Math 271: Mathematical Methods Tridip Ray
Semester I, 2025-26 ISI, Delhi

Second Midterm Exam (12 October 2025)

- Answer *all* the questions. You have 3 hours to write this exam.

1. [20 marks]

   Let $\{x_n\}$ be a sequence of real numbers. Prove that the following two alternative definitions of a *limit point* (or an *accumulation point*) of the sequence $\{x_n\}$ are equivalent.

   Definition 1: A real number L is a *limit point* of $\{x_n\}$ if for every $\varepsilon > 0$ and for every $N \in \mathbb{N}$ (the set of natural numbers), there exists some $n > N$ such that $|x_n - L| < \varepsilon$. [This formulation means that infinitely many terms of the sequence lie arbitrarily close to L.]

   Definition 2: A real number L is a *limit point* of $\{x_n\}$ if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k \to \infty} x_{n_k} = L$.

2. [20 marks]

   Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function on $\mathbb{R}_+$ which is twice continuously differentiable on $\mathbb{R}_{++}$ with $f'(x) > 0$ and $f''(x) < 0$ for all x in $\mathbb{R}_{++}$.

   Prove that for every x in $\mathbb{R}_{++}$, $\dfrac{f(x)}{x} > f'(x)$.

3. [60 marks]

- *Bijective Map* and *Inverse Function*: Let $f : A \to B$ be a function between two sets A and B. We say f is *bijective* if it is both (i) *injective* (*one-to-one*), that is, for all a_1, $a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$, and (ii) *surjective* (*onto*), that is, for every $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$. When f is bijective, f has an *inverse function*, $f^{-1} : B \to A$, defined by $f^{-1}(f(a)) = a$, for all $a \in A$.

- *Inverse Function Theorem*: Let $f : \mathbb{R} \to \mathbb{R}$ be C^1. Suppose $f'(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. Then there exist open intervals U containing x_0 and V containing $f(x_0)$ such that

  - $f : U \to V$ is a bijection (and hence the inverse $f^{-1} : V \to U$ exists),

- the inverse $f^{-1} : V \to U$ is C^1,
- for every $y \in V$,

$$\left(f^{-1}\right)'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

- In what follows, we develop, step by step, the proof of the inverse function theorem. We will prove the theorem assuming $f'(x_0) > 0$. The proof when $f'(x_0) < 0$ is the same after replacing "increasing" by "decreasing".

(a) [6 marks] Prove that there exists $\delta > 0$ and $c > 0$ such that $f'(x) \geq c > 0$, for all $x \in (x_0 - \delta, x_0 + \delta) \equiv U$.

(b) [5 marks] Prove that f is *strictly* increasing on U, that is, for any $x_1, x_2 \in U$, with $x_2 > x_1$, we have $f(x_2) > f(x_1)$.

(c) [6 marks] Prove that the following inequality holds for all $x_1, x_2 \in U$,

$$|f(x_2) - f(x_1)| \geq c\,|x_2 - x_1|\,.$$

(d) [6 marks] Define $V = f(U)$, the image of f on U. Prove that V is an interval.

[Recall that an *interval* I is a set of real numbers with the property that, if $x \in I$ and $y \in I$ and $x \leq z \leq y$, then $z \in I$.]

(e) [8 marks] Prove that V is an *open* interval.

[Hint: The inequality in part (c) might help.]

- *Since f is strictly increasing, it follows that $f : U \to V$ is a bijection between open intervals U and V and the inverse $f^{-1} : V \to U$ exists.*

(f) [12 marks] Let $A \subset \mathbb{R}$ and $g : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that $|g(u) - g(v)| \leq K\,|u - v|\,,\ \forall\ u, v \in A$, then g is said to be a *Lipschitz function* on A.

  (i) Prove that f^{-1} is a Lipschitz function on V.
  (ii) Prove that $f^{-1} : V \to U$ is continuous.

(g) [12 marks] Prove that for every $y \in V$, f^{-1} is differentiable at y with the formula of the derivative given in the statement of the theorem.

(h) [5 marks] Prove that f^{-1} is C^1 on V.

[Hint: Composition of continuous functions is continuous.]