Math 271: Mathematical Methods
Semester I, 2024-25

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Second Midterm Exam (27 October 2024)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [25 marks]

   Consider an arbitrary set $A \subset \mathbb{R}^n$ and an arbitrary point $x \in \mathbb{R}^n$. Then one of the following three possibilities must hold.

   (1) There is an open ball $B_\epsilon(x)$ such that $B_\epsilon(x) \subset A$. These points $x \in \mathbb{R}^n$ constitute the **interior** of A, denoted by *Int* A.

   (2) There is an open ball $B_\epsilon(x)$ such that $B_\epsilon(x) \subset A^c$. These points $x \in \mathbb{R}^n$ constitute the **exterior** of A, denoted by *Ext* A.

   (3) For every $\epsilon > 0$, $B_\epsilon(x)$ contains points of both A and A^c. These points $x \in \mathbb{R}^n$ constitute the **boundary** of A, denoted by *Bnd* A.

   Question: For an arbitrary set $A \subset \mathbb{R}^n$, prove whether each of these sets – *Int* A, *Ext* A, and *Bnd* A – is open, closed, or neither open nor closed in $\mathbb{R}^n$.

2. [25 marks]

   Nearest Point Theorem: *Let A be a non-empty, closed subset of $\mathbb{R}^n$, and let x be a point in $\mathbb{R}^n$ which does not belong to A. Then there exists a point y in A such that*

   $$\|z - x\| \geq \|y - x\|, \text{ for all } z \text{ in } A.$$

   Question: Prove the Nearest Point Theorem showing your steps clearly.

3. [25 marks]

   Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly concave function. Let a, b, c, d be arbitrary positive real numbers, satisfying

   (i) $a < b < c < d$, and

   (ii) $a + d = b + c$.

   Prove that $f(a) + f(d) < f(b) + f(c)$.

4. [25 marks: $10+15$]

Suppose $f : \mathbb{R}^2_{++} \to \mathbb{R}$ is twice continuously differentiable and *homogeneous of degree one* on its domain.

(a) Show that, for every $x \in \mathbb{R}^2_{++}$, the Hessian matrix of f evaluated at x is non-invertible.

(b) Suppose further that f is *quasi-concave* on $\mathbb{R}^2_{++}$ with $D_1 f(x_1, x_2) > 0$ and $D_2 f(x_1, x_2) > 0$ for all (x_1, x_2) in $\mathbb{R}^2_{++}$. Prove that then f is also *concave* on $\mathbb{R}^2_{++}$.