Workshop on
Applied Optimization Models and Computation

(WAOMC15)

Organized by

Indian Statistical Institute,

SQC & OR Unit, Delhi Centre

Contents

Welcome to WAOMC15

On behalf of the organizers of WAOMC15, I welcome you in the *workshop on Applied Optimization Models and Computation during* January 28-30, 2015 at Indian Statistical Institute, Delhi Centre. This workshop aims to promote research and applications in the area of Optimization and related topics. Leading scientists, experienced researchers and practitioners, as well as younger researchers will come together to exchange knowledge and to build scientific contacts. In this event, faculties and research scholars from academic institutes and other organizations will present their respective research topics/problems. In addition, we also plan for tutorial session on relevant topics to be organized by the domain experts. This will be an event under the project on *Optimization & Reliability Modeling* and it will take place at Indian Statistical Institute Delhi Centre. This event will provide an excellent opportunity to disseminate the latest major achievements and to explore new directions and perspectives, and is expected to have a broad international appeal, dealing with topics of fundamental importance in applied optimization and other related sciences (Economics, Physics, Engineering). In this workshop we mainly focus on classical and modern optimization theory, algorithms (local and global aspects), as well as related topics in applied mathematics, including game theory. This workshop also seeks for applied contributions on modeling and large scale optimization in the fields of engineering, finance and Economics, management, manufacturing, supply-chain and other branches of science where robust and/or stochastic models are used to provide decision support.

The workshop topics include (but not limited to):

- Linear and Nonlinear Programming
- Multi-Objective Optimization
- Nonsmooth Optimization
- Complementarity problem & Variational inequalities
- Combinatorial Optimization

- New developments in Classical Combinatorial Optimization Problems (Knapsack, Vehicle Routing & Aircraft Scheduling, Traveling salesman problem)
- Optimization techniques for game problems
- Application of Optimization Models to finance and Economics
- Optimization software (Matlab, Mathematica and Maple)

Information about social events will be available to you at the time of registration.

S. K. Neogy
Organizing Committee Chair

Committees

Organizing Committee

S. K. Neogy (Organizing Committee Chair), Anup Dewanji, Amitava Bandyopadhyay, Debasis Sengupta, Dipak K. Manna, Biswabrata Pradhan, Abhijit Gupta, Arup K. Das (Convener)

Programme Co-ordinating Committee

R. Chakraborty, Dipti Dubey and Simmi Marwah

Facilities Committee

R. C. Satija, Simmi Marwah

Workshop on
Applied Optimization Models and Computation

Program Overview

Inaugural Session Details

January 28, 2015   Time: 10:00 -10:30   Venue: Auditorium

Welcome address, Opening Remarks, About Workshop

Tea Break: 10:30-11:00

Sessions Details

January 28, 2015   Time: 11:00 -13:00   Venue: Auditorium

Invited Session I

Chairman : S. K. Neogy, Indian Statistical Institute Delhi Centre

1.	**Lina Mallozzi** (University of Naples Federico II via Claudio 21, 80125 Naples, Italy) A Bilevel Location-Allocation Problem in the Planar Region
2.	**Sandeep Juneja** (Tata Institute of Fundamental Research, Mumbai) Ordinal Optimization in Simulation and Pure Exploration Multi-Armed Bandit Methods
3.	**T. E. S. Raghavan** (University of Illinois at Chicago, USA) Legal Disputes Resolved Via Game Theoretic Methods

Lunch: Guest House Lawn   Time 13:00 – 14:00

January 28, 2015 Time: 14:00 -15:15 Venue: Auditorium

Invited Session II
Chairman : David Bartl, University of Ostrava, Czech Republic.

1.	**S. K. Mishra** (Banaras Hindu University, Varanasi, India) On Minty Variational Principle for Nonsmooth Vector Optimization Problems with Approximate Convexity
2.	**B.K.Mohanty** (Indian Institute of Management, Lucknow), Multiple Attribute Decision Making in e-Business- A Fuzzy Approach

Tea Break: 15:15-15:45

January 28, 2015 Time: 15:45 -18:00 Venue: Auditorium

Parallel session-I

Chairman : T parthasarathy, ***Chennai Mathematical Institute and Indian Statistical Institute Chennai***

1.	**Dipti Dubey** (Indian Statistical Institute Delhi Centre) On Linear Complementarity Problem with a Hidden-Z Matrix
2.	C.S. Lalitha and **Mansi Dhingra** (University of Delhi) Approximate Lagrangian duality for set-valued optimization problem
3.	**Syeda Darakhshan Jabeen** (Indian Institute of Technology Kanpur) Designing vehicle parameters using Split and discard decision making strategy
4	**Deepmala,** Indian Statistical Institute, Kolkata, Solving Optimization Problems using Mathematica
5	**J.K. Verma** and C.P. Katti (Jawaharlal Nehru University/School of Computer & Systems Sciences, New Delhi, 110067, India), Optimized Resource Utilization Techniques for Cloud Computing Environment

January 28, 2015 Time: 15:45 -18:15 Venue: Conference Room

Parallel session-II

Chairman : B. K. Mohanty Indian Institute of Management, Lucknow

1.	**Mahima Gupta** (Great Lakes Institute of Management, Chennai, India) Opinion mining using Internet Reviews: A Fuzzy MADM approach
2.	**Anjali Singh,** Anjana Gupta, Aparna Mehra An AHP-PROMETHEE II Method for 2-tuple Linguistic Multi-criteria Group Decision Making**,** Delhi Technological University, Delhi 110042, India
3.	**Mamata Sahu,** Anjana Gupta, Aparna Mehra Interval Valued Intuitionistic Fuzzy Multiple Criteria Decision Making Problem, Delhi Technological University, Delhi 110042, India
4	**Pankaj Kumar** (Shiv Nadar University, India) Convex Optimization for Big Data in Finance
5	**Shreya Khosla** (Shiv Nadar University, India) Task Scheduling in Cloud Computing Environments using Large Scale Linear Programming
6	**Ashish Bhayana** (Shiv Nadar University, India) Classification of URLs based on malign/benign: An optimization approach
7	**Akhilesh Kumar**, Anjana Gupta, Aparna Mehra (Delhi Technological University, Delhi, India) Multiobjective Vendor's Decision Problem on Contingent Demand Satisfaction
8	**Shivi Agarwal (**BITS, Pilani) Fuzzy BCC Data Envelopment Analysis Model: A Credibility Approach

Cultural Programme: Classical Music

Dinner : Guest House Lawn Time 20:30 – 21:30

January 29, 2015 Time: 10:00 -11:20 Venue: Auditorium

Invited Session III

Chairman : R. B. Bapat, Indian Statistical Institute Delhi

1.	**N. Hemachandra** (Indian Institute of Technology Bombay) When are Some Queues (Not) in Equilibrium?
2.	**David Bartl,** University of Ostrava, Czech Republic, Farkas Lemma and Linear optimization in abstract spaces: the infinite case

Tea Break: 11:20-11:45

January 29, 2015 Time: 11:45 -13:00 Venue: Auditorium

Invited Session IV

Chairman : **Lina Mallozzi,** University of Naples Federico II via Claudio 21, 80125 Naples, Italy

1.	**Reshma Khemchandani,** (South Asian University, New Delhi) Support Vector Machines and its Extension.
2.	**Reshma Khemchandani,** (South Asian University, New Delhi) Solving Optimization Models Using MATLAB

Lunch: Guest House Lawn Time 13:00 – 14:00

January 29, 2015 Time: 14:00 -15:15 Venue: Auditorium

Invited Session V

Chairman : David Bartl, University of Ostrava, Czech Republic.

1.	**Suresh Chandra** (Indian Institute of Technology Delhi) Application of Quadratic Programming in Portfolio Optimization
2.	**Valeriu Ungureanu** (State University of Moldova) Mathematical Theory of Pareto-Nash-Stackelberg Game-Control Models

Tea Break: 15:15-15:45

January 29, 2015 Time: 15:45 -18:30 Venue: Auditorium

Parallel session-III

Chairman : T. E. S. Raghavan, University of Illinois at Chicago, USA

1.	**Prasenjit Mondal** & Sagnik Sinha (Jadavpur University, Kolkata-700032, India) One Player Control Semi-Markov Games With Limiting Average Payoffs
2.	**Kalpana Shukla** (GLA University, Mathura-281406 India) Optimality and Duality of Variational Programming Problems
3.	S. K. Mishra and **B. B. Upadhyay** (Banaras Hindu University, Varanasi-221005, India) On Relations between Vector Variational Inequality and Nonsmooth Vector Pseudolinear Optimization Problems
4	**Amit K. Bardhan** (Faculty of Management Studies, University of Delhi) On Computation of Minimal Forecast Horizon for a Stochastic Dynamic Lot-Size Problem
5.	**Debasish Ghorui** (Jadavpur University, Kolkata) Use of Maple to Solve Optimization Problem
6.	**Rwitam Jana**, (Jadavpur University, Kolkata), On computation using MATLAB

January 29, 2015 Time: 15:45 –18:30 Venue: Conference Room

Parallel session-IV

Chairman : T parthasarathy, Chennai Mathematical Institute and Indian Statistical Institute Chennai

1.	**R. K. Arora,** Amit Sachdeva, V Ashok, Abhay Kumar, S Pandian,(Vikram Sarabhai Space Centre, Trivandrum) Multi-objective Shape Optimization of a Re-entry Capsule
2.	**Sadia Samar Ali** (New Delhi Institute of Management, New Delhi) Exploring Green Manufacturing antecedents: A MICMAC Analysis
3.	**M. Upmanyu**, R. R. Saxena (University of Delhi) On Solving a Multiobjective Fixed Charge Problem with Imprecise Fractional Objectives
4	**Pulkit Dwivedi** (Shiv Nadar University, India) Portfolio Optimization Problem involving Big Data Analytics
5	**Abhinav Banerjee** (Shiv Nadar University, India) Pricing Decision Optimisation Using Data for Online Retailers
6	**Rupakshi Bhatia** (Shiv Nadar University, India) On Optimizing Resource Consumption and Crowd-based pick-up and delivery for a distribution network
7.	**Premanjali Rai** and Kunwar P. Singh, (CSIR-Indian Institute of Toxicology Research, Lucknow) Optimization of Tetracycline Adsorption By Magnetic Carbon from Water Using Box-Behnken Design and Response Surface Modeling

January 30, 2015 Time: 10:00 -11:20 Venue: Auditorium

Invited Session VI

Chairman : R. B. Bapat, Indian Statistical Institute Delhi

1.	**T. Parthasarathy** (Chennai Mathematical Institute, Chennai, India & Indian Statistical Institute, Chennai, India) Completely Mixed Stochastic Games
2.	**Pranab Muhuri** (South Asian University, New Delhi) Optimization Under Fuzzy Uncertainty For Time And Safe-Critical Systems

Tea Break: 11:20-11:45

January 30, 2015   Time: 11:45 -13:00   Venue: Auditorium

Invited Session VII

Chairman : Valeriu Ungureanu, State University of Moldova

1.	**Aparna Mehra** (Indian Institute of Technology Delhi) Data Envelopment Analysis Approach to 'Green' Efficiency
2.	**A. K. Das** (Indian Statistical Institute, Kolkata) Role of Principal Pivot Transform in Optimization

Lunch: Guest House Lawn   Time 13:00 – 14:00

January 30, 2015   Time: 14:00 -15:15   Venue: Auditorium

Invited Session VIII

Chairman : S. K. Neogy, Indian Statistical Institute Delhi Centre

1.	**C. S. Lalitha (**University of Delhi) Solution Concepts in Vector and Set Optimization
2.	**Pankaj Gupta** (University of Delhi) Portfolio Optimization: Some Recent Advances

Tea Break: 15:15-15:30

January 30, 2015 Time: 15:30 -17:30 Venue: Auditorium

Invited Session IX

Chairman : A. K. Das, Indian Statistical Institute, Kolkata

1.	**P. C. Jha (**University of Delhi) Sustainable Supply Chain Management
2.	**Sagnik Sinha, (**Jadavpur University, Kolkata 700032, India) Semi-Markov Decision Processes with Limiting Average Rewards

January 30, 2015 Time: 15:45 -17:30 Venue: Conference Room

Parallel session-V

Chairman : T parthasarathy, ***Chennai Mathematical Institute***
and Indian Statistical Institute Chennai

1.	B.S. Panda, **Arti Pandey** (Indian Institute of Technology Delhi New Delhi) Open Neighborhood Locating-Dominating Set In Graphs: Complexity And Algorithms
2.	**Amita Sharma** (Indian Institute of Technology Delhi New Delhi) An overview of Index Tracking and Enhanced Index Tracking
3.	**Meenal Chauhan** (Visva-Bharati University, Shantiniketan) On Solving Optimization problem in R
4.	**Desai Trunil Shamrao** (ICGEB, New Delhi) Metabolic Modelling through Optimization Strategies
5.	**S. Krishnakumar** (ICGEB, New Delhi) Genome Scale Metabolic Model Development and Flux Analysis of Thermophilic Organism: An *In Silico* Optimization Approach

High Tea: 17:30-18:00

ABSTRACT OF THE PAPERS

A Bilevel Location-Allocation Problem in the Planar Region

Egidio D'Amato ⋆, Elia Daniele †, **Lina Mallozzi** ‡

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A distribution of citizens in an urban area (rectangular region in the plane), where a given number of services must be located, is given. Citizens are partitioned in service regions such that each facility serves the costumer demand in one of the service regions. For a fixed location of all the services, every citizen chooses the service minimizing the total cost, i.e. the capacity acquisition cost plus the distribution cost (depending on the travel distance).

In our model there is a fixed cost of each service depending on its location and an additional cost due to time spent being in the queue for a service, depending on the amount of people waiting for the same service, but also on the characteristics of the service itself (for example, its dimension). The objective is to find the optimal location of the services in the urban area and the related costumers partition.

We consider a two-stage optimization model to solve this location-allocation problem. The social planner minimizes the social costs, i.e. the fixed costs plus the waiting time costs, taking into account that the citizens are partitioned in the region according to minimizing the capacity acquisition costs plus the distribution costs in the service regions.

This model is studied from a theoretical and a numerical point of view. Existence results of solutions to the bilevel problem have been proved by using optimal transport theory. We find also a solution of the bilevel problem numerically, by means of a genetic algorithm procedure.

Ordinal Optimization in Simulation and Pure Exploration Multi-Armed Bandit Methods

Sandeep Juneja*
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Consider the ordinal optimization problem of finding a population amongst many with the largest mean when these means are unknown but population samples can be generated via simulation. Typically, by selecting a population with the largest sample mean, it can be shown that the false selection probability decays at an exponential rate. Lately researchers have sought algorithms that guarantee that this probability is restricted to a small δ in order $\log (1/\delta)$ computational time by estimating the associated large deviations rate function via simulation. We show that such guarantees are misleading. We then adapt methods from multi-armed bandit literature to devise algorithms that provide these computational guarantees on the probability of false selection. ***jointly with Peter Glynn, Stanford University***

On Minty Variational Principle For Nonsmooth Vector Optimization Problems with Approximate Convexity

S. K. Mishra
Department of Mathematics
Banaras Hindu University
Varanasi-221005, India

In this paper, we consider a vector optimization problem involving locally Lipschitz approximately convex functions and give several concepts of approximate efficient solutions. We formulate approximate vector variational inequalities of Stampacchia and Minty type and use these inequalities as a tool to characterize an approximate efficient solution of the vector optimization problem.

Keywords: Nonsmooth vector optimization; Approximately convex functions; Clarke subdifferentials; Approximate vector variational inequalities

Application of Quadratic Programming in Portfolio Optimization

Suresh Chandra

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Hauz Khas, New Delhi - 110016, India

The celebrated Mean-Variance Theory of Markowitz for Portfolio optimization is discussed and it is shown that Quadratic Programming plays a major role in this development. Certain limitations of this theory along with some recent developments are also presented.

Solving Optimization Models Using MATLAB

Reshma Khemchandani

Department of Computer Science

South Asian University, New Delhi

Matlab Optimization Toolbox provides functions for finding parameters that minimize or maximize objectives while satisfying constraints. The toolbox includes solvers for linear programming, mixed-integer linear programming, quadratic programming, nonlinear optimization, and nonlinear least squares. These solvers can be used to find optimal solutions to continuous and discrete problems, to perform tradeoff analyses, and to incorporate optimization methods into algorithms and applications. In my talk, I would be discussing

-Solvers for nonlinear least squares, data fitting, and nonlinear equations

-Quadratic and linear programming

-Optimization app for defining and solving optimization problems and monitoring solution progress

Support Vector Machines and its Extension.

Reshma Khemchandani

Department of Computer Science

South Asian University, New Delhi

The last decade has witnessed the evolution of Support Vector Machines (SVMs) as a powerful paradigm for pattern classification and regression. SVMs emerged from research in statistical learning theory on how to regulate the tradeoff between structural complexity and empirical risk. One of the most popular SVM classifiers is the maximum margin one, that attempts to reduce generalization error by maximizing the margin between two disjoint half planes. The resultant optimization task involves the minimization of a convex quadratic function subject to linear inequality constraints. In this talk main topic would be theory of SVM and its applications and extensions.

