Math 271: Mathematical Methods — Tridip Ray
Semester I, 2022-23 — ISI, Delhi

Second Midterm Exam (06 November 2022)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [15 marks]

   Let $f^1, f^2, ..., f^m : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions. Consider the system of equations

$$\begin{aligned} f^1(x_1, ..., x_n) &= c_1 \\ f^2(x_1, ..., x_n) &= c_2 \\ &\vdots \\ f^m(x_1, ..., x_n) &= c_m \end{aligned}$$

   where $n > m$ and c_j is a scalar, $j = 1, 2, ..., m$. Suppose that $(x_1^*, ..., x_n^*)$ is a solution to the system of equations and the *rank* of the $m \times n$ matrix

$$\begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n} \end{pmatrix}$$

   evaluated at $(x_1^*, ..., x_n^*)$ is m.

   Prove that there exists an open interval $(-\epsilon, \epsilon)$ and C^1 functions $x_1(t), ..., x_n(t)$ defined for $t \in (-\epsilon, \epsilon)$ such that $x_i(0) = x_i^*$, $i = 1, 2, ..., n$, and

$$f^j(x_1(t), ..., x_n(t)) = c_j + t, \ \ j = 1, 2, ..., m,$$

   for all $t \in (-\epsilon, \epsilon)$.

2. [22 marks: $12+10$]

(a) Let $A \subset \mathbb{R}^n$ and $g : A \rightarrow \mathbb{R}$.

(i) Define $V(a_1, a_2, ..., a_n) = \max\limits_{\{x_1, x_2, ..., x_n\}} \{a_1 x_1 + a_2 x_2 + ... + a_n x_n : g(x_1, x_2, ..., x_n) \leq 0\}$, where a_i is a scalar, $i = 1, 2, ..., n$. Prove carefully whether $V(\cdot)$ *concave* or *convex* in $a_1, a_2, ..., a_n$.

(ii) Define $w(a_1, a_2, ..., a_n) = \min\limits_{\{x_1, x_2, ..., x_n\}} \{a_1 x_1 + a_2 x_2 + ... + a_n x_n : g(x_1, x_2, ..., x_n) \geq 0\}$, where a_i is a scalar, $i = 1, 2, ..., n$. Prove carefully whether $w(\cdot)$ *concave* or *convex* in $a_1, a_2, ..., a_n$.

(iii) For parts (i) and (ii), how does your answer depend on the nature of the function $g(x_1, x_2, ..., x_n)$?

(b) For $(p_1, p_2, ..., p_n, M) \gg 0$, define

$$v(p_1, p_2, ..., p_n, M) = \max_{\{x_1, x_2, ..., x_n\}} \{u(x_1, x_2, ..., x_n) : p_1 x_1 + p_2 x_2 + ... + p_n x_n \leq M\},$$

where $u : \mathbb{R}^n_+ \rightarrow \mathbb{R}$ is a continuous utility function. Prove that $v(p_1, p_2, ..., p_n, M)$ is *quasiconvex* in $p_1, p_2, ..., p_n, M$.

3. [22 marks: $16+6$]

Continuum Property: Every non-empty set of real numbers which is bounded above has a smallest upper bound. Every non-empty set of real numbers which is bounded below has a largest lower bound. The smallest upper bound is called the *supremum* of the set. The largest lower bound is called the *infimum* of the set.

(a) In this part we will prove the following *theorem*:

Let f be a real-valued and continuous function on a compact interval $[\alpha, \beta]$ in $\mathbb{R}$, and suppose that $f(\alpha)$ and $f(\beta)$ have opposite signs. Then there is at least one point γ in the open interval (α, β) such that $f(\gamma) = 0$.

(i) For definiteness, assume $f(\alpha) > 0$ and $f(\beta) < 0$. Define the set

$$A = \{x \colon \; x \in [\alpha, \beta] \text{ and } f(x) \geq 0\}.$$

Argue clearly that the *supremum* of A exists.

(ii) Let $\delta =$ supremum of A. Prove the theorem by showing that $f(\delta) = 0$.

(b) Prove the following version of the *intermediate value theorem*:

Let f be a real-valued continuous function defined on an interval containing the real numbers a and b, with say $f(a) < f(b)$. Then, given any $\psi \in \mathbb{R}$ such that $f(a) < \psi < f(b)$, there exists $c \in (a, b)$ such that $f(c) = \psi$.

4. [41 marks: $5 + 10 + 13 + 13$]

Recall that a symmetric $n \times n$ matrix A is *positive definite* if $x^T Ax > 0$ for all x in $\mathbb{R}^n$, $x \neq 0$.

(a) Prove that A is *positive definite* if and only if it satisfies $y^T Ay > 0$ for all y in the *unit sphere* $U = \{u \in \mathbb{R}^n : \|u\| = 1\}$.

(b) Let A be a *positive definite* $n \times n$ matrix. Prove that there exists $z \in U$ (the unit sphere) such that

$$z^T Az \leq x^T Ax, \text{ for all } x \in U.$$

Argue that there exists $\epsilon > 0$ such that $x^T Ax \geq \epsilon > 0$, for all $x \in U$.

(c) Define $\beta = \dfrac{\epsilon}{2n^2} > 0$. Let B be any symmetric $n \times n$ matrix such that $|b_{ij} - a_{ij}| < \beta$, for all $i, j = 1, 2, ..., n$, where a_{ij} and b_{ij} are the elements corresponding to the ith row and jth column of matrices A and B, respectively.

Prove that

$$\left| x^T (B - A) x \right| < \frac{\epsilon}{2}, \text{ for all } x \in U.$$

Argue that B is also a *positive definite* matrix.

(d) Suppose D is an open subset of $\mathbb{R}^n$, and $f : D \to \mathbb{R}$ is twice continuously differentiable on D such that at some point $x^* \in D$, the Hessian matrix of f evaluated at x^*, $H_f(x^*)$, is *positive definite.*

Prove that there exists an open ball $B_\delta(x^*)$ such that for all $x \in B_\delta(x^*)$, $H_f(x)$ is also a *positive definite* matrix.