the consumption set $X = \mathbb{R}^n_+$. The consumer's problem of choosing her most preferred consumption bundle given prices $p \gg 0$ and income $M > 0$ can be stated as the following *utility maximization problem* (UMP):

$$\max_{x \geq 0} \; u(x)$$
$$\text{subject to, } p \cdot x \leq M.$$

If $p \gg 0$ and $u(\cdot)$ is continuous, then the UMP has a solution. The rule that assigns the set of optimal consumption vectors in the UMP to each price-income situation $(p, M) \gg 0$ is denoted by $x(p, M) \in \mathbb{R}^n_+$ and is known as the *Walrasian demand correspondence.* When $x(p, M)$ is single-valued for all (p, M), we refer to it as the *Walrasian demand function.*

Assume that $u(\cdot)$ is *twice continuously differentiable* and that $\nabla u(x) \neq 0$ for all x. Consider that $x(p, M) >> 0$ is single-valued, that is, $x(p, M) >> 0$ is, for some $\lambda > 0$, the unique solution to the first-order conditions:

$$M - p \cdot x = 0,$$

$$\nabla u(x) - \lambda p = 0.$$

- **Question:** Consider the following statement: "*If u is quasiconcave on X, then $x(p, M)$ is a continuously differentiable function of (p, M).*"
  - If you agree with the statement, prove it.
  - If you do not agree with the statement, provide an alternative statement about the condition on u (for $x(p, M)$ to be a continuously differentiable function of (p, M)) with a rigorous proof.

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Second Midterm Exam: Question 4 (26 December 2021)

- Maximum marks: **30**
- Time allotted (including uploading on Moodle): **65 minutes**

- A noncooperative normal-form game has three components. There is a set $N = \{1, 2, ..., n\}$ of players. Each player $i \in N$ has a nonempty set of strategies, $S_i \subset \mathbb{R}^n$, available to him. We use S to denote the Cartesian product $S_1 \times S_2 \times ... \times S_n$. Each player i has a pay-off function $\pi_i : S \to \mathbb{R}$. We denote $(\pi_1, \pi_2, ..., \pi_n)$ by π. A game is simply an ordered triple (N, S, π).

  Given a game $G = (N, S, \pi)$, and given $s \in S$ and $t_i \in S_i$, we use (s/t_i) to denote the following strategy combination: $(s_1, ..., s_{i-1}, t_i, s_{i+1}, ..., s_n)$.

  In the game G, the strategy combination s^* is a *Nash equilibrium* if and only if,

  $$\pi_i(s^*) \geq \pi_i(s^*/s_i), \text{ for all } i \in N, \text{ and for all } s_i \in S_i.$$

  Assume that, for all $i \in N$, S_i is *compact* and *convex*, π_i is *continuous*, and $\pi_i(s/t_i)$ is *strictly quasi-concave* with respect to $t_i \in S_i$, for all $s \in S$. Since S_i is compact and convex for all i, it follows that S is compact and convex.

  Brouwer's Fixed Point Theorem: Let $B \subset \mathbb{R}^m$ and $f : B \to B$. If B is compact and convex and f is continuous, then there exists $x^* \in B$ such that $f(x^*) = x^*$.

  (a) [20 marks: $7 + 13$]

  For each player $i \in N$, define the *best response function* for i, $r_i : S \to S_i$, as follows: for all $s \in S$, $r_i(s) = \underset{t_i \in S_i}{\operatorname{argmax}}\, \pi_i(s/t_i)$.
  [Let $f(x_1, ..., x_m)$ be a real-valued function on $\mathbb{R}^m$. The $\underset{x_i}{\operatorname{argmax}}\, f(x_1, ..., x_m)$ is the set of values of the i-th argument at which $f(x_1, ..., x_m)$ attains its maximum given that $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_m)$ is fixed.]

(i) Prove that, for any $s \in S$, $r_i(s)$ exists and is unique.

(ii) Prove that, for all i, r_i is a continuous function.

(b) [10 marks]

Prove that the game $G = (N, S, \pi)$ has a Nash equilibrium.