Optimization Under Fuzzy Uncertainty For Time And Safe-Critical Systems

Pranab Muhuri

Department of Computer Science

South Asian University, New Delhi

During the systems designing phase, decision variables are mostly approximated estimations by the designers. Considerations of crisp approximations often results with wrong decisions. Models are accepted or rejected based on mere compliance of these estimated decision variables. This indicates that there are underlying uncertainties in these decision variables. Therefore fuzzy numbers are considered for modelling these decision variables. Fuzzy numbers can model these decision variables very well offering wider options for the system designers in choosing a right model for a particular application. The talk shall highlight several techniques of ‘optimization under fuzzy uncertainty’ with applications in the areas of scheduling, energy efficiency, reliability and redundancy optimizations especially for time and safety critical systems.

Solution Concepts in Vector and Set Optimization

C. S. Lalitha
Department of Mathematics
University of Delhi South Campus

The main aim of the presentation is to focus on some of the existing solution concepts in vector and set optimization. The well-known notions of efficient and weak efficient solutions in vector optimization have been extended to set-valued problems by many researchers. In this approach the solution concept requires just one point of the image set of the solution to satisfy the vector criterion definition. Another approach referred to as the set criterion approach involves the comparison of the entire image set rather than just a single element of this set. Even though the order relations are quasi-orders, the solution concept using set criterion is more appropriate for set-valued optimization problems.

Completely Mixed Stochastic Games

T. Parthasarathy
Chennai Mathematical Institute, Chennai, India
&
Indian Statistical Institute, Chennai, India

Consider a finite stochastic games. In this talk we try to address the following question: Suppose the stochastic game is completely mixed. Can we say individual matrix games corresponding to each state completely mixed ? Under some conditions we answer the question in the affirmative in the discounted case. Converse of this result is true if in each state each player has only two actions but it is not true if each has three actions. At the end we give some examples to show the sharpness of our results.

(This is a joint work with Sujatha Babu and Nagarajan Krishnamurthy).

Multiple Attribute Decision Making in e-Business- A Fuzzy Approach

B.K. Mohanty

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LUCKNOW - 226 013

In any business traditional or online a buyer normally develops in his/her mind some sort of ambiguity, given the choice of similar alternative products. For example, in a CAR purchasing problem a buyer always in dilemma given a choice of CARs like , Maruti Alto, Santro, i20 etc. The ambiguity or dilemma is mainly due to two reasons. Firstly how to make a final product choice and secondly on what basis the other products will be rejected. In order to answer the above questions, the customer may like to classify the products in different preference levels, preferably through a numerical strength of preference. For example, one can say an i20 may be 20% better than Maruti 800 or vice versa. Achievement of this classification will serve as a decision aid to the customer in the sense that while purchasing a product he/she will come to know how far he/she is compromising with reference to the best available product (zero compromise if the best product is purchased) and to what extent the buyer's choice is inferior to the best available products in the Internet.

In real terms, the buyer expects this product classification and final product recommendation from the e-business system itself. This task is next to impossible in the e-business as there is no direct interaction takes place between the sales personal and the buyer. This difficulty is further multiplied when the buyers express their product specifications in day to day linguistic terms. For instance; in any business situation normally a buyer express his/her desire linguistically or fuzzily defined terms. Another difficulty in successful implementation of e-business is to consider buyers' multiple desires or attributes, which are conflicting; non commensurable and fuzzy in nature, while making his/her product choices.

For example while purchasing car a buyer may express his/ her desires in multiple number of attributes which are conflicting, non commensurable and in fuzzy or linguistic terms in the following way.

- The price of the car should be *around $20000.*
- Resale value should be *high*
- *More or less* mileage should be *around 20Kms/gallon*
- The CAR should be *comfortable*
- Maintenance cost must be *low*.

In the above statements italic words are fuzzy.

The above statements are vague for computation reasons but they are the realistic day to day language of the buyers for business purposes, traditional or online. At times, the customer wants to make some trade-offs in the attribute specifications, mostly in a situation when there is a conflict amongst the product attributes. For example, in the CAR purchasing problem, the conflict may arise between the attributes "price and "mileage". The buyer may like to compromise a little amount of price in order to get a better mileage. As the attribute "price" and 'Mileage" are non commensurable and defined imprecisely (as shown in the above statements), for the e-business system it becomes more complicated to assess the buyers' needs and to finally recommend a product. Before the product recommendation, the e-business system needs to understand the buyers' above requirements. Further, the e-business system requires representing and incorporating the linguistic or fuzzily defined terms of the buyer into the system to make the e-business more customers' focused. Fuzzy logic helps in solving the above complex problems and arriving at a solution as per the buyers' requirements.

In general it is observed that while making a product choice a buyer normally assigns some weights to the product attributes implicitly. For example in a CAR purchasing problem a buyer's hidden linguistic weights to attributes price as " very important", to mileage "more or less OK", to maintenance cost " moderate" and to resale value as "high" etc. Not only have that, to another product the same customer's weighting pattern changes. In traditional markets the above implicitly defined hidden weights of the buyers can be somewhat explicated through the buyers' body language and style of talking. However, in the e-business it is next to impossible to articulate these underlying weights of the attributes when the customers make subjective judgements and gives an overall rating of the product. Enunciation of these hidden weights will not only make the e-business more customers focused, but also help in analyzing the needs of the customers in their product requirements. This work addresses this issue by using the concept OWA (ordered weighted average) operator and the linguistic quantifier.

Legal Disputes Resolved Via Game Theoretic Methods

T. E. S. Raghavan

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Mathematical foundations of conflict resolutions are deeply rooted in the theory of cooperative and non-cooperative games. While many elementary models of conflicts are formalized, often one raises the question whether game theory and its mathematically developed tools are applicable to actual legal disputes in practice. We choose an example from union management conflict on hourly wage dispute and how zero sum two person game theory can be used by a judge to bring about the need for realistic compromises between the two parties. We choose another example from the 2000 year old Babylonian Talmud to describe how certain debt problem was resolved. While they may be unaware of cooperative game theory, their solution methods are fully consistent with the solution concept called the nucleolus of a TU game.

Portfolio Optimization: Some Recent Advances

Pankaj Gupta

Department of Operational Research

University of Delhi, Delhi, India

Optimization models have been widely used in financial decisions and are globally accepted as one of the finest approaches to arrive at the optimal investment decision. We consider some classes of portfolio optimization problems treated through various portfolio optimization models. Most basic portfolio optimization problems are based on mean-variance optimization models corresponding to return and risk preferences of the investor. These mathematical portfolio optimization problems are either the quadratic programming or linear parametric programming problems. The early major contributions of the field are contributed by **Markowitz (1952, 1959)** and **Roy (1952)**. The term "optimization" in portfolio selection problem aims to find an optimal portfolio, which provides the lowest level of risk for a required level of return or, conversely, the highest return for a specified level of risk. By varying the values of risk /return level, one can obtain a set of optimal portfolios collectively called as the efficient frontier. There are many critical issues that require major attention while arriving at the optimal investment decision. One can note that besides the return and risk preferences there could be other preferences of the investor based on more important criteria. Further, the issue of transaction costs is critical to the construction and management of portfolios and there can be major impact of these costs on performance of the portfolio. Furthermore, the extensions of the classical mean-variance portfolio optimization model have been proposed by considering alternative measures of risk and many realistic constraints other than the capital budget constraint in order to arrive at the better decision making in the financial decisions. A common assumption in most of early portfolio optimization models is that we have enough historical data of assets (securities) to construct distributions for return and risk and that the market situation in future can be correctly reflected by asset data in the past. Such an assumption ignores, for example, the appearance of new assets in the market, or the uncertain situations arising from social and economical conditions. To deal with uncertainty, the major emphasis has been given to fuzzy set theory concepts for building portfolio optimization models using fuzzy variables for returns instead of random variables (**Gupta et al. (2014)**). We will focus on certain issues highlighted as above to present some major contributions of the recent literature from the field.

References

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2. H. M. Markowitz (1959). ***Portfolio Selection***. John Wiley & Sons, New York.
3. A. D. Roy (1952). **Safety First and the Holding of Assets,** Econometrica, 20, 431–449.
4. Pankaj Gupta, Mukesh Kumar Mehlawat, Masahiro Inuiguchi and Suresh Chandra (2014). ***Fuzzy Portfolio Optimization: Advances in Hybrid Multi-criteria Methodologies***, Studies in Fuzziness and Soft Computing, Vol. 316, Springer, Heidelberg, Germany.

One Player Control Semi-Markov Games With Limiting Average Payoffs

Prasenjit Mondal & Sagnik Sinha

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Zero-sum two-person finite state and action spaces semi-Markov games with limiting average (undiscounted) payoffs are considered where the transition probabilities and the transition times are controlled by a fixed player in all states. We prove the existence of value and optimal semi-stationary strategy (i.e., a semi-Markov strategy independent of decision epoch counts) for both the players. Some of the results obtained in this paper can easily be extended to nonzero-sum undiscounted semi-Markov games.

Keywords. Semi-Markov games, limiting average payoffs, one player control semi-Markov games, semi-stationary strategies.

Sustainable Supply Chain Management

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Department of Operational Research
University of Delhi

Due to customer's initiative and government legislative, companies are under legal and social pressures to redesign the logistics network in order to achieve sustainability. The need of the hour is for companies to take cognizance of the alarming existing situation and act accordingly. Sustainability can be attained by restricted use of natural resources, waste minimization and by reducing the negative social and environmental impact of supply chain practices and decisions. Organizations can contribute to sustainable development by integrating environmentally, socially and financially viable practices into the complete supply chain lifecycle, from product design and development, to material selection, (including raw material extraction or agricultural production), manufacturing, packaging, transportation, warehousing, distribution, consumption, return and disposal. Environmentally sustainable supply chain management and practices can assist organizations in not only reducing their total carbon footprint, but also in optimizing their end-to-end operations to achieve greater cost savings and profitability. Reverse logistics is inherently associated with sustainability which has prompted manufacturers in many countries to be financially and organizationally responsible for the take-back of their products when they reach the end of their life cycle. Instead of carting products to landfills, the value can be retrieved through a variety of other paths, such as refurbishing, remanufacturing, donations, secondary market sales and recycling, thus simultaneously reducing harmful effect on the environment while increasing profitability, product utilization and social impact. Major issues of all three dimensions will be addressed separately as well as simultaneously.

When are Some Queues (Not) in Equilibrium?

N. Hemachandra

Indian Institute of Technology Bombay, India

Consider a $M/M/1$ queue where admitting an arrival gives queue manager a reward of r Rupees, but, incurs a holding cost of h Rupees for each time unit the admitted arrival spends waiting for service. Suppose the manager has an option of either admitting an arrival or declining admission to it. For discounted reward structure, it is known that threshold policies are optimal: it is optimal to admit an arrival if and only if the number in the system is not more than a suitable R^*; this means that at optimality only a fraction of arrivals are admitted which we view as the Quality of Service, QoS, offered by the queue. Suppose now that the Poisson arrival rate of the queue, λ, is a function of the fraction of customers admitted, QoS; the QoS in turn depends on the arrival rate. Under mild conditions, we argue that if there is no equilibrium in such queues, there is an equilibrium set. We identify the role of the multiple optimal threshold policies in the existence of these equilibrium sets. Change in finite support of discrete valued inter-arrival times also has a similar role. We illustrate the above with numerical examples bringing out the role of symbolic computation tools. We also indicate some typos in the known algorithm to compute the optimal thresholds, which may be of independent interest. (Based on an on going joint work with Kishor Patil and Sandhya Tripathi.)

On Linear Complementarity Problem with a Hidden-Z Matrix

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Matrix classes plays an important role in the theory and algorithms of linear complementarity problem (LCP). The class of hidden-Z matrices was studied by Mangasarian and Pang in 80s and LP formulations are given to solve LCP for various special cases. We revisit the classes of hidden-Z matrices and discuss various properties.

On Solving Optimization problem in R

Meenal Chauhan

Visva-Bharati University

Shantiniketan, West Bengal

In this presenatation the use of R programming language (an "Open soure" package) is explained in the context of solving linear and nonlinear optimization problems. We intend to present the use of lp () function in the lpSolve package and solve() function in the lpSolve API package for solving the linear problems while using solnp() function in the Rsolnp package in order to obtain the solutions of nonlinear programming problems. We observe that R is a one of the highly efficient software for dealing with optimization problems.

Data Envelopment Analysis Approach to 'Green' Efficiency

Aparna Mehra

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Since its inception by Charnes, Cooper and Rhodes in 1978, data envelopment analysis (DEA) has come a long way in measuring relative efficiency score of homogeneous decision making units (DMUs). DMUs are entities which consume multiple inputs to produce multiple outputs. DEA is a popular multicriteria decision making (MCDM) aid capable to distinguish the benchmark entities based on an efficiency score and also identify the sources and amounts of inefficiency in the inefficient DMUs. The latter one is an important feature of DEA approach for it not only identify the inefficient units among the compared ones but also set targets for these DMUs. A sensible and realistic target-setting is important for success of every system to improve its efficiency in due course. Another significant aspect of DEA approach is that, unlike many other widely used MCDM techniques, it does not require supply of weights for decision makers or decision criteria involved in the problem.

In the thirty years of DEA history, several mathematical programming models have been prescribed in literature to measure different types of efficiency scores, viz. technical efficiency, ecological efficiency, quality efficiency, to name a few. Through this talk, we shall aim to explore some of the existing optimization models for DEA and their applications more specifically to problems with environment concerns; for instance green supply chain, green transportation, green

product designs. With society becoming more aware on environmental issues, green management has emerged as a key approach for organizations to become environmentally sustainable. In last couple of years DEA approach and its associated optimization models have contributed in this direction. We shall be understanding and highlighting these points in details.

Semi-Markov Decision Processes with Limiting Average Rewards

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Limiting average (Undiscounted) reward finite (state and action spaces) semi-Markov decision processes (SMDPs) are considered. Existence of an optimal semi-stationary strategy (i.e. a semi-Markov strategy independent of decision epoch count) is proved. All the work in the field of Markov decision processes considered under limiting average reward criterion can be viewed as special cases of the developments of this paper.

Multi-objective Shape Optimization of a Re-entry Capsule

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The aerodynamic shape optimization problem of a re-entry body has conflicting multi-objectives: minimization of its weight and maximizing its stability. The shape of a ballistic re-entry body is typically a spherical nose-cone-flare configuration and the design parameters for the multi-objective optimization are nose radius, first conical flare angle and its length and second conical flare angle and its length. A response surface model is generated which provide aerodynamic coefficients as a function of these parameters. The model is generated using modified Newtonian flow which is valid for hypersonic flows. Particle Swarm Optimization technique is used to solve the multi-objective problem.

Key words: Particle swarm optimization, multi objective optimization, hypersonic flow

Optimality and Duality of Variational Programming Problems

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We consider the following multi-objective variational problem as the following form:

$$\min \int_a^b f^1\left(t, x(t), \dot{x}(t), \ddot{x}(t)\right)dt, ..., \min \int_a^b f^k\left(t, x(t), \dot{x}(t), \ddot{x}(t)\right)dt$$

$$subject\ to\ \ x(a) = 0 = x(b),$$

$$\dot{x}(a) = 0 = \dot{x}(b)$$

$$h^j\left(t, x(t), \dot{x}(t), \ddot{x}(t)\right) \le 0,\ j \in M \equiv \{1, 2, ..., m\}$$

$$x \in PS\left(T, R^n\right),$$

Where function $f^i, i \in K = \{1, 2, ..., k\}$ and $h^j,\ j \in M = \{1, 2, ..., m\}$ are continuously differentiable function defined on $I \times R^n \times R^n \times R^n$.

In this paper we have established some optimality conditions for the multiobjective variational programming problems with generalized convexity of higher orders. A higher order dual is associated and weak and strong duality results are established under generalized convexity assumptions.

Keywords: Multiobjective programming, Variational problems, Duality

Role of Principal Pivot Transform in Optimization

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The principal pivot transform (PPT) is an operation of a linear system exchanging unknowns with the corresponding entries of the right-hand side of the system. The concept of principal pivot transform helps to develop many theories and algorithms in optimization theory and plays an important role in the study of matrix classes. The notion of PPT is encountered in mathematical programming, statistics and numerical analysis among other areas.

One of the main matrix classes discussed in association with PPTs is the class of P-matrices of whose principal minors are positive. Tucker asserts that principal pivot transform preserves the class of P-matrices. However, it is interesting to note that if the diagonal entries for every PPT are nonnegative, then the matrix need not be a Po-matrix. The notion of PPT is originally motivated by the well-known linear complementarity problem (LCP). The linear complementarity problem is normally identified as a problem of mathematical programming and provides a unifying framework for several optimization problems. In particular, the problem of computing a Karush-Kuhn-Tucker (KKT) point of a convex quadratic programming problem can be formulated as a linear complementarity problem. Matrix classes play an important role for studying the theory and algorithms of LCP. Several algorithms have been designed for the linear complementarity problem which is matrix class dependent, i.e. the algorithms work only for LCPs with some special classes of matrices and can give no information otherwise.

In the context of linear complementarity problem, some of the matrix classes are defined based on principal pivot transform. For example, Cottle and Stone introduced the notion of a fully semimonotone matrix by requiring that every PPT of such a matrix is a semimonotone matrix. For the class of fully semimonotone matrix with some additional conditions, LCP(q,A) has a unique solution. Stone studied various properties of fully semimonotone matrix and conjectured that fully semimonotone matrix with Qo-property are contained in Po. Parthasarathy et al. introduced fully copositive matrices, a subclass of fully semimonotone matrix. The conjecture was shown to be true when fully semimonotone matrix was replaced by fully copositive matrices. Neogy and Das introduced two new classes of matrices based on principal pivot transform. One of these classes has the property that its PPTs are either copositive or almost copositive with at least one PPT almost copositive and the other class has the property that its PPTs are either semimonotone or almost copositive with at least one PPT almost copositive.

