Math 271: Mathematical Methods
Semester I, 2023-24

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Second Midterm Exam (28 October 2023)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [10 marks]

   In Homework 6 you have proved the following version of the intermediate value theorem:

   Let f be a real-valued continuous function defined on an interval containing the real numbers a and b, with say $f(a) < f(b)$. Then, given any $y \in \mathbb{R}$ such that $f(a) < y < f(b)$, there exists $x \in (a, b)$ such that $f(x) = y$.

   Question: Prove the following version of the intermediate value theorem:

   Suppose A is a convex subset of $\mathbb{R}^n$, and $f : A \to \mathbb{R}$ is a continuous function on A. Suppose x^1 and x^2 are in A, and $f(x^1) > f(x^2)$. Then, given any $c \in \mathbb{R}$ such that $f(x^1) > c > f(x^2)$, there exists $0 < \theta < 1$ such that $f(\theta x^1 + (1 - \theta) x^2) = c$.

2. [20 marks]

   Prove the following theorem:

   Let $f : I \to \mathbb{R}$ be a C^3 function defined on an open interval I in $\mathbb{R}$. For any two points a and $a + h$ in I, there exists a point c between a and $a + h$, i.e., $a < c < a + h$, such that

   $$f(a+h) = f(a) + f'(a)h + \frac{1}{2!}f''(a)h^2 + \frac{1}{3!}f'''(c)h^3.$$

   [**Hints:**

   Consider the following theorem (Mean Value Theorem): *Let $f : I \to \mathbb{R}$ be a C^1 function defined on an open interval I in $\mathbb{R}$. For any two points a and $a + h$ in I, there exists a point c between a and $a + h$, i.e., $a < c < a + h$, such that*

   $$f(a+h) = f(a) + f'(c)h.$$

   Recall that to prove this theorem we considered the function

   $$g_0(x) = f(x) - f(a) - M_0(x - a)$$

and chose M_0 appropriately to our advantage.

Now consider the following theorem: *Let $f : I \to \mathbb{R}$ be a C^2 function defined on an open interval I in $\mathbb{R}$. For any two points a and $a+h$ in I, there exists a point c between a and $a+h$, i.e., $a < c < a+h$, such that*

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2!}f''(c)h^2.$$

Recall that to prove this theorem we considered the function

$$g_1(x) = f(x) - f(a) - f'(a)(x-a) - M_1(x-a)^2$$

and chose M_1 appropriately to our advantage.]

3. [20 marks]

Contractive Sequence: A sequence of real numbers $\{x_n\}$ is a contractive sequence if there exists a constant c, $0 < c < 1$, such that $|x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$.

Prove that *every contractive sequence of real numbers is a convergent sequence.*

4. [20 marks]

Limit Point: Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is a limit point of A if for every $\delta > 0$ there exists at least one point $x \in A$, $x \neq c$, such that $|x - c| < \delta$.

(a) Prove that *a point $c \in \mathbb{R}$ is a limit point of A if and only if there exists a sequence $\{x_n\}$ in A such that $\{x_n\} \to c$ and $x_n \neq c$ for all $n \in \mathbb{N}$.*

(b) Prove that *a subset of $\mathbb{R}$ is closed if and only if it contains all of its limit points.*

5. [30 marks]

Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$.

Continuity: f is *continuous* at every point $y \in A$ if given any $\epsilon > 0$ and any $y \in A$, there is a $\delta(\epsilon, y) > 0$ such that for all x if $x \in A$ and $|x - y| < \delta(\epsilon, y)$, then $|f(x) - f(y)| < \epsilon$. By writing δ as a function of ϵ and y, $\delta(\epsilon, y)$, it is emphasized that, in general, δ depends on both $\epsilon > 0$ and $y \in A$.

Now it often happens that the function f is such that the number δ can be chosen to be independent of the point $y \in A$ and to depend only on ϵ. For example, if $f(x) = 2x$ for all $x \in \mathbb{R}$, then

$$|f(x) - f(y)| = 2|x - y|,$$

and so we can choose $\delta(\epsilon, y) = \frac{\epsilon}{2}$ for all $\epsilon > 0$ and all $y \in \mathbb{R}$.

Uniform Continuity: We say that f is *uniformly continuous* on A if given any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $x, y \in A$ are any numbers satisfying $|x - y| < \delta(\epsilon)$, then $|f(x) - f(y)| < \epsilon$.

(a) Argue that if f is *not* uniformly continuous on A, then there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ and y_δ in A such that $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$.

(b) Argue that if f is *not* uniformly continuous on A, then there exists an $\epsilon_0 > 0$ and two sequences $\{x_n\}$ and $\{y_n\}$ in A such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$.

(c) Prove the following theorem:

Let I be a closed and bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.