These PPT based matrix classes are important with respect to Lemke's algorithm. In fact, these classes extend the class processable by Lemke's algorithm. It is well-known that Lemke's algorithm finds solution for a linear complementarity problem for the class of Po with Qo-property. However, it is difficult to verify whether a matrix class belongs to Po with Qo-property or not. The class mentioned above is a subclass of Po with Qo-property and the membership of this class is easy to verify. The PPT based matrix classes motivate further study and applications in matrix theory.

Interval Valued Intuitionistic Fuzzy Multiple Criteria Decision Making Problem

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A Multi-criteria Decision Making(MCDM) problem for interval valued intuitionistic fuzzy(IVIF) information is considered. Hierarchical clustering approach is applied using equivalence relation. Entropy method is applied for calculating weights of the criteria. Method is illustrated via an example.

Keywords: Interval-valued intuitionistic fuzzy set, Hierarchy, Cluster analysis, correlation, equivalence relation, relation, Multi-criteria Decision Making (MCDM), entropy method.

Optimization of Tetracycline Adsorption By Magnetic Carbon from Water Using Box-Behnken Design and Response Surface Modeling

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Frequent detection of pharmaceuticals in the aquatic matrices, including drinking water has become a cause of serious concern from human and ecological health perspectives. Adsorption based treatment process by a novel magnetic carbon is an efficient technique for water decontamination. However, the conventional adsorption assays based on one-factor at a time approach is cumbersome, fails to reveal the interactive effects of the process variables and the area of true optimum is seldom reached from the investigated experimental domain. This largely depreciates any efficient treatment technique. Hence, the need of an optimal treatment process which yields the desired response within less time, energy and cost is explicit. The present study involves statistical optimization of three process variables (temperature (ºC), pH and adsorbent dose (g/l)) in aqueous phase adsorption of tetracycline, a ubiquitous broad spectrum antibiotic by plastic waste derived magnetic carbon using Box-Behnken Design (BBD) and Response Surface Modeling (RSM). The optimization modeling of the process variables involved the following steps: (1) performing the statistically designed experiments according to the design, factors, and levels selected; (2) estimating the coefficients of the second-order polynomial model to predict the response (adsorption capacity of the magnetic carbon (mg/g)); and (3) checking suitability of the selected model. Results showed that the highest response obtained from the three-factor, three-level BBD corresponded to an adsorption capacity of 51.41 mg/g at temperature of 50ºC, pH 7 and adsorbent dose of 1 g/l. After performing a quadratic fit between the design factors and response, the model predicted an adsorption capacity of 53.12 mg/g at optimized factor settings of temperature 49.69 ºC, pH 6.9 and adsorbent dose of 0.4 g/l at a constant initial tetracycline concentration of 100 mg/l. This was experimentally verified to be 49.34 mg/g under laboratory conditions. The analysis of variance (ANOVA) was performed to evaluate the statistical significance of the model and its components. The Fisher's F test and associated probability (p) values in ANOVA demonstrated overall significance of the model ($p<0.05$). The coefficient of determination (R^2) was computed to be as high as 0.992 while the Adj R^2 was close at 0.979, indicating inclusion of adequate number of model terms.

The goodness of fit of the regression model was further adjudged from the Root Mean Square Error of Prediction (RMSEP), Relative Square Error of Prediction (RSEP) and Chi-Square ($\chi2$) values. There was good agreement between the model predicted and experimental response as the RMSEP, RSEP and χ^2 values were as low as 1.00, 3.29 and 0.33 respectively. The fitted polynomial equation was illustrated in the form of three dimensional graphical surface plots. The interactive effects of pH with temperature (ºC) and pH with adsorbent dose (g/l) synergistically influenced the adsorption of tetracycline by the prepared magnetic carbon. The selected model was also validated using an external dataset generated from experimental runs performed at random combinatorial levels of the process variables. The R^2 and RMSEP values determined from the model validation set were 0.970 and 2.26 respectively, indicating robustness and good predictive ability of the model. Hence, BBD combined with RSM can be suitably applied as a predictive and process optimization tool in adsorption of emerging pollutants such as pharmaceuticals from water.

An overview of Index Tracking and Enhanced Index Tracking

Amita Sharma, Shubhada Aggrawal and Aparna Mehra

In this paper, we aim to overview some of the existing optimization techniques to track and enhance the benchmark index. Subsequently, we propose a model for enhanced indexing using relaxed second order stochastic dominance criterion, a concept derived from almost second order stochastic dominance.

Keywords Index tracking, enhanced indexing, second order stochastic dominance, almost second order stochastic dominance.

Approximate Lagrangian duality for set-valued optimization problem

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In this paper, we study approximate Lagrangian duality for set-valued optimization problem where the solutions are defined using set relations introduced by Kuroiwa

(Kuroiwa, D.: The natural criteria in set-valued optimization. Sūrikaisekikenkyūsho Kōkyūroku **1031**, 85-90 (1998)).

Keywords: Set-valued optimization, Approximate solutions, Lagrangian duality

Optimization of Farm Income through Farming Systems on Tribal Farms in Udham Singh Nagar district of Uttarakhand

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The main aim of this study is to examine the potentiality to increase the farm income through farming system. To analyze the same, linear programming technique has been used. The study is conducted in Udham Singh Nagar district of Uttarakhand, based on the primary data collected from 60 tribal farmers for the agricultural year 2008-09. Farming systems practiced by more than 90 per cent of tribal farmers in the study area is considered as major farming systems. The farming systems selected are Crop + Livestock (FS-I), only Crop (FS-II), Crop + Livestock + Orchard (FS-III) and only livestock (FS-IV) farming systems. Across the farming systems, potential to increase Net Return Over Variable Cost (NROVC) over existing plan is highest in Farming System (FS)-IV (livestock) followed by FS-I (Crop+Livestock) and FS-II (Crop). Whereas in case of FS-III (Crop+Livestock+Orchard), potential to increase the income is only 0.27 per cent. Major policy implications emerged from the study are; more attention is required towards the improvement of orchard and livestock rearing. Income of the tribal farmers can be increased through increasing the

area under orchard and by rearing improved breed of livestock. Potato cultivation should be encouraged in the study area, however, its cold storage requirement for the crop produce needs to keep in mind. Therefore, it is possible to increase the farm income of tribal farmers through reallocation of existing farm resources optimally under all the farming systems.

Keywords: Optimization, Farm Income, Farming System and Tribal farms

Designing vehicle parameters using Split and discard decision making strategy

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In this paper, we present a mathematical model of a half car with two passengers. The model under study has important features such as; non-linearity of suspension spring and damper, tire damping with non-linear spring stiffness. These features validate the model to real application. The response of the dynamical system running over a road with series of irregular shaped bumps has been studied by simulation. These bumps have been mathematically expressed. The suspension and tire parameters have been determined optimally using a new hybrid algorithm in time domain. The developed algorithm is based on Split and Discard Strategy (SDS) and advanced real coded genetic algorithm (ARCGA). To find these parameters we have formulated a constrained non-linear optimization problem to minimize the vibration experienced by the passenger as well as to enhance road holding performance during riding. For this purpose the weighted sum of sprung mass jerk and tire deflections are minimized under technological constraints. Moreover, the results obtained from simulations of model with original and optimized suspension parameters are compared.

Keywords: Suspension, half car, optimization, genetic algorithm

Exploring Green Manufacturing antecedents: A MICMAC Analysis

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With rapid change in technology, customer needs and globalization, manufacturing itself is evolving. Due to the global climate, humans have started to realize the vulnerability of nature and disasters it may bring as response of our negligence. Enduring developments pose challenges for supply chain managers on the strategic management expertise of today’s companies. These trends embrace ongoing globalisation and the increasing passion of competition, the emergent demands of security, environmental protection and resource scarcity as the need for flexible, trustworthy and cost-efficient business systems capable of supporting customer diversity. Manufacturing plays a strategic role in an organisation to improve performance. Green manufacturers try to make products that have a lower environmental impact than other products. India has a huge consumer base and a tremendous market and efforts are needed to catalyse waste reduction efforts. Critical things needed is the sort of lean technologies, green chemical and life cycle assessment and engineering to make processes more scalable in the presence of variable demand. Nevertheless, this study has been specifically undertaken to explore pressures which motivates manufacturing industries to increasingly adopt green manufacturing prominent contribution. Hence, the study benefits the researchers by providing bright outlook for future.

Keywords: Green Manufacturing, practices, ISM and MICMAC analysis

Genome Scale Metabolic Model Development and Flux Analysis of Thermophilic Organism: An *In Silico* Optimization Approach.

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In our day-to-day life, the liquid fuels that are commonly used in transportation and other commodities are the various forms of petroleum products derived from fossil fuels. However, due to the reduction in the availability of fossil fuels and its potential risk of scarcity in the future, there is a huge demand for producing biofuels from sustainable energy sources. In an attempt to that, the first generation biofuels such as bioethanol and biodiesel was produced from food crops, however as it was directly affecting the food supply chain, the second generation biofuels from agricultural

wastes or lignocellulosic biomass was implemented for bioenergy applications. Rice-straw is one such lignocellulosic waste material, which can be readily hydrolyzed into fermentable sugars through appropriate pre-processing treatment. Ethanol producing microbes, which are ubiquitous in nature, can avail these hydrolyzed sugars for its growth and metabolism and can convert this into ethanol. As India is the third largest producer of rice crop in the world, the readily available rice-straw wastes can be used as a potential feedstock for bioethanol production.

Thermophilic organism, a well-known species to grow at high temperature and known to efficiently hydrolyze starch material was used as a reference model strain, in order to understand its metabolic potential to produce ethanol. A large scale metabolic model was constructed for this organism and *in silico* analysis was performed by maximizing the cell growth as an objective function to produce ethanol through linear programming approach. During the model development process, all the pre-requisite metabolic information of the organism was used and systematically organized in a mathematical format, such that representing S.V=0, wherein S is a m x n stoichiometric matrix comprising of 'm' number of metabolites and 'n' number of reactions, and V denoting the fluxes to be calculated for each metabolite. A flux balance analysis was performed for the set of these formulated equations and intracellular metabolic fluxes was computed for each metabolites, by considering the system at steady state with the input of known constraints and flux values.

Furthermore, the gaps in the model were identified and corrected using 'gap-filling' approach. To overproduce the yield of ethanol, the possible efficient metabolic routes to engineer the strain was identified using 'Optknock' and 'GDLS' (Genetic Design through Local Search) methods, which are heuristic algorithms and it uses bilevel optimization and mixed-integer linear programming(MILP) approaches to solve the optimization problem. The developed metabolic model would find large application in suggesting all the possible *in silico* predictions to engineer the strain such that to produce increased amount of biofuels.

Metabolic Modelling through Optimization Strategies

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Genome scale metabolic models (GSMMs) are convenient way of representing known information about the metabolism of an organism in the form of a metabolic network. They contain all the known biochemical reactions in an organism; transport reactions for metabolites and biomass reaction based of biochemical composition of the organism in question. GSMMs can be used to predict the metabolic behaviour of the organism using flux balance analysis (FBA).

GSMM is essentially a stoichiometric matrix (**S**) of size '*m x n*', where *m* equals to number of metabolites and *n* equals to number of reaction in the model. FBA solves a system of linear equations, $\mathbf{S}\cdot\mathbf{v} = \mathbf{0}$, for unknown **v** which is column vector of length '*n*' representing the flux values (flux distribution) of individual reactions. The '*zero*' in the right hand side of the equation assumes a metabolic steady state i.e. the rate of formation of any metabolite is equal to the rate of consumption of the metabolite in the network. The solution is calculated subject to an objective function (e.g. maximization of biomass reaction flux) and constraints on the flux values (elements of **v**) individual reactions can take. Constraining reaction fluxes to experimentally measured values make the model behave close to the actual organism.

Although FBA can roughly predict the metabolite secretion/uptake profiles of organisms, the internal reaction rates are quite difficult to predict. Also, the prediction of metabolic behaviour of engineered organism (perturbed system) is not possible with simple FBA. This poses serious limitations in finding knockout/knock in strategies for the production of industrially important metabolite from an organism.

Various FBA based optimization methods are used to predict the perturbed system's flux distribution. They differ in the objective function/functions and the way they impose constrains on individual flux values. I work on finding knockout strategies for the production of biofuel molecules in bacteria. I have used Genetic Design through Local Search (GDLS) method to identify knockout strategies and Relative Change (RELATCH) method to predict flux distribution of organisms after knockouts. GDLS is heuristic approach which uses local search with multiple search paths and mixed integer linear programming to identify best knockout strategy for metabolite production with at least some defined minimum growth rate (biomass reaction flux). RELATCH predicts the flux distribution of the engineered organism by minimizing the relative change in flux values with respect to the wild type organism's flux values.

Myriad optimization methods are available on different programming platforms for metabolic modelling. Some are based on bilevel (e.g. OptKnock) or even multilevel (e.g. RobustKnock) optimizations. Each one has its own strengths and limitations. In order to develop new optimization methods which can predict metabolic behaviour of organisms more accurately, the understanding of mathematics behind optimization problems, their formulation and finally coding in familiar programming platform is necessary. The workshop on Applied Optimization Models and Computation can help me in this purpose.

On Relations between Vector Variational Inequality and Nonsmooth Vector Pseudolinear Optimization Problems

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This paper deals with the relations between a solution of a nonsmooth Stampacchia type vector variational inequality problem and efficient and properly efficient solutions of a nonsmooth vector optimization problem. We derive a characterization for the Clarke generalized gradient of locally Lipschitz pseudolinear functions. This characterization is employed to establish that a variant of nonsmooth Stampacchia type vector variational inequality is a necessary as well as sufficient optimality condition for a solution to be efficient for a nonsmooth pseudolinear vector optimization problem. To the best of our knowledge, such results have not been established till now.

Keywords: Nonsmooth vector optimization; Vector variational inequality; Locally Lipschitz function, Pseudolinearity.

Opinion mining using Internet Reviews: A Fuzzy MADM approach

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With the emergence of Web 2.0, the large number of customers' reviews is available to be read by the potential buyers. It is a common practice of the buyers to assess popularity of the products after reading other users' views about them on the internet. These opinions about the products' multiple features are scattered over on the net and are expressed linguistically in day to day terms. Further, the expressions regarding the product's performance in multiple features vary in multiple degrees. Thus it becomes difficult for a buyer(user) to get a comprehensive view about a product's performance relative to other products available in the market considering all the features simultaneously. In our paper, we propose a methodology to calculate the popularity score of the products as per opinions or reviews of the other buyers in the internet. The views are taken across multiple features that may be important for a buyer in that product category. The methodology is based on the techniques of Fuzzy Multiple Attribute Decision Making. The views regarding the products' multiple features are taken from different web sites. The views regarding a product feature, expressed linguistically, is represented in a linguistic 2-tuple using a basic linguistic term set (F. Herrera 2000). The numerical equivalents of these linguistic preferences are obtained using the technique given in (Herrera and Martinez 2000). These values across multiple attributes are aggregated using PROMETHEE that gives us ranking of the products considering their multiple features. The methodology is illustrated with the help of an example in product category

Solving Optimization Problems using Mathematica

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Mathematica is a software system and computer language for use in mathematical applications. The three classes of Mathematica computations are: numerical, symbolic, and graphical. Mathematica can be used as a calculator with a much higher degree of precision than traditional calculators. It is an advanced software system that enables symbolic computing, numerics, program code development, model visualization and professional documentation in a unified framework. The first version of Mathematica was released by Wolfram Research, USA in 1988. Mathematica includes an internal compiler to optimize the performance of numerical code and compilation to external C code. Both systems can utilize an external C compiler for optimized performance of users' numerical code and automatic performance enhancement for key computations such as the numeric solution of differential equations.

Mathematica is an interactive program with a vast range of uses:

- Numerical calculations to required precision
- Symbolic calculations/ simplification of algebraic expressions
- Matrices and linear algebra
- Graphics and data visualisation
- Calculus
- Equation solving (numeric and symbolic)
- Optimization
- Statistics
- Polynomial algebra
- Discrete mathematics
- Number theory
- Logic and Boolean algebra
- Computational systems e.g. cellular automata

Mathematica Versions are available for:

- Windows XP, Vista
- Mac OS-X
- Unix/Linux

Mathematica provides a very powerful and flexible environment both for carrying out optimization techniques and for developing appropriate optimization algorithms.

The aim of this talk is to explore how technology-based teaching with Mathematica helps students in understanding optimization methods. Students/Researchers solve the optimizations problems by using Mathematica.

In optimization we discuss the facilities for numeric and symbolic, global and local constrained and unconstrained optimisation as:

- Numeric:
  - ✓ local – FindMinimum, FindMaximum
  - ✓ fitting - FindFit
  - ✓ global - NMinimize, NMaximize

- Symbolic:
  - ✓ Minimize
  - ✓ Maximize

- The above functions have been updated for Mathematica 10.0.2.

The numerical capabilities of the Mathematica are illustrated by simple and more advanced examples, pointing towards a broad range of potential applications. We demonstrate some simple examples of optimization problems. Some examples and case study are also used to discuss the comparative study among Mathematica, Matlab and Maple in the context of Optimizations techniques.

A rich variety of real world optimization problems can be cast as integer linear programming. In this talk, we show how Mathematica tools can be used to solve these programming problems when some or all the decision variables must be integer.

The methods used to solve local and global optimization problems depend on specific problem types. Optimization problems can be categorized according to several criteria. Depending on the type of functions involved there are linear and nonlinear (polynomial, algebraic, transcendental etc.) optimization problems. Mathematica functions for constrained optimization include Minimize, Maximize, NMinimize and NMaximize for global constrained optimization, FindMinimum for local constrained optimization, and LinearProgramming for efficient and direct access to linear programming methods.

Linear programming problems are optimization problems where the objective function and constraints are all linear. Mathematica has a collection of algorithms for solving linear optimization problems with real variables, accessed via LinearProgramming, FindMinimum, FindMaximum, NMinimize, NMaximize, Minimize, and Maximize. LinearProgramming gives direct access to linear programming algorithms, provides the most flexibility for specifying the methods used, and is the most efficient for large-scale problems. FindMinimum, FindMaximum, NMinimize, NMaximize, Minimize, and Maximize are convenient for solving linear programming problems in equation and inequality form.

The simplex and revised simplex algorithms solve linear programming problems by constructing a feasible solution at a vertex of the polytope defined by the constraints, and then moving along the edges of the polytope to vertices with successively smaller values of the objective function until the minimum is reached. Although the sparse implementation of simplex and revised algorithms are quite efficient in practice, and are guaranteed to find the global optimum, they have a poor worst-case behavior: it is possible to construct a linear programming problem for which the simplex or revised simplex method takes a number of steps exponential in the problem size. Mathematica implements simplex and revised simplex algorithms using dense linear algebra. The unique feature of this implementation is that it is possible to solve exact/extended precision problems.

Numerical algorithms for constrained nonlinear optimization can be broadly categorized into gradient-based methods and direct search methods. Gradient search methods use first derivatives (gradients) or second derivatives (Hessians) information. Examples are the sequential quadratic programming (SQP) method, the augmented Lagrangian method, and the (nonlinear) interior point method. Direct search methods do not use derivative information. Examples are genetic algorithm and differential evolution, and simulated annealing.

Direct search methods tend to converge more slowly, but can be more tolerant to the presence of noise in the function and constraints. Typically, algorithms only build up a local model of the problems. Furthermore, to ensure convergence of the iterative process, many such algorithms insist on a certain decrease of the objective function or of a merit function which is a combination of the objective and constraints. Such algorithms will, if convergent, only find the local optimum, and are called local optimization algorithms.

In Mathematica local optimization problems can be solved using FindMinimum. Global optimization algorithms, on the other hand, attempt to find the global optimum, typically by allowing decrease as well as increase of the objective/merit function. Such algorithms are usually computationally more expensive. Global optimization problems can be solved exactly using Minimize or numerically using NMinimize.

In addition to a demonstration of how to use (and how not to use) Mathematica's built-in general procedure Find Minimum for doing unconstrained nonlinear programming, the package MultiplierMethod.m is provided for accomplishing a wide variety of more complex optimization problems, including nonlinear programming with nonlinear equality and inequality constraints.

Mathematica can also make use of MathLink to interact with numerous other familiar programs such as Excel. It is worth noting that Mathematica contains close variants of
all the standard iterative and logical (branching) commands of other well-known languages, Do's, For's, If 's, etc., we discuss these in details during the talk.

Mathematica, then, is an environment that one can start using quickly, easily, and productively for many standard needs. But it is also an environment that can, with learning and effort, accommodate almost any computational research need.

To solve optimizations problems, we can use other software namely Matlab and Maple. Matlab, Mathematica, and Maple have a full-featured command line interface alternative to their GUIs and devoted adherents. All three are available for Windows, Mac OS X, Linux, and most flavors of Unix (Solaris, AIX, IRIX, HP-UX, Tru64).

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Convex Optimization for Big Data in Finance

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We present review of some representative convex optimization algorithms/techniques for big data in finance, especially High Frequency Trading (HFT). Our selective review outlines that with the ever increasing availability of data in finance, trading per se, there is need to solve ever larger instances of data science and machine learning problems, many of which turn out to be convex optimization problems in huge dimensions. This demand the need for efficient algorithms, which can benefit from distributed computing, where only a part of the input is stored on each of the nodes of a cluster and both the computation and communication are designed accordingly.

Keywords: Convex Optimization, High Frequency Trading , Big Data Trading Strategies

On computation using MATLAB

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MATLAB is both a computer programming language and a software environment for technical computing. It integrates computation, visualization, and programming in an easy-to-use environment where problems and solutions are expressed in familiar mathematical notation. It includes

- Math and Numerical computation
- Algorithm development
- Modeling, simulation, and prototyping
- Data analysis, exploration, and visualization
- Scientific & engineering graphics
- Application development, including graphical user interface building

MATLAB is an interactive system whose basic data element is an array that does not require dimensioning. Every number can be represented as matrix. It allows solving many technical computing problems. It would take to write a program in a scalar non-interactive language such as C and FORTRAN.

The name MATLAB stands for matrix laboratory. MATLAB was originally written to provide easy access to matrix software developed by the LINPACK and EISPACK projects. Today, MATLAB engines incorporate the LAPACK and BLAS libraries, embedding the state of the art in software for matrix computation. Normally numerical computation and simulation can be done with the help of MATLAB. Actually there are various toolboxes in MATLAB that perform more specialized computations, dealing with applications. MATLAB have the following features:

- User friendly(GUI)
- Easy to work with
- Powerful tool for real and complex mathematics
- Platform independence
- Plotting

Normally researchers and scientists in the field of Mathematical sciences, Physical sciences and Medical sciences are the main user of this software. As this is a user friendly language large computation can be done with the help of MATLAB.

The MATLAB system consists of five main parts:

Development Environment: This is the set of tools and facilities that help you use MATLAB functions and files. Many of these tools are graphical user interfaces. It includes the MATLAB desktop and Command Window, a command history, an editor and debugger, and browsers for viewing help, the workspace, files, and the search path.

MATLAB Mathematical Function Library: This is a vast collection of computational algorithms ranging from elementary functions, like sum, sine, cosine, and complex arithmetic, to more sophisticated functions like matrix inverse, matrix Eigen-values, Bessel functions, and fast Fourier transforms.

MATLAB Language: This is a high-level matrix/array language with control flow statements, functions, data structures, input/output, and object-oriented programming features. It allows both "programming in the small" to rapidly create quick and dirty throw-away programs, and "programming in the large" to create large and complex application programs.

Graphics: MATLAB has extensive facilities for displaying vectors and matrices as graphs, as well as annotating and printing these graphs. It includes high-level functions for two-dimensional and three-dimensional data visualization, image processing, animation, and presentation graphics. It also includes low-level functions that allow you to fully customize the appearance of graphics as well as to build complete graphical user interfaces on your MATLAB applications.

MATLAB Application Program Interface (API): This is a library that allows you to write C and FORTRAN programs that interact with MATLAB. It includes facilities for calling routines from MATLAB (dynamic linking), calling MATLAB as a computational engine, and for reading and writing MAT-files.

There is a toolbox for optimization in MATLAB. Optimization Toolbox is a collection of functions that extend the capability of the MATLAB numeric computing environment. There are four general categories of Optimization Toolbox solvers:

Minimizers: This group of solvers attempts to find a local minimum of the objective function near a starting point x_0. They address problems of unconstrained optimization, linear programming, quadratic programming, and general nonlinear programming.

Multi-objective minimizers: This group of solvers attempts to either minimize the maximum value of a set of functions (fminimax), or to find a location where a collection of functions is below some prespecified values (fgoalattain).

Equation solvers: This group of solvers attempts to find a solution to a scalar- or vector-valued nonlinear equation $f(x) = 0$ near a starting point x_0. Equation-solving can be considered a form of optimization because it is equivalent to finding the minimum norm of $f(x)$ near x_0.

Least-Squares (curve-fitting) solvers: This group of solvers attempts to minimize a sum of squares. This type of problem frequently arises in fitting a model to data. The solvers address problems of finding nonnegative solutions, bounded or linearly constrained solutions, and fitting parameterized nonlinear models to data.

All the toolbox functions are MATLAB M-files, made up of MATLAB statements that implement specialized optimization algorithms.

Running the optimization: There are two ways to run the optimization namely Using “Optimization app” (Start the optimization app by typing **optimtool** at the command line.) and using command line functions.

Simulink is an interactive tool for modeling, simulating, and analyzing dynamic, multidomain systems. It lets you accurately describe, simulate, evaluate, and refine a system's behaviour through standard and custom block libraries. Simulink models have ready access to MATLAB, providing you with flexible operation and an extensive range of analysis and design tools. You can use your models for many tasks beyond modeling and simulation via other products. The Simulink Report Generator extracts design information in models into technical documents, and the Real-Time Workshop and Real-Time Workshop Embedded Coder generate highly portable ANSI C and ISO C code from models for use in embedded systems, rapid prototyping, model deployment, and hardware-in-the-loop applications.

Importing data in MATLAB means loading data from an external file. The **importdata** function allows loading various data files of different formats. When you run the file, MATLAB displays the image file. However, you must store it in the current directory. Now consider the case of export data. Data export in MATLAB means to write into files. MATLAB allows you to use your data in another application that reads ASCII files. For this, MATLAB provides several data export options.

Use of Maple to Solve Optimization Problem

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Maple is technical computing software for engineers, mathematicians, and scientists. Maple's world-leading computation engine offers the breadth, depth, and performance to handle every type of mathematics. Maple is a computer program for doing a variety of symbolic, numeric, and graphical computations. Such a program is commonly called a CAS, short for Computer Algebra System, originally developed as a joint research project centered at the University of Waterloo and ETH Zurich. In Mathematics we can work with a vast range of uses:

- Algebra
- Basic Mathematics
- Calculus
- Calculus of Variation
- Differential Equations
- Differential Geometry
- Discrete Mathematics
- Factorization and Solving Equation
- Financial Functions
- Geometry
- Group Theory
- Linear Algebra
- Logic
- Mathematical Functions
- Number Theory
- Numerical Computations
- **Optimization**
- Power Series
- Special Functions
- Statistics
- Tensor Analysis
- Vector Calculus

The first concept of Maple arose from a meeting in November 1980 at the University of Waterloo. Researchers at the university wished to purchase a computer powerful enough to run Macsyma. Instead, it was decided that they would develop their own computer algebra system that would be able to run on lower cost computers. The first limited version appearing in December 1980 with Maple demonstrated first at conferences beginning in 1982. The name is a reference to

Maple's Canadian heritage. By the end of 1983, over 50 universities had copies of Maple installed on their machines.

In 2005, Maple 10 introduced a new “document mode”, as part of the standard interface. The main feature of this mode is that mathematics is entered using two dimensional input. In 2008, Maple 12 added additional user interface features found in Mathematica, including special purpose style sheets, control of headers and footers, bracket matching, auto execution regions, command completion templates, syntax checking and auto-initialization regions. Additional features are added for making Maple easier to use as a MATLAB toolbox. Maple 13 introduced a new flythrough feature for graphing a new way to visualize graphing. In September 2009 Maple and Maplesoft were acquired by the Japanese software retailer Cybernet Systems. Maple 16's performance was being undercut by Mathematica when it compared its newest version to Maple 15. Many of Maple16's performance enhancements were actually much better than Mathematica's hence Wolfram's decision to compare it to an earlier version. Maple 16's graphical environment is much improved over the past. Maple performs best on problems involving symbolic, as opposed to numerical computation. However, it is generally easier to use Maple on numerical problems rather than write programs in FORTRAN or C, for numerical calculations that are not too involved. Maple also provides the user with a lot of graphical power.

Maplesoft offers a suite of products designed for online placement testing, homework delivery, drill and practice, exam questions and assignments, high stakes testing, standards and gateway testing, and “just in time” teaching. The Möbius project is the biggest academic initiative in Maplesoft’s 25 year history. Create math apps, share them with everyone, and grade them to assess understanding. MapleNet offers a suite of mathematical services that let you use Maple in interactive web and desktop applications, share solutions over the web through interactive Maple documents, and develop rich technical web content. Maplesoft’s Professional Services can help you implement your modeling and simulation strategy in a timely and cost effective way. Our team of highly experienced engineers, mathematicians, and computing experts are the ideal complement to your team. Maple software consists of two distinct parts namely user interface and computation engine.

User Interface: We can use the Maple *user interface* to enter, manipulate, and analyze mathematical expressions and commands. The user interface communicates with the Maple computation engine to solve mathematical problems and display their solutions.

Computation Engine: The Maple computation engine is the command processor, which consists of two parts namely kernel and math library.

The *kernel* is the core of the Maple computation engine. It contains the essential facilities required to run and interpret Maple programs, and manage data structures.

The Maple kernel also consists of *kernel extensions*, which are collections of external compiled libraries that are included in Maple to provide low-level programming functionality. The math *library* contains most of the Maple commands. It includes functionality for numerous mathematical domains, including calculus, linear algebra, number theory, and combinatorics. Also, it contains commands for numerous other tasks, including importing data into Maple, XML processing, graphics, and translating Maple code to other programming languages. All library commands are implemented in the high-level Maple programming language, so they can be viewed and modified by users. By learning the Maple programming language, one can create custom programs and packages, and extend the Maple library.

Optimization: The **Optimization** package is a collection of commands for numerically solving optimization problems, which involve finding the minimum or maximum of an objective function possibly subject to constraints. The package takes advantage of built-in library routines provided by the Numerical Algorithms Group (NAG). The package solves linear programs (LPs), quadratic programs (QPs), nonlinear programs (NLPs), and both linear and nonlinear least-squares problems. Both constrained and unconstrained problems are accepted. In general, variables are assumed to be continuous, and local solutions are computed for problems that are not convex. However, the LPSolve command does accept integer programs and the NLPSolve command provides a global search algorithm for limited situations. The following is a list of commands available in the **Optimization** package:

(i) ImportMPS

(ii) Interactive

(iii) Maximize

(iv) Minimize

Minimize(obj,constr,bd,opts)

Maximize(obj,constr,bd,opts)

Minimize(opfobj,ineqcon,eqcon,opfbd,opts)

Maximize(opfobj,ineqcon,eqcon,opfbd,opts)

obj=algebraic; objective function

constr = set or list of relation; constraints

bd=sequence of name=range; bounds for one or more variables

opfobj=procedure; objective function

ineqcon=set or list of procedure; inequality constraints

eqcon=set or list of procedure; equality constraints

opfbd= sequence of ranges; bounds for all variables

opts= equation of the form: option=value, where option is one of assume, feasibilitytolerance, infinitebound, initialpoint, iterationlimit; specify options for the Minimize or Maximize command

The opts argument can contain one or more of the following options. The list below contains the options applicable to most or all of the Optimization package.

assume=*nonnegative*; assume that all the variables are nonnegative
feasibilitytolerance = *realcons(positive)* -- Set the maximum absolute allowable constraint violation.
infinitebound = *realcons(positive)* -- Set any value greater than the infinitebound value to be equivalent to infinity during the computation.
initialpoint = *set(equation)*, *list(equation)*, or *list(numeric)* -- Use the provided initial point, which is a set or list of equations *varname = value* (for algebraic form input) or a list of exactly n values (for operator form input).

iterationlimit = *posint* -- Set the maximum number of iterations performed by the algorithm. variables = *list(name)* or *set(name)* -- Specify the problem variables when the objective function is in algebraic form.

(v) LPSolve
(vi) LSSolve
(vii) NLPSolve
(viii) QPSolve

Calling sequence

LPSolve(c, lc, bd, opts)
NLPSolve(obj, constr, bd, opts)
NLPSolve(opfobj, ineqcon, eqcon, opfbd, opts)
QPSolve(obj, constr, bd, opts)

Optimization[LPSolve](Matrix Form) - solve a linear program in Matrix Form

Parameters:
c = Vector; linear objective function
lc = list; linear constraints
bd = list; bounds
opts = equation(s) of the form option = value where option is one of assume, binaryvariables, depthlimit, feasibilitytolerance, infinitebound, initialpoint, integertolerance, integervariables, iterationlimit, maximize, nodelimit or output; specify options for the LPSolve command.

Task Scheduling in Cloud Computing Environments using Large Scale Linear Programming

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Cloud computing is a recently evolved computing terminology based on utility and consumption of computing resources rather than having local servers or personal devices to handle applications. It involves deploying groups of remote servers and software networks that allow centralized data storage and online access to computer services or resources. A simple example of cloud computing is Yahoo email, Gmail, or Hotmail etc. All you need is just an internet connection and you can start sending emails.
While handling the complex applications there are problems faced and thus we need task scheduling in order to deal with them. The problems can be divided into two classes.

The one is computing intensive, the other is data intensive. As far as the data intensive application is considered, our scheduling strategy should decrease the data movement which means decreasing the transferring time; but for the computing intensive tasks, our scheduling strategy should be to schedule the data to the high performance computer.
Our main contributions are as follows:
(1) We formulate a model for task scheduling in cloud computing to minimize the overall executing cost and the transforming time.
(2) We propose an interior point method(IPM) for linear programming-which has gained extraordinary interest in the past 10 years i.e the infeasible-primal-dual algorithm to solve the task scheduling which is widely considered the most efficient general purpose IPM.
Interior point methods are now very reliable optimization tools. Sometimes only for the reason of inertia, the operations research community keeps using the simplex method in applications that could undoubtedly benefit from the new interior point technology. This is particularly important in those applications which require the solution of very large linear programs (with tens or hundreds of thousand constraints and variables).The model formulated in our case has a linear objective function and linear constraints with large number of variables.
The most efficient interior point method today is the infeasible-primal-dual algorithm. It is computationally most attractive IPM, indeed has been implemented in all commercial software packages.
The major work in a single iteration of any IPM consists of solving a set of linear equations, the so-called Newton equation system. This system reduces all IPMs to the problem that is equivalent to an orthogonal projection of a vector on the null space of the scaled linear operator.

There are two effective direct approaches for solving the Newton equations: the augmented system approach (M. Arioli, J. W. Demmel, and I. S. Du. Solving sparse linear systems with sparse backward error. SIAM J. Mat. Anal. Appl. , 10(2):165-190, 1989., M. Arioli, I. S. Du, and P. P. M. de Rijk. On the augmented system approach to sparse least-squares problems. Numer. Math. , 55:667-684, 1989). and the normal equations approach. The former requires factorization of a symmetric indefinite matrix, the latter works with a smaller positive definite matrix.
Another approach discussed is a hybrid interior point/simplex approach. It has been shown in literature that the combination of interior point/simplex is a powerful tool for very large scale linear programs. We can apply this method using CPLEX OPTIMIZER.

Classification of URLs based on malign/benign: An optimization approach

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Optimization provides a powerful toolbox for solving data analysis and learning problem. Today millions of rogue websites advance a wide variety of scams including marketing counterfeit goods,perpetrating financial fraud(e.g.,'phishing') and propgating malware. All these activities have in common the use of the URL(Universe Resource Locator) as an agent to bring internet users into their inuence. It is di_cult for user to figure out the associated risk with the click of an unfamiliar URL, as this can be a difficult judgment to make. In this paper, we consider a large scale problem of classification, consider a problem of identifying/labeling a URL on the basis,whether it is malicious or nonmalicious. We consider the method of SVM(Support Vector Machine) in classifying the labels(malign/benign). The mathematical model is framed as:

Given a training set $\{(x_i, y_i)\}_{i=1}^{n}$ *of training examples where* $y_i \in \{\pm 1\}$, *the hyperplane parametrized by normal vector* w *that balances the goal of seperating the data and maximizing margin can be found by solving the following optimization problem:*

$$\min_{w} \frac{\lambda}{2}\|w\|^2 + \frac{1}{n}\sum_{i=1}^{n} max(0, 1 - y(w.x)),$$

where $\lambda \geq 0$ *is called the regularization parameter.*

We can think of the problem as requiring the minimization of the empirical loss, plus a regularization term that limits the complexity of the solution[Shalev-Shwartz et al., 2007]. In addition,small-literature survey has been done on some approaches, such as (Gradient Descent, Stochastic Gradient Descent,Second-order gradient methods, sub-gradient descent and projection) algorithms to optimize the above function.

Portfolio Optimization Problem involving Big Data Analytics

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Within the changing business scenarios, IT applications and assets are getting more and more pivotal for the maximized business performance of an organization. This results in the production of various types of data being produced every day. Huge amount of structured and unstructured data are produced in financial markets. The growing importance financial market has pushed the organizations to lookout for ways to optimize their portfolio by processing these data. This could help and investor to make an informed investment decision. Based on the previous works, there were some gaps identified. None of the previous frameworks could handle the both structured data and unstructured data for portfolio optimization. In this paper, a framework has been reviewed to incorporate both structured and unstructured data for portfolio optimization. This framework considers both stock price data and current affairs data (i.e. news articles, market sentiments) to help investors to make an informed investment decision. All listed firms at a particular stock exchange are considered as the initial input to the framework and the output would be a set of stocks that would maximize the return and minimize the risk. Portfolio optimization consists of three processes: *Asset selection*, *Asset weighting and Asset management*. This framework proposes to achieve the first two processes using a 5-stage methodology. The stages include shortlisting stocks using Data Envelopment Analysis (DEA), incorporation of the qualitative factors using text mining, stock clustering, stock ranking and optimizing the portfolio using optimization heuristics. DEA is used to narrow the sample space of firms by identifying the efficient firms. Productive efficiency of decision making units (or DMUs) can be measured with this. Decision making units (DMU) represents the attempts to formalize marketing decision-making in complex environments. The key factors influencing the DMU's activities include: Buy class (e.g. straight rebuy, new task or modified rebuy), Product type (e.g. materials, components, plant and equipment and MRO (maintenance, repair and operation) and Importance of the purchase. DEA calculates the efficiency score of a DMU based on the given set of inputs and outputs.

We will review CCR DEA Model, where we assume that there are n DMUs to be evaluated. Each DMU consumes varying amounts of m different inputs to produce s different outputs. Specifically, DMU_j consumes amount x_{ij} of input i and produces amount y_{rj} of output r. We assume that $x_{ij} \geq 0$ and $y_{rj} \geq 0$ and further assume that each DMU has at least one positive input and one positive output value. We now turn to the "ratio-form" of DEA. In this form, as introduced by Charnes, Cooper, and Rhodes, the ratio of outputs to inputs is used to measure the relative efficiency of the $DMU_j = DMU_o$ to be evaluated relative to the ratios of all of the $j = 1$, 2, …, n DMU_j . We can interpret the CCR construction as the reduction of the multiple-output /multiple-input situation (for each DMU) to that of a single 'virtual' output and 'virtual' input. For a particular DMU the ratio of this single virtual output to single virtual input provides a measure of efficiency that is a function of the multipliers. In mathematical programming parlance, this ratio, which is to be maximized, forms the objective function for the particular DMU being evaluated, so that symbolically

$$max\ h_o\,(u,\,v) = \sum_r u_r\, y_{ro} \,/\, \sum_i v_i\, x_{io} \qquad (1.1)$$

where the variables are the u_r 's and the v_i's and the y_{ro}'s and x_{io}'s are the observed output and input values, respectively, of DMU_o , the DMU to be evaluated. Of course, without further additional constraints (developed below) (1) is unbounded.

A set of normalizing constraints (one for each DMU) reflects the condition that the virtual output to virtual input ratio of every DMU, including $DMU_j = DMU_o$, must be less than or equal to unity. The mathematical programming problem may thus be stated as

$$max\ h_o\,(u,\,v) = \sum_r u_r\, y_{ro} \,/\, \sum_i v_i\, x_{io} \qquad (1.2)$$

subject to

$$\sum_r u_r\, y_{ro} \,/\, \sum_i v_i\, x_{io} \leq 1 \text{ for } j=1\ldots.n,$$

$$u_r,\ vi \geq 0 \text{ for all } i \text{ and } r.$$

After short listing the stocks, in order to validate the firms as potential candidates for portfolio optimization, the latest information about the company is retrieved and processed from online news articles and tweets using text mining to the sentiments about the company in current context. The validated efficient firms are clustered into different groups to aid the diversification of portfolio. This is further followed by ranking of the stocks within each cluster and followed by asset weighting using optimization algorithms. Ranked stocks should be optimized to maximize returns and to minimize risk. Various optimization heuristics like Particle Swarm Optimization (PSO) or Ant Colony Optimization (ACO) can be used. This framework would help the investors to choose the proportions of various assets to be held in a portfolio, in such a way as to the portfolio better than any other according to some criterion.

Keywords: Portfolio Optimization, Big Data, Hadoop, Data Envelopment Analysis (DEA)

Pricing Decision Optimisation Using Data for Online Retailers

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This work serves to show how an online retailer can use its wealth of data to optimize pricing decisions on a daily basis. One of retailers' biggest challenges is pricing and predicting demand for products that it has never sold before, which account for a large part of sales and revenue. Having done a literature review, we present a two-fold approach, wherein we first try and devise a demand prediction model using regression trees and other non-parametric structural models, along with the dependence of a product's demand on the price of competing products. We then try and develop an efficient algorithm using Linear and Integer Programming Models for multi-product price optimization. Together, these can potentially can be created and implemented into a real time pricing decision support tool.

On Optimizing Resource Consumption and Crowd-based pick-up and delivery for a distribution network,

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Optimal utilization of resources is a key competitive advantage for logistics providers. Excess capacities lower profitability (which is critical for low-margin forwarding services), while capacity shortages impact service quality and put customer satisfaction at risk. In this presentation we discuss how Logistics providers perform resource planning, both at strategic and operational levels.

On Solving a Multiobjective Fixed Charge Problem with Imprecise Fractional Objectives

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The fixed charge problem is a special type of nonlinear programming problem which forms the basis of many industry problems wherein a charge is associated with performing an activity. In real world situations, the information provided by the decision maker regarding the coefficients of the objective functions may not be of a precise nature. This paper aims to describe a solution algorithm for solving such a fixed charge problem having multiple fractional objective functions which are all of a fuzzy nature. The enumerative technique developed not only finds the set of efficient solutions but also a corresponding fuzzy solution, enabling the decision maker to operate in the range obtained. A numerical example is presented to illustrate the proposed method.

Keywords: Fixed Charge Problem, Multiobjective Programming, Fractional Programming, Fuzzy Objective Function

Open Neighborhood Locating-Dominating Set In Graphs: Complexity And Algorithms

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A set $D \subseteq V$ of a graph $G = (V, E)$ is called an open neighborhood locating-dominating set (OLD-set) if (i) $N_G(v) \cap D \neq \emptyset$ for all $v \in V$, and (ii) $N_G(u) \cap D \neq N_G(v) \cap D$ for every pair of distinct vertices $u, v \in V(G)$. Given a graph $G = (V, E)$, the Min OLD-set problem is to find an OLD-set of minimum cardinality. The cardinality of a minimum OLD-set of G is called the open neighborhood location-domination number of graph G, and is denoted by $\gamma_{old}(G)$. Given a graph G and a positive integer k, the Decide OLD-set problem is to decide whether G has an OLD-set of cardinality at most k. The Decide OLD-set problem is known to be NP-complete for general graphs. In this paper, we strengthen this NP-complete result by showing that the Decide OLD-set problem remains NP-complete for perfect elimination bipartite graphs, a subclass of bipartite graphs. Then, we show that the Min OLD-set problem can be solved in polynomial time in chain graphs, a subclass of perfect elimination bipartite graphs. We show that for a graph G, $\gamma_{old}(G) \geq \frac{2n}{\Delta(G)+2}$, where n denotes the number of vertices in graph G, and $\Delta(G)$ denotes the maximum degree of G. As a consequence we obtain a $\frac{\Delta(G)+2}{2}$-approximation algorithm for the Min OLD-set problem. Finally, we prove that the Min OLD-set problem is APX-complete for bounded degree graphs.

Keywords: Domination, Open neighborhood location domination, chain graph, perfect elimination bipartite graph, NP-completeness, APX-completeness

An AHP-PROMETHEE II Method for 2-tuple Linguistic Multi-criteria Group Decision Making

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The paper aims to propose a hybrid AHP-PROMETHEE II method for multicriteria group decision

making problems where the criteria values take the 2-tuple linguistic information. The method can be seen as an extension of the traditional analytical hierarchy process (AHP) to extract the exact weights of criteria which are then supplied to the PROMETHEE II method to rank various alternatives.

The PROMETHEE II method is also extended to work with 2-tuple linguistic information. A graphical representation is provided to demonstrate the aggregated evaluation of optimal alternative. A prototype example is presented to illustrate the practicability of the proposed method.

Keywords: Multicriteria decision making, PROMETHEE-II, Analytical hierarchy process, 2-tuple Linguistic variables.

On Computation of Minimal Forecast Horizon for a Stochastic Dynamic Lot-Size Problem

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This paper presents a computational procedure for identifying minimal forecast horizon for dynamic lot-sizing problem with stochastic demand. All future demands assumed to be discrete random variables with integers values. An integer programming approach is suggested based on a stochastic-programming formulation of the problem. On an extensive test bed, the computational tractability of this approach is demonstrated.

Multiobjective Vendor’s Decision Problem on Contingent Demand Satisfaction

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The planning of production and other operations in a firm is done based on demand forecasts. But production decisions get affected by actual realization of the demand. These decisions are taken keeping into consideration production capacities. Sometimes sudden and unforeseen demands from customers arise. This happens generally in the case when customers of a vendor firm get some new contracts or there is a short term but high fluctuation in their demands. When this sudden or contingent demand arises on customers' side, they create a demand for corresponding spare parts for their vendors. The vendor's decision to satisfy this demand fully or partially is restricted by its own available resources, as processes like production capacity expansion can't be carried out in such a short response time.

In this paper, we consider the problem of a vendor firm which manufactures products that are used by its customer firms as spare parts to manufacture some particular products. We will try to address the decision making on the extent of satisfaction of contingent demand of each customer. While deciding on the extent of demands of its customers, the vendor firm doesn't always concentrate just on the objective of profit, but also on the future business and relation with its customers.

Fuzzy BCC Data Envelopment Analysis Model: A Credibility Approach

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This paper presents Fuzzy Data Envelopment Analysis (FDEA) model under variable returns to scale using credibility approach. FDEA is used to measure the relative efficiencies of a set of

decision making units (DMUs) under fuzzy environment in which the input and output data can be represented as linguistic variable characterized by fuzzy variables. The FDEA is solved using the concept of chance-constrained programming and credibility approach. In a special case, when fuzzy inputs and fuzzy outputs are independent trapezoidal or triangular fuzzy variables, the model can be transformed into crisp linear programming. Finally, numerical illustration is presented to illustrate the FDEA model to measure the efficiency of DMUs with fuzzy data as well as the effectiveness of the presented method. By extending to fuzzy environment, the DEA approach is made more powerful for application.
Keywords: Data Envelopment Analysis, Efficiency, Fuzzy LPP, Credibility Theory, Chance-constrained programming (CCP).

Optimized Resource Utilization Techniques for Cloud Computing Environment

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Digital revolution led to shift from the industrial age to Information and Communication Technology (*ICT*) age. Cloud computing is one of the important innovation of the digital revolution. It is a new

paradigm for hosting and delivering services using standard protocols of the Internet. It refers to the service-oriented architecture that offers ubiquitous and convenient computing, greater flexibility, on-demand services, minimized total cost of ownership, reduced burden of Information Technology (*IT*) overhead for the end-user, and many other things which offers computing any time and anywhere. Cloud computing allows access to the large computing power in a fully virtualized mode by aggregating configurable pool of resources that are provisioned and released rapidly in dynamic fashion. It also provides the capability to utilize scalable, distributed computing environments within the confines of Internet with minimal management effort while keeping a single coherent system view.

Last few decades are witness of steeper growth in demand of high computational power. Such trend in demand of computing power caused the establishment of large-scale data centers that are situated at geographically apart locations. However, these large-scale data centers consume an enormous amount of electrical energy which results into very high operating cost and large amount of carbon dioxide (CO_2) emission due to heavy resource underutilization. We present optimized resource utilization techniques to overcome the problems such as resource underutilization, high energy consumption, and large CO_2 emissions. Further, we present a comparative study among the presented and existing techniques showing that proposed methodologies outperforms over the existing one in terms of energy consumption and the number of VM migrations.

Mathematical Theory of Pareto-Nash-Stackelberg Game-Control Models

Valeriu Ungureanu

Abstract. We expose the current evolution of Pareto-Nash-Stackelber game-control theory by referring various real dynamic processes with particular features and parameters. Among different concrete examples, we present analyses and investigation for the problem of linear discrete-time Pareto-Nash-Stackeberg control of decision processes that evolve as Pareto-Nash-Stackeberg games with constraints (a mixture of hierarchical and simultaneous games) under the influence of echoes and phenomena. We present mathematical models, solution notions, conditions for Pareto-Nash-Stackeberg control existence and method for Pareto-Nash-Stackeberg control computing. Wolfram Mathematica applications, demonstrations and benchmarks are exposed, too.

1 Introduction

Interactive decisions situations, which involve both sequential decisions and simultaneous decisions made by independent and interdependent players with one or more objectives, can be modelled by means of strategic games (Stackelberg game [4, 15], Nash game [5, 8, 10–14], Pareto-Nash game [2, 3, 6, 9], Pareto-Nash-Stackelberg game). At every stage (level) of the Nash-Stackelberg game a Nash game is played. The stage profiles (joint decisions) are executed sequentially throughout the hierarchy as a Stackelberg game. At every stage of the multiobjective Pareto-Nash-Stackelberg game a multiobjective Pareto-Nash game is played. Stage profiles are executed sequentially throughout the hierarchy. Via notion of best response mapping graph we define unsafe and safe Stackelberg equilibria for Stackelberg games, pseudo and multi-stage Nash-Stackelberg equilibria for Nash-Stackelberg games, and Pareto-Nash-Stackelberg equilibria for multiobjective Pareto-Nash-Stackelberg games.

When the players in such games have to control a system, we obtain a new kind of problems, which are as game problems, as control problems. The second part of this work deals with game-control problems. A direct-straightforward method for solving linear discrete-time optimal control problem is applied to solve control problem of a linear discrete-time system as a mixture of multi-criteria Stackelberg and Nash games. For simplicity, the exposure starts with the simplest case of linear discrete-time optimal control problem and, by sequential considering of more general cases, investigation finalizes with the highlighted Pareto-Nash-Stackelberg and set valued control problems. Different principles of solving are compared and their equivalence is proved.

The third part of the work investigates linear discrete-time Pareto-Nash-Stackelberg control problems with echoes and retroactive future.

2 Strategic Games

Consider the noncooperative strategic game

$$\Gamma = \langle N, \{X_p\}_{p \in N}, \{f_p(x)\}_{p \in N} \rangle,$$

where

- $N = \{1, 2, ..., n\}$ is a set of players,
- $X_p \subseteq R^{k_p}$ is a set of strategies of player $p \in N$,
- $k_p < +\infty, p \in N$,
- and $f_p(x)$ is a p^{th} player cost function defined on the Cartesian product $X = \underset{p \in N}{\times} X_p$ — profiles set of the game. Without loss of generality **suppose that all players minimize the values of their cost functions.**

Suppose that the players make their moves hierarchically:

first player chooses his strategy $x_1 \in X_1$ and communicates it to the second player,

the second player chooses his strategy $x_2 \in X_2$ after observing the moves x_1 of the first player and communicates x_1, x_2 to the third player,

and so on

at the last the n^{th} player selects his strategy $x_n \in X_n$ after observing the moves $x_1, ..., x_{n-1}$ of the preceding players.

On the resulting profile $x = (x_1, ..., x_n)$ every player computes the value of his cost function.

When player $p \in N$ moves, players $1, 2, ..., p-1$ are leaders or predecessors of player p and players $p+1, ..., n$ are followers or successors of the player p. Players have all the information about the predecessors choices and doesn't have information about the choices of the successors, but the p^{th} player ($p < n$) has all the information about the all strategy sets and the cost functions of the players $p, p+1, ..., n$.

By backward induction, every player $n, n-1, ..., 2$ determines his best move mapping and the first player determines the set of his best moves:

$$B_n(x_1, ..., x_{n-1}) = \operatorname*{Arg\,min}_{y_n \in X_n} f_n(x_1, ..., x_{n-1}, y_n),$$

$$B_{n-1}(x_1, ..., x_{n-2}) = \operatorname*{Arg\,min}_{y_{n-1},\, y_n:\, (x_1, ..., x_{n-2},\, y_{n-1},\, y_n) \in Gr_n} f_{n-1}(x_1, ..., x_{n-2},\, y_{n-1}, y_n),$$

$$\dots$$

$$B_2(x_1) = \operatorname*{Arg\,min}_{y_2, ..., y_n:\, (x_1, y_2, ..., y_n) \in Gr_3} f_2(x_1, y_2, ..., y_n),$$

$$\hat{X} = \operatorname*{Arg\,min}_{(y_1, ..., y_n) \in Gr_2} f_1(y_1, ..., y_n)$$

where

$$Gr_n = \{x \in X : x_1 \in X_1, ..., x_{n-1} \in X_{n-1}, x_n \in B_n(x_1, ..., x_{n-1})\},$$
$$Gr_{n-1} = \{x \in Gr_n : x_1 \in X_1, ..., x_{n-2} \in X_{n-2}, (x_{n-1}, x_n) \in B_{n-1}(x_1, ..., x_{n-2})\},$$
$$\dots$$
$$Gr_2 = \{x \in Gr_3 : x_1 \in X_1, (x_2, ..., x_n) \in B_2(x_1)\}.$$

Evidently, $Gr_2 \subseteq Gr_3 \subseteq \cdots \subseteq Gr_n$, and forms a family of nested sets.

Definition. *Any profile $\hat{x} \in \hat{X}$ of the game is called unsafe (optimistic, strong) Stackelberg equilibrium.*

This definition of the unsafe Stackelberg equilibrium is equivalent with respective [4] definition. For $n = 2$ the unsafe Stackelberg equilibrium notion and original Stackelberg equilibrium [15] notion are equivalent.

3 Unsafe Stackelberg Equilibrium. Existence and Properties

1 Theorem. *For every finite hierarchical game the set $\hat{X}$ of unsafe Stackelberg equilibria is non empty.*

2 Theorem. *If every strategy set $X_p \subset R^{k_p}, p = \overline{1,n}$ is compact and every cost function $f_p(x_1, ..., x_p, ..., x_n), p = \overline{1,n}$ is continuous by $(x_p, ..., x_n)$ on $X_p \times \cdots \times X_n$ for every fixed $x_1 \in X_1, ..., x_{p-1} \in X_{p-1}$, then the unsafe Stackelberg equilibria set $\hat{X}$ is non empty.*

3 Theorem. *If every strategy set $X_p \subseteq R^{k_p}, p = \overline{1,n}$ is convex and every cost function $f_p(x_1, ..., x_p, ..., x_n), p = \overline{1,n}$ is strict convex by $(x_p, ..., x_n)$ on $X_p \times \cdots \times X_n$ for every fixed $x_1 \in X_1, ..., x_{p-1} \in X_{p-1}$, then the game has a unique unsafe Stackelberg equilibrium with the "guaranteed" realization property.*

4 Safe Stackelberg Equilibrium

The safe Stackelberg equilibrium notion is equivalent with respective notion in [4].

By backward induction, every player $2, ..., n$ determines his best move mapping and the first player determines the set of his best moves:

$$B_n(x_1, ..., x_{n-1}) = \operatorname*{Arg\,min}_{y_n \in X_n} f_n(x_1, ..., x_{n-1}, y_n),$$

$$\tilde{B}_{n-1}(x_1, ..., x_{n-2}) = \operatorname*{Arg\,min\,max}_{\substack{y_{n-1} \quad y_n \\ (x_1,...,x_{n-2},y_{n-1},y_n) \in Gr_n}} f_{n-1}(x_1, ..., x_{n-2}, y_{n-1}, y_n),$$

$$\tilde{B}_{n-2}(x_1, ..., x_{n-3}) = \operatorname*{Arg\,min\,\max}_{\substack{y_{n-2} \; y_{n-1}, y_n \\ (x_1,...,x_{n-3},y_{n-2},...,y_n) \in \tilde{G}r_{n-1}}} f_{n-1}(x_1, ..., x_{n-3}, y_{n-2}, ..., y_n),$$

...

$$\tilde{B}_2(x_1) = \operatorname*{Arg\,min\,\max}_{\substack{y_2 \quad y_3,...,y_n \\ (x_1,y_2,...,y_n) \in \tilde{G}r_3}} f_2(x_1, y_2, ..., y_n),$$

$$\tilde{X} = \operatorname*{Arg\,min\,\max}_{\substack{y_1 \quad y_2,...,y_n \\ (y_1,...,y_n) \in \tilde{G}r_2}} f_1(y_1, ..., y_n)$$

where

$$Gr_n = \{x \in X : x_1 \in X_1, ..., x_{n-1} \in X_{n-1}, x_n \in B_n(x_1, ..., x_{n-1})\},$$
$$\tilde{G}r_{n-1} = \left\{x \in Gr_n : x_1 \in X_1, ..., x_{n-2} \in X_{n-2}, (x_{n-1}, x_n) \in \tilde{B}_{n-1}(x_1, ..., x_{n-2})\right\},$$
...
$$\tilde{G}r_2 = \left\{x \in Gr_3 : x_1 \in X_1, (x_2, ..., x_n) \in \tilde{B}_2(x_1)\right\}.$$

Evidently, $\tilde{G}r_2 \subseteq \tilde{G}r_3 \subseteq \cdots \subseteq \tilde{G}r_{n-1} \subseteq Gr_n$, too.

Definition. *The profile $\tilde{x} \in \tilde{X}$ of the game is named safe (pessimistic, weak) Stackelberg equilibrium.*

In general, the unsafe Stackelberg equilibria set is not equivalent to a safe Stackelberg equilibria set, *i.e.* $\hat{X} \neq \tilde{X}$.

Theorems 1–3 analogs for safe Stackelberg equilibrium may be formulated and proved. In the theorem 3 conditions the unsafe and safe Stackelberg equilibria are identical.

5 Pseudo-Equilibrium. Nash-Stackelberg Equilibrium

Consider the strategic game

$$\Gamma = \langle N, \{X_p^l\}_{l \in S, p \in N_l}, \{f_p^l(x)\}_{l \in S, p \in N_l}\rangle,$$

where

- $S = \{1, 2, ..., s\}$ is a set of stages,

- $N_l = \{1, 2, ..., n_l\}$ is a set of players at stage (level) $l \in S$,
- $X_p^l \subseteq R^{k_p^l}$ is a set of strategies of player $p \in N_l$ at stage $l \in S$,
- $s < +\infty, \quad n_l < +\infty, l \in S$,
- and $f_p^l(x)$ is a l^{th} stage p^{th} player cost function defined on the Cartesian product $X = \underset{p \in N_l, l \in S}{\times} X_p^l$.

Elements $x = (x_1^1, x_2^1, ..., x_{n_1}^1, x_1^2, x_2^2, ..., x_{n2}^2, ..., x_1^s, x_2^s, ..., x_{n_s}^s) \in X$ form profiles of the game.

Suppose that the players make their moves hierarchically:

at the first stage players $1, 2, ..., n_1$ selects their strategies $x_1^1 \in X_1^1$, $x_2^1 \in X_2^1$, ..., $x_{n_1}^1 \in X_{n_1}^1$ simultaneously and communicate it to the second stage players $1, 2, ..., n_2$,

the second stage players $1, 2, ..., n_2$ select simultaneously their strategies $x_1^2 \in X_1^2$, $x_2^2 \in X_2^2, ..., x_{n_2}^2 \in X_{n_2}^2$ after observing the moves $(x_1^1, x_2^1, ..., x_{n_1}^1)$ of the first stage players and communicate two stages result to the third stage players,

and so on

the s^{th} stage players $1, 2, ..., n_s$ select simultaneously their strategies $x_1^s \in X_1^s, x_2^s \in X_2^s, ..., x_{n_s}^s \in X_{n_s}^s$ at the last after observing the moves $(x_1^1, x_2^1, ..., x_{n_1}^1, x_1^2, x_2^2, ..., x_{n_2}^2, ..., x_1^{s-1}, x_2^{s-1}, ..., x_{n_{s-1}}^{s-1})$ of the precedent stages players.

On the resulting profile $x = (x_1^1, x_2^1, ..., x_{n_1}^1, x_1^2, x_2^2, ..., x_{n_2}^2, ..., x_1^s, x_2^s, ..., x_{n_s}^s)$ every player computes the value of his cost function.

Suppose that the l^{th} stage p^{th} player has all information about all strategy sets and the cost functions of the players of stages $l, l+1, ..., s$. Without loss of generality **suppose that all players minimize the values of their cost functions.**

Definition. *The profile $\hat{x} \in X$ of the game is pseudo-equilibrium if for every $l \in S$ there exist $y^{l+1} \in X^{l+1}, ..., y^n \in X^n$ such that*

$$f_p^l(\hat{x}^1, ..., \hat{x}^{l-1}, x_p^l \| \hat{x}_{-p}^l, y^{l+1}, ..., y^n) \geq f_p^l(\hat{x}^1, ..., \hat{x}^l, y^{l+1}, ..., y^n), \forall x_p^l \in X_p^l, \ \forall p \in N_l,$$

where $\hat{x}_{-p}^l = (\hat{x}_1^l, ..., \hat{x}_{p-1}^l, \hat{x}_{p+1}^l, ..., \hat{x}_{n_l}^l)$.

Accordingly the definition, players $1, 2, ..., n_l$, $l = 1, 2, ..., s-1, s$ select their pseudo-equilibrium strategies:

$$
\begin{array}{l}
B_p^1(\chi_{-p}^1) = \underset{x_p^1 \in X_p^1}{\operatorname{Arg\,min}}\ f_p^1\left(x_p^1 \| \chi_{-p}^1\right), p \in N_1, \\
(\hat{x}^1, x^2, ..., x^s) \in PE^1 = \bigcap\limits_{p \in N_1} Gr_p^1, \\
B_p^2(\hat{x}^1, \chi_{-p}^2) = \underset{x_p^2 \in X_p^2}{\operatorname{Arg\,min}}\ f_p^2\left(\hat{x}^1, x_p^2 \| \chi_{-p}^2\right), p \in N_2, \\
(\hat{x}^1, \hat{x}^2, x^3, ..., x^s) \in PE^2 = \bigcap\limits_{p \in N_2} Gr_p^2, \\
\dots \\
B_p^s(\hat{x}^1, \hat{x}^2, ..., \hat{x}^{s-1}, \chi_{-p}^s) = \underset{x_p^s \in X_p^s}{\operatorname{Arg\,min}}\ f_p^s\left(\hat{x}^1, ..., \hat{x}^{s-1}, x_p^s \| \chi_{-p}^s\right), p \in N_s, \\
(\hat{x}^1, \hat{x}^2, ..., \hat{x}^s) \in PE^s = \bigcap\limits_{p \in N_s} Gr_p^s,
\end{array}
$$

where

$$
\begin{array}{l}
Gr_p^1 = \left\{(x^1, ..., x^s) : x_p^1 \in B_p^1(\chi_{-p}^1)\right\}, p \in N_1, \\
Gr_p^2 = \left\{(\hat{x}^1, x^2, ..., x^s) : x_p^2 \in B_p^2(\hat{x}^1, \chi_{-p}^2)\right\}, p \in N_2, \\
\dots \\
Gr_p^s = \left\{(\hat{x}^1, ..., \hat{x}^{s-1}, x^s) : x_p^s \in B_p^s(\hat{x}^1, ..., \hat{x}^{s-1}, \chi_{-p}^s)\right\}, p \in N_s, \\
\chi_{-p}^l = (x_{-p}^l, x^{l+1}, ..., x^s) \in X_{-p}^l \times X^{l+1} \times \cdots \times X^s, \\
X_{-p}^l = X_1^l \times ... \times X_{p-1}^l \times X_{p+1}^l \times ... \times X_{n_l}^l.
\end{array}
$$

Surely, **the set of all pseudo-equilibria is $PE = PE^s$.**

The pseudo-equilibrium definition does not used the information that at the following stage the stage players will choose the strategy accordingly the pseudo-equilibrium statement. As the result, the profiles do not safe the required statement at all stages.

For excluding this inconvenient, it's reasonable to choose strategies by backward induction procedure and thus we obtain a new equilibrium notion.

By stage backward induction, players $1, 2, ..., n_l$, $l = s, s-1, ..., 2, 1$ select their equilibrium strategies:

$$B_p^s(x^1, ..., x^{s-1}, x_{-p}^s) = \operatorname*{Arg\,min}_{y_p^s \in X_p^s} f_p^s\left(x^1, ..., x^{s-1},\ y_p^s \| x_{-p}^s\right),\ p \in N_s,$$

$$NSE^s = \bigcap_{p \in N_s} Gr_p^s,$$

$$B_p^{s-1}(x^1, ..., x^{s-2}, x_{-p}^{s-1}) = \\ = \operatorname*{Arg\,min}_{\substack{y_p^{s-1},\, y^s: \\ (x^1,...,x^{s-2},\, y_p^{s-1} \| x_{-p}^{s-1},\, y^s) \in NSE^s}} f_p^{s-1}\left(x^1, ..., x^{s-2}, y_p^{s-1} \| x_{-p}^{s-1}, y^s\right),\ p \in N_{s-1},$$

$$NSE^{s-1} = \bigcap_{p \in N_{s-1}} Gr_p^{s-1},$$

$$B_p^{s-2}(x^1, ..., x^{s-2}, x_{-p}^{s-3}) = \\ = \operatorname*{Arg\,min}_{\substack{y_p^{s-2},\, y^{s-1},\, y^s: \\ (x^1,...,x^{s-3},\, y_p^{s-2} \| x_{-p}^{s-2},\, y^{s-1},\, y^s) \in NSE^{s-1}}} f_p^{s-2}\left(x^1, ..., x^{s-3}, y_p^{s-2} \| x_{-p}^{s-2}, y^{s-1}, y^s\right),$$

$$p \in N_{s-2},$$

$$NSE^{s-2} = \bigcap_{p \in N_{s-2}} Gr_p^{s-2},$$

$$\dots$$

$$B_p^1(x_{-p}^1) = \operatorname*{Arg\,min}_{y_p^1, y^2, ..., y^s : (y_p^1 \| x_{-p}^1,\, y^2, ..., y^s) \in NSE^2} f_p^1\left(y_p^1 \| x_{-p}^1,\ y^2, ..., y^s\right),\ p \in N_1,$$

$$NSE^1 = \bigcap_{p \in N_1} Gr_p^1,$$

where

$$Gr_p^s = \left\{ x \in X :\ \begin{array}{c} x^l \in X^l,\ l = \overline{1, s-1}, \\ x_{-p}^s \in X_{-p}^s, \\ x_p^s \in B_p^s(x^1, ..., x^{s-1}, x_{-p}^s) \end{array} \right\},\ p \in N_s,$$

$$Gr_p^{s-1} = \left\{ x \in NSE^s :\ \begin{array}{c} x^l \in X^l, l = \overline{1, s-2}, \\ x_{-p}^{s-1} \in X_{-p}^{s-1}, \\ x_p^{s-1} \in B_p^{s-1}(x^1, ..., x^{s-2}, x_{-p}^{s-1}) \end{array} \right\},\ p \in N_{s-1},$$

$$\dots$$

$$Gr_p^1 = \left\{ x \in NSE^2 :\ \begin{array}{c} x_{-p}^1 \in X_{-p}^1, \\ x_p^1 \in B_p^1(x_{-p}^1) \end{array} \right\},\ p \in N_1.$$

Of course, $NSE^1 \subseteq NSE^2 \subseteq \cdots \subseteq NSE^s$.

Definition. *Every element of the NSE^1 is called Nash-Stackelberg equilibrium.*

The set of all Nash-Stackeberg equilibria NSE^1 is denoted by NSE also.

If $s = 1$ and $n_1 > 1$, then every Nash-Stackelberg equilibrium is the Nash equilibrium. If $s > 1$ and $n_1 = n_2 = ... = n_s = 1$, then every equilibrium is an unsafe Stackelberg equilibrium. Thus, the Nash-Stackelberg equilibrium notion generalizes the both Stackelberg and Nash equilibria notions.

By stage backward induction, players $1, 2, ..., n_l,\ l = s, s-1, ..., 2, 1$ select their equilibrium strategies:

$$
\begin{aligned}
&B_p^s(x^1, ..., x^{s-1}, x_{-p}^s) = \operatorname*{Arg\,min}_{y_p^s \in X_p^s} f_p^s\left(x^1, ..., x^{s-1},\ y_p^s \| x_{-p}^s\right),\ p \in N_s,\\
&SNSE^s = NSE^s = \bigcap_{p \in N_s} Gr_p^s,\\
&\tilde{B}_p^{s-1}(x^1, ..., x^{s-2}, x_{-p}^{s-1}) =\\
&= \operatorname*{Arg\,min\,max}_{\substack{y_p^{s-1} \quad y^s \\ (x^1, ..., x^{s-2}, y_p^{s-1} \| x_{-p}^{s-1}, y^s) \in SNSE^s}} f_p^{s-1}\left(x^1, ..., x^{s-2}, y_p^{s-1} \| x_{-p}^{s-1}, y^s\right), p \in N_{s-1},\\
&SNSE^{s-1} = \bigcap_{p \in N_{s-1}} \tilde{G}r_p^{s-1},\\
&\tilde{B}_p^{s-2}(x^1, ..., x^{s-2}, x_{-p}^{s-3}) =\\
&= \operatorname*{Arg\,min\ \ max}_{\substack{y_p^{s-2} \quad y^{s-1}, y^s \\ (x^1, ..., x^{s-3}, y_p^{s-2} \| x_{-p}^{s-2}, y^{s-1}, y^s) \in NSE^{s-1}}} f_p^{s-2}\left(x^1, ..., x^{s-3}, y_p^{s-2} \| x_{-p}^{s-2}, y^{s-1}, y^s\right),\\
&p \in N_{s-2},\ SNSE^{s-2} = \bigcap_{p \in N_{s-2}} \tilde{G}r_p^{s-2},\\
&\dots\\
&\tilde{B}_p^1(x_{-p}^1) = \operatorname*{Arg\,min\ \ max}_{\substack{y_p^1 \quad y^2, ..., y^s \\ (y_p^1 \| x_{-p}^1, y^2, ..., y^s) \in NSE^2}} f_p^1\left(y_p^1 \| x_{-p}^1,\ y^2, ..., y^s\right), p \in N_1,\\
&SNSE^1 = \bigcap_{p \in N_1} \tilde{G}r_p^1,
\end{aligned}
$$

where

$$
\begin{aligned}
&Gr_p^s = \left\{ x \in X : \begin{array}{l} x^l \in X^l,\ l = \overline{1, s-1}, \\ x_{-p}^s \in X_{-p}^s, \\ x_p^s \in B_p^s(x^1, ..., x^{s-1}, x_{-p}^s) \end{array} \right\},\ p \in N_s,\\
&\tilde{G}r_p^{s-1} = \left\{ x \in NSE^s : \begin{array}{l} x^l \in X^l, l = \overline{1, s-2}, \\ x_{-p}^{s-1} \in X_{-p}^{s-1}, \\ x_p^{s-1} \in \tilde{B}_p^{s-1}(x^1, ..., x^{s-2}, x_{-p}^{s-1}) \end{array} \right\}, p \in N_{s-1},\\
&\dots\\
&\tilde{G}r_p^1 = \left\{ x \in NSE^2 : \begin{array}{l} x_{-p}^1 \in X_{-p}^1, \\ x_p^1 \in \tilde{B}_p^1(x_{-p}^1) \end{array} \right\}, p \in N_1.
\end{aligned}
$$

Surely, $SNSE^1 \subseteq SNSE^2 \subseteq \cdots \subseteq SNSE^s$.

Definition. *Elements of $SNSE^1$ are called safe Nash-Stackelberg equilibria.*

The set of all safe Nash-Stackeberg equilibria $SNSE^1$ is denoted by $SNSE$ also.

In the same manner as for Nash-Stackelberg games the equilibrium principles can be introduced. An essential difference in corresponding definitions is the strong requirement that every minimization or maximization operator must be interpreted as Pareto maximization or minimization operator. Evidently, the Pareto optimal response mapping and the graph of the Pareto optimal response mapping are considered for every player. An intersection of the graphs of Pareto optimal response mappings is considered in every definition as the stage profile.

6 Linear discrete-time optimal control problem

Consider the following problem [1]:

$$
\begin{aligned}
f(x,u) = \sum_{t=1}^{T} (c^t x^t + b^t u^t) \quad &\to \max, \\
x^t = A^{t-1} x^{t-1} + B^t u^t, \quad & t = 1, ..., T, \\
D^t u^t \le d^t, \quad & t = 1, ..., T,
\end{aligned}
\tag{1}
$$

where x^0, x^t, $c^t \in R^n$, u^t, $b^t \in R^m$, $A^{t-1} \in R^{n\times n}$, $B^t \in R^{n\times m}$, $d^t \in R^k$, $D^t \in R^{k\times n}$, $c^t x^t = \langle c^t, x^t \rangle$, $b^t u^t = \langle b^t, u^t \rangle$, $t = 1, ..., T$, $u = (u^1, \dots, u^T)$.

4 Theorem. *Let (1) be solvable. The sequence $\bar u^1, \bar u^2, \dots, \bar u^T$ forms an optimal control if and only if $\bar u^t$ is the solution of linear programming problem*

$$
\begin{aligned}
&\left(c^t B^t + c^{t+1} A^t B^t + \dots + c^T A^{T-1} A^{T-2} \dots A^t B^t + b^t\right) u^t \to \max, \\
&D^t u^t \le d^t,
\end{aligned}
$$

for $t = 1, \dots, T$.

5 Theorem. *If $A^0 = A^1 = \dots = A^{T-1} = A$, $B^1 = B^2 = \dots = B^T = B$, and (1) is solvable, then the sequence $\bar u^1, \bar u^2, \dots, \bar u^T$ forms an optimal control if and only if $\bar u^t$ is the solution of linear programming problem*

$$
\begin{aligned}
&\left(c^t B + c^{t+1} AB + c^{t+2} (A)^2 B + \dots + c^T (A)^{T-t} B + b^t\right) u^t \to \max, \\
&D^t u^t \le d^t,
\end{aligned}
$$

for $t = 1, \dots, T$.

Theorem 4 establishes a principle for solving (1). By considering Hamiltonian functions

$$H_t(u^t) = \langle p^t B^t + b^t, u^t \rangle, t = T, \dots, 1,$$

where $p^t, t = T, \dots, 1$ are defined by recurrently, as it is conjectured in [1] and proved above by two ways, the maximum principle of Pontryagin [7] holds.

6 Theorem. *Let (1) be solvable. The sequence $\bar u^1, \bar u^2, \dots, \bar u^T$ forms an optimal control if and only if*

$$H_t(\bar u^t) = \max_{u^t : D^t u^t \le d^t} H_t(u^t), t = T, \dots, 1.$$

Theorems 4 and 6 are equivalent.

7 Linear discrete-time Stackelberg control problem

Let us modify the problem (1) by considering the control of Stackelberg type [15], that is Stackelberg game with T players [**?**, 4, 5, 15]. In such game, at each stage t $(t = 1, \dots, T)$ the player t selects his strategy and communicates his and all precedent selected strategies to the following $t+1$ player. After all stage strategy selections, all the players compute their gains on the resulting profile. Let us name such type of system control as Stackelberg control, and the corresponding problem – linear discrete-time Stackelberg control problem. The described decision process may be formalized as it follows:

$$
\begin{aligned}
&f_1(x,u) = \sum_{t=1}^{T} \left(c^{1t}x^t + b^{1t}u^t\right) \underset{u^1}{\longrightarrow} \max,\\
&f_2(x,u) = \sum_{t=1}^{T} \left(c^{2t}x^t + b^{2t}u^t\right) \underset{u^2}{\longrightarrow} \max,\\
&\dots\\
&f_T(x,u) = \sum_{t=1}^{T} \left(c^{Tt}x^t + b^{Tt}u^t\right) \underset{u^T}{\longrightarrow} \max,\\
&\\
&x^t = A^{t-1}x^{t-1} + B^t u^t, t = 1, ..., T,\\
&D^t u^t \le d^t, t = 1, ..., T,
\end{aligned}
\tag{2}
$$

where $x^0, x^t, c^{\pi t} \in R^n$, $u^t, b^{\pi t} \in R^m$, $A^{t-1} \in R^{n\times n}$, $B^t \in R^{n\times m}$, $d^t \in R^k$, $D^t \in R^{k\times n}$, $c^{\pi t}x^t = \left\langle c^{\pi t}, x^t \right\rangle$, $b^{\pi t}u^t = \left\langle b^{\pi t}, u^t \right\rangle$, $t, \pi = 1, ..., T$.

In fact, as we can find out, the strategy sets of the players are interconnected and the game is not a simple normal form game. A situation similar with that in optimization theory may be established – there are problems without constraints and with constraints. So, the strategy (normal form) game may be named strategy game without constraints. Game which contains commune constraints on strategies may be named strategy game with constraints.

7 Theorem. *Let (2) be solvable. The sequence $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^T$ forms a Stackelberg equilibrium control in (2) if and only if $\bar{u}^\pi$ is optimal solution of linear programming problem*

$$
\begin{aligned}
&f_\pi(u^\pi) = \big(c^{\pi\pi}B^\pi + c^{\pi\pi+1}A^\pi B^\pi + c^{\pi\pi+2}A^{\pi+1}A^\pi B^\pi + \dots +\\
&+ c^{\pi T}A^{T-1}A^{T-2}\dots A^\pi B^\pi + b^{\pi\pi}\big)\, u^\pi \underset{u^\pi}{\to} \max,\\
&D^\pi u^\pi \le d^\pi,
\end{aligned}
$$

for $\pi = 1, \dots, T$.

8 Theorem. *If $A^0 = A^1 = \dots = A^{T-1} = A$, $B^1 = B^2 = \dots = B^T = B$, and (2) is solvable, then the sequence $\bar{u}^1, \bar{u}^2, \dots, \bar{u}^T$ forms a Stackelberg equilibrium control if* *and only if $\bar{u}^{\pi}$ is the solution of linear programming problem*

$$\begin{array}{l} \left(c^{\pi\pi}B + c^{\pi\pi+1}AB + \cdots + c^{\pi T}(A)^{T-\pi}B + b^{\pi\pi}\right)u^{\pi} \underset{u^{\pi}}{\rightarrow} \max, \\ D^{\pi}u^{\pi} \leq d^{\pi}, \end{array}$$

for $\pi = 1, \ldots, T$.

Theorem 7 establishes a principle for solving (2). The maximum principle of Pontryagin may be applied for solving (2) too. Let us consider the following recurrent relations

$$\begin{array}{rcl} p^{\pi T} & = & c^{\pi T}, \\ p^{\pi t} & = & p^{\pi t+1}A^{t} + c^{\pi t}, \quad t = T-1, ..., 1, \end{array} \tag{3}$$

where $\pi = 1, \ldots, T$. Hamiltonian functions are defined as

$$H_{\pi t}\left(u^{t}\right) = \left\langle p^{\pi t}B^{t} + b^{\pi t}, u^{t}\right\rangle, t = T, \ldots, 1, \pi = 1, \ldots, T,$$

where $p^{\pi t}, t = T, \ldots, 1, \pi = 1, \ldots, T$, are defined by (3).

9 Theorem. *Let (2) be solvable. The sequence of controls $\bar{u}^{1}, \ldots, \bar{u}^{T}$ forms a Stackelberg equilibrium control if and only if*

$$H_{\pi\pi}\left(\bar{u}^{\pi}\right) = \max_{u^{\pi}: D^{\pi}u^{\pi} \leq d^{\pi}} H_{\pi\pi}\left(u^{\pi}\right),$$

for $\pi = 1, \ldots, T$.

8 Conclusions

The reasonable volume of an extended abstract doesn't permit to expose entirely the presented work, which includes mathematical models of Pareto-Nash-Stackelberg game-control types and principles for their solving.

The examined processes of decision making are very often phenomena of real life. Their mathematical moddeling as Pareto-Nash-Stackelberg games and control processes gives an powerful tool for investigation, analysis and solving hierarchical decision problems. Nevertheless, the problem of equilibrium principle choosing in real situations is a task for both a decision maker and a game theorist.

There are different types of control: optimal control, Stackelberg control, Pareto-Stackelberg control, Nash-Stackelberg control, Pareto-Nash-Stackelberg control, etc.

The direct-straightforward, dual and classical principles (Pontryagin and Bellman) may be applied for determining the desired control of dynamic processes. These principles are the bases for pseudo-polynomial methods, which are exposed as a consequence of theorems for linear discrete-time Pareto-Nash-Stackelberg control problems and to the problems with echoes and retroactive future.

The direct-straightforward principle is applied for solving the problem of determining the optimal control of set-valued linear discrete-time processes.

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Farkas' Lemma and Linear Optimization in Abstract Spaces: the infinite case

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Abstract

We study the problem of linear programming in the setting of a vector space over a linearly ordered (commutative or skew) field. The dimension of the space may be infinite. The objective function is a linear mapping into another linearly ordered vector space over the same field. In that algebraic setting, we recall known results: Farkas' Lemma, Gale's Theorem of the alternative, and the Duality Theorem for linear programming with finite number of linear constraints. Given that "semi-infinite" case, i.e. results for finite systems of linear inequalities in an infinite-dimensional space, we are motivated to consider the infinite case: infinite systems of linear inequalities in an infinite-dimensional space. Given such a system, we assume that only a finite number of the left-hand sides is non-zero at a point. We shall also assume a certain constraint qualification (CQ), presenting counterexamples violating the (CQ). Then, in the described setting, we formulate an infinite variant of Farkas' Lemma along with an infinite variant of Gale's Theorem of the alternative; we also formulate the problem of an infinite linear programming, its dual problem, and the Duality Theorem for the problems. Finally, we show an application to the problems of semi-infinite linear programming and put a question for further research.

Extended Abstract

1 Introduction

There are many generalizations of Farkas' Lemma and Duality Theorem for linear programming in the literature (Anderson & Nash, 1987; Goberna & López, 1998). In the following section, we recall a particular generalization due to Bartl (2007). We shall introduce some notation and concepts first.

Let F be a linearly ordered (commutative or skew) field. (A field is skew if and only if it is not commutative.) The ordering of the field F is a binary relation "$\leq$" such that, for all $\lambda, \mu \in F$,

$$\lambda \leq \mu \iff \lambda - \mu \leq 0$$

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and

$$\lambda \leq 0 \ \vee \ \lambda \geq 0 \,,$$
$$\lambda \leq 0 \ \wedge \ \lambda \geq 0 \ \implies \lambda = 0 \,,$$
$$\lambda \geq 0 \ \wedge \ \mu \geq 0 \ \implies \lambda + \mu \geq 0 \,,$$
$$\lambda \geq 0 \ \wedge \ \mu \geq 0 \ \implies \lambda\mu \geq 0 \,,$$

where we have used the usual convention that $\lambda \geq \mu$ if and only if $\mu \leq \lambda$. The field of the real numbers $\mathbb{R}$ or that of the rational numbers $\mathbb{Q}$ with the usual ordering is an example of a linearly ordered commutative field. Linearly ordered skew fields also exists; an example of such a field was given already by Hilbert in 1901, see Cohn (1995, Notes and comments to Chapter 1, p. 45, with Sections 2.1 and 2.3, pp. 47–50 and 66) and Lam (1991, Example 1.7, p. 10, and above Proposition 18.7, p. 288).

Let W be a vector space over the field F. No additional structure (such as topology) is assumed on the space W, whose dimension may be finite or infinite. For example, if $F = \mathbb{R}$, then W can be $\mathbb{R}^n$, finite-dimensional, or $\mathcal{C}_{[0,1]}$, the space of real continuous functions on the closed interval $[0, 1]$, or another functional space. Considering a problem of linear optimization (or programming), the space W will be the *primal variable space* (the "base space") in which we shall work:

Let $A\colon W \to F^m$ be a linear mapping and let $\boldsymbol{b} \in F^m$ be a column vector. Then

$$Ax \leq \boldsymbol{b}$$

is a finite system of linear inequalities, which circumscribes the set of the feasible solutions. For example, if $F = \mathbb{R}$ and $W = \mathbb{R}^n$, then A is induced by a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. Considering such a problem of linear programming, the space F^m will be the *primal constraint space*.

Let V be a linearly ordered vector space over the linearly ordered (commutative or skew) field F. The ordering of the space is a binary relation "$\preceq$" such that, for all $u, v \in V$,

$$u \preceq v \iff u - v \preceq 0$$

and, for all $\lambda \in F$ and $u, v \in V$, it holds

$$u \preceq 0 \ \vee \ u \succeq 0 \,,$$
$$u \preceq 0 \ \wedge \ u \succeq 0 \ \implies u = 0 \,,$$
$$u \succeq 0 \ \wedge \ v \succeq 0 \ \implies u + v \succeq 0 \,,$$
$$\lambda \geq 0 \ \wedge \ u \succeq 0 \ \implies \lambda u \succeq 0 \,,$$

where, again, we have used the usual convention that $u \succeq v$ iff $v \preceq u$. The space F^1 or, more generally, the space F^N with the lexicographical ordering is an example of a linearly ordered vector space. (Given two vectors $\boldsymbol{u} = (u_i)_{i=1}^N$, $\boldsymbol{v} = (v_i)_{i=1}^N \in F^N$, recall that $\boldsymbol{u}$ is *lexicographically less than or equal to* $\boldsymbol{v}$, writing $\boldsymbol{u} \preceq \boldsymbol{v}$, iff, for some $i_0 \in \{1, \ldots, N, N+1\}$, we have $u_i = v_i$ for $i = 1, \ldots, i_0 - 1$ and $u_{i_0} < v_{i_0}$ if $i_0 \leq N$.) Considering a problem of linear programming, the space V will be the space of the "objective values" of a linear mapping $\gamma\colon W \to V$ whose value is to be maximized subject to the given constraints.

Let $A\colon W \to F^m$ be a linear mapping, let $\boldsymbol{b} \in F^m$ be a column vector, and let $\gamma\colon W \to V$ be a linear mapping. Then the *primal problem* of linear programming, which we consider, is to

$$\begin{aligned} &\text{maximize} \quad \gamma x \\ &\text{subject to} \ \ Ax \le \boldsymbol{b} \,. \end{aligned}$$

For example, when $F = \mathbb{R}$ and $W = \mathbb{R}^n$ and $V = \mathbb{R}^1$, then A corresponds to a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and γ corresponds to a row vector $\boldsymbol{c}^T \in \mathbb{R}^{1\times n}$. Note that the case when $V = \mathbb{R}^N$ with the lexicographical ordering has some applications in the multiobjective optimization.

The symbol "ι" (Greek letter iota) transposes the next two elements; the elements are to be multiplied in the new order. For a vector $u \in V$ and a scalar $\lambda \in F$, we have

$$\iota u\lambda = \lambda u \,,$$

the λ-multiple of the vector u. If $\boldsymbol{u} = (u_i)_{i=1}^m \in V^m$ is an m-component column vector of vectors, then its transpose $\boldsymbol{u}^T$ is a row vector, which can be multiplied by the symbol ι from the left and by another column vector $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^m \in F^m$ of scalars from the right. We have

$$\iota\boldsymbol{u}^T\boldsymbol{\lambda} = (\,\iota u_1 \quad \ldots \quad \iota u_m\,)\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} = \iota u_1\lambda_1 + \cdots + \iota u_m\lambda_m = \lambda_1 u_1 + \cdots + \lambda_m u_m \,.$$

Note that, actually, the vector $u \in V$ induces a linear mapping

$$\begin{aligned} \iota u &\colon F \longrightarrow V, \\ \iota u &\colon \lambda \longmapsto \iota u\lambda = \lambda u \end{aligned}$$

for $\lambda \in F$. If $\alpha\colon W \to F$ is a linear form, then $\iota u\alpha\colon W \to V$ is the composition of the mappings. For an $x \in W$, we have

$$\iota u\alpha x = \iota u(\alpha x) = (\alpha x)u \,.$$

Analogously, the vector $\boldsymbol{u} \in V^m$ induces a linear mapping

$$\begin{aligned} \iota\boldsymbol{u}^T &\colon F^m \longrightarrow V, \\ \iota\boldsymbol{u}^T &\colon \ \boldsymbol{\lambda} \ \longmapsto \iota\boldsymbol{u}^T\boldsymbol{\lambda} \end{aligned}$$

for $\boldsymbol{\lambda} \in F^m$. If $A = (\alpha_i)_{i=1}^m\colon W \to F^m$ is a linear mapping, which is made up of m linear forms $\alpha_1, \ldots, \alpha_m\colon W \to F$, then $\iota\boldsymbol{u}^T A\colon W \to V$ is the composition of the mappings. For an $x \in W$, we have

$$\iota\boldsymbol{u}^T Ax = \iota\boldsymbol{u}^T(Ax) = \iota u_1(\alpha_1 x) + \cdots + \iota u_m(\alpha_m x) = (\alpha_1 x)u_1 + \cdots + (\alpha_m x)u_m \,.$$

Conventions analogous to those above also apply when $u \in F$ or $\boldsymbol{u} \in F^m$.

Finally, the symbol o shall denote the zero linear form $o\colon W \to F$ on W with $ox = 0$ for all $x \in W$. The symbol $\boldsymbol{o}$ shall denote a column vector of zeros of the field F or the vector space V; the meaning will always be clear from the context. Inequalities between column vectors – like $Ax \le \boldsymbol{b}$, $\boldsymbol{y} \le \boldsymbol{b}$, $Ax \le \boldsymbol{o}$, $\boldsymbol{\lambda} \ge \boldsymbol{o}$ or $\boldsymbol{u} \succeq \boldsymbol{o}$ – are understood componentwise.

2 Algebraic linear programming

Let V be a linearly ordered vector space over a linearly ordered (commutative or skew) field F, let W be a vector space over the field F, let $A\colon W \to F^m$ and $\gamma\colon W \to V$ be linear mappings, and let $\boldsymbol{b} \in F^m$ be a column vector. The following three results – Farkas' Lemma 1, Gale's Theorem 2 of the alternative, and Duality Theorem 3 for linear programming – were proved by Bartl (2007):

Lemma 1 (Farkas' Lemma)**.** *It holds*

$$\forall x \in W\colon\ \ Ax \leq \boldsymbol{o} \implies \gamma x \preceq 0 \tag{1}$$

if and only if

$$\exists \boldsymbol{u} \in V^m,\ \boldsymbol{u} \succeq \boldsymbol{o}\colon\ \ \iota\boldsymbol{u}^{\mathrm{T}}A = \gamma\,. \tag{2}$$

Remark 1. Formula (2) essentially means that, given linear mappings $A\colon W \to F^m$ and $\gamma\colon W \to V$, there exists a non-negative linear mapping $\iota\boldsymbol{u}^{\mathrm{T}}\colon F^m \to V$ which makes the following diagram commute:

$$\begin{tikzcd} W \arrow[r, "A"] \arrow[d, "\gamma"'] & F^m \arrow[dl, dashed, "\iota\boldsymbol{u}^{\mathrm{T}}"] \\ V & \end{tikzcd}$$

We say that a linear mapping $L\colon F^m \to V$ is *non-negative* iff it preserves also the ordering, i.e. for all $\boldsymbol{\lambda} \in F^m$, if $\boldsymbol{\lambda} \geq \boldsymbol{o}$, then $L\boldsymbol{\lambda} \succeq 0$. Shorter algebraic proofs of Farkas' Lemma can be found in Bartl (2008, 2012a, and 2012b).

Theorem 2 (Gale's Theorem)**.** *It holds that*

$$\nexists x \in W\colon\ \ Ax \leq \boldsymbol{b} \tag{3}$$

if and only if

$$\exists \boldsymbol{\lambda} \in F^m,\ \boldsymbol{\lambda} \geq \boldsymbol{o}\colon\ \ \iota\boldsymbol{\lambda}^{\mathrm{T}}A = o \ \wedge\ \iota\boldsymbol{\lambda}^{\mathrm{T}}\boldsymbol{b} < 0\,. \tag{4}$$

Remark 2. Formula (3) says that the system of linear inequalities $Ax \leq \boldsymbol{b}$ has no solution. Formula (4) means in words that, in the space F^m, there exists a hyperplane that separates the subspace $\operatorname{Rng} A = \{\, Ax : x \in W \,\}$ from the shifted cone $\{\, \boldsymbol{y} \in F^m : \boldsymbol{y} \leq \boldsymbol{b} \,\}$. (See Fig. 1.) Indeed: The system $Ax \leq \boldsymbol{b}$ has no solution if and only if the set $\{\, \boldsymbol{y} \in F^m : \boldsymbol{y} \leq \boldsymbol{b} \,\}$ does not intersect the range $\operatorname{Rng} A$ of the mapping A. The column vector $\boldsymbol{\lambda}$ induces the linear form $\iota\boldsymbol{\lambda}^{T}\colon F^m \to F$ with $\iota\boldsymbol{\lambda}^{T}\colon \boldsymbol{y} \to \iota\boldsymbol{\lambda}^{T}\boldsymbol{y}$ for $\boldsymbol{y} \in F^m$. The equality $\iota\boldsymbol{\lambda}^{T}A = o$ means in words that the linear form is zero on the subspace $\operatorname{Rng} A$, i.e. the range $\operatorname{Rng} A$ is contained in the kernel of the form. Observe that $\iota\boldsymbol{\lambda}^{T}\boldsymbol{y} \leq \iota\boldsymbol{\lambda}^{T}\boldsymbol{b}$ for all $\boldsymbol{y} \in \{\, \boldsymbol{y} \in F^m : \boldsymbol{y} \leq \boldsymbol{b} \,\}$ if and only if $\boldsymbol{\lambda} \geq \boldsymbol{o}$. We can see hence that, choosing a constant $c \in F$ so that $\iota\lambda^{T}\boldsymbol{b} < c < 0$, the hyperplane $\{\, \boldsymbol{y} \in F^m : \iota\boldsymbol{\lambda}^{T}\boldsymbol{y} = c \,\}$ separates the subspace $\operatorname{Rng} A$ from the shifted cone $\{\, \boldsymbol{y} \in F^m : \boldsymbol{y} \leq \boldsymbol{b} \,\}$. See also Bartl (2012c).

Theorem 3 (Duality Theorem)**.** *Consider the following primal and dual problem of linear programming:*

$$\begin{array}{lll} \text{(P)} & \text{maximize} & \gamma x \\ & \text{s.t.} & Ax \leq \boldsymbol{b}\,, \end{array} \qquad\qquad \begin{array}{lll} \text{(D)} & \text{minimize} & \iota\boldsymbol{u}^{\mathrm{T}}\boldsymbol{b} \\ & \text{s.t.} & \iota\boldsymbol{u}^{\mathrm{T}}A = \gamma\,, \\ & & \boldsymbol{u} \succeq \boldsymbol{o}\,, \end{array}$$

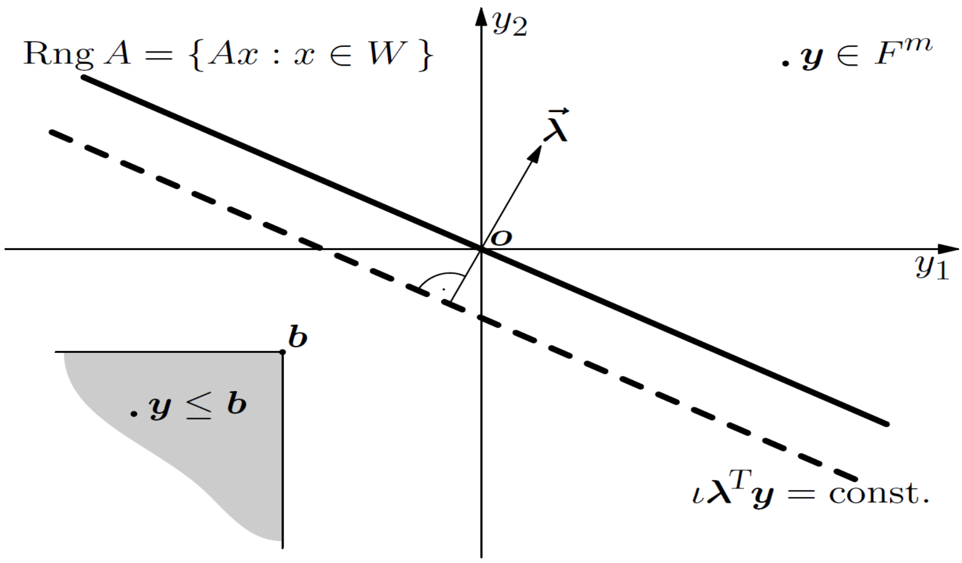

Figure 1: Illustration of Gale's Theorem of the alternative. The system $Ax \le \boldsymbol{b}$ has no solution, i.e. the subspace $\operatorname{Rng} A = \{ Ax : x \in W \}$ does not intersect the shifted cone $\{ \boldsymbol{y} \in F^m : \boldsymbol{y} \le \boldsymbol{b} \}$ so that both sets are separated by a hyperplane $\{ \boldsymbol{y} \in F^m : \iota\boldsymbol{\lambda}^T\boldsymbol{y} = \text{const.}\}$ with $\iota\boldsymbol{\lambda}^T\boldsymbol{b} < \text{const.} < 0$.

where $x \in W$ and $\boldsymbol{u} \in V^m$ are variable. Then:

I. *If $x^* \in W$ is an optimal solution to the primal problem* (P), *then there is an optimal solution $\boldsymbol{u}^* \in V^m$ to the dual problem* (D) *with $\gamma x^* = \iota{\boldsymbol{u}^*}^T\boldsymbol{b}$.*

II. *If $\boldsymbol{u}^* \in V^m$ is an optimal solution to the dual problem* (D) *and the vector space V is non-trivial, then there is an optimal solution $x^* \in W$ to the primal problem* (P) *with $\gamma x^* = \iota{\boldsymbol{u}^*}^T\boldsymbol{b}$.*

Remark 3. Farkas' Lemma 1 essential to prove Part I and Gale's Theorem 2 is necessary to prove Part II of Duality Theorem 3 (Bartl, 2007).

We have recalled three general results (Bartl, 2007). When we put $F = \mathbb{R}$, the field of the real numbers, take $W = \mathbb{R}^n$, a space of a finite dimension, and $V = \mathbb{R}^1$, the real axis, in Farkas' Lemma 1, Gale's Theorem 2, and Duality Theorem 3, then we obtain the classical version of Farkas' Lemma (Farkas, 1902), Gale's Theorem of the alternative (Fan, 1956; Gale, 1960), and Duality Theorem for linear programming (Gale et al., 1951), respectively. See Bartl (2007, 2008, 2012a, and 2012b) for a more detailed discussion. Recall that the case when $V = \mathbb{R}^N$ with the lexicographical ordering has some applications in the multiobjective optimization.

3 Infinite algebraic linear programming

3.1 Motivation

In the preceding section, we recalled some results with a finite system of linear inequalities $Ax \le \boldsymbol{o}$ (Farkas' Lemma 1) or $Ax \le \boldsymbol{b}$ (Gale's Theorem 2 and Duality Theorem 3). It has been an interesting question whether it is possible to obtain analogous generalized results with an infinite system of linear inequalities $A x \leq \boldsymbol{o}$ or $A x \leq \boldsymbol{b}$.

Farkas' Lemma 1 is a cornerstone in the theory due to Bartl (2007): all the results (Gale's Theorem 2, other theorems of the alternative, and Duality Theorem 3) follow from it. That is why we shall deal with an infinite version of Farkas' Lemma first. Thus, let M be an infinite index set and let $\alpha_i \colon W \to F$, for $i \in M$, be linear forms. We consider the infinite system of linear inequalities $A x \leq \boldsymbol{o}$, or $\alpha_i x \leq 0$ for $i \in M$. Assuming that $\gamma x \preceq 0$ for all $x \in W$ such that $A x \leq \boldsymbol{o}$, we should have

$$\gamma = \iota \boldsymbol{u}^T A = \sum_{i \in M} \iota u_i \alpha_i$$

for some non-negative $\boldsymbol{u} \in V^M$, i.e. some non-negative vectors $u_i \in V$ for $i \in M$. That is, for an $x \in W$, we should have

$$\gamma x = \iota \boldsymbol{u}^T A x = \sum_{i \in M} \iota u_i \alpha_i x \,.$$

However, the sum $\sum_{i \in M} \iota u_i \alpha_i x$ must be correct – we do not consider any additional concept such as topology or convergence here – whence, only a finite number of the terms can be non-zero. In addition, as in Remark 1, we should have the commutative diagram

$$\begin{tikzcd} W \arrow[r, "A"] \arrow[d, "\gamma"'] & ? \arrow[dl, dashed, "\iota \boldsymbol{u}^T"] \\ V & \end{tikzcd}$$

perhaps with $\boldsymbol{u} \in V^M$, meaning that possibly all of the u_i can be non-zero or positive. Hence, we can guess that we should have $? = F^{(M)}$, the space of all infinite sequences with only a finite number of non-zero entries.

3.2 Definitions, counterexamples, and the constraint qualification

Let V be a linearly ordered vector space over a linearly ordered (commutative or skew) field F and let W be a vector space over the field F.

Let M be a (finite or infinite) index set. Formally, a column vector $\boldsymbol{u} = (u_i)_{i \in M} \in V^M$ of vectors of the space V is a sequence or mapping

$$\begin{aligned} \boldsymbol{u} \colon M &\longrightarrow V \,, \\ \boldsymbol{u} \colon \ i &\longmapsto u_i \,. \end{aligned}$$

Now, for a set X, we write $\operatorname{Fin} X$ iff the set X is finite. Analogously then, a column vector $\boldsymbol{\lambda} = (\lambda_i)_{i \in M} \in F^{(M)}$ of scalars of the field F with a finite number of non-zero entries is a sequence of mapping

$$\begin{aligned} \boldsymbol{\lambda} \colon M &\longrightarrow F \,, \\ \boldsymbol{\lambda} \colon \ i &\longmapsto \lambda_i \end{aligned}$$

with

$$\operatorname{Fin}\{\, i \in M : \lambda_i \neq 0 \,\} \,.$$

To conclude, we have the two spaces

$$\begin{aligned} V^M &= \{\, \boldsymbol{u}\colon M \to V \,\}\,, \\ F^{(M)} &= \{\, \boldsymbol{\lambda}\colon M \to F : \mathrm{Fin}\{\, i \in M : \lambda_i \neq 0 \,\}\,\}\,, \end{aligned}$$

where M is an index set. Now, we conjecture that the following version of Farkas' Lemma could hold:

Hypothesis 1 (An infinite version of Farkas' Lemma)**.** *Let V be a linearly ordered vector space over a linearly ordered (commutative or skew) field F, let W be a vector space over the field F, let M be an index set, and let $A\colon W \to F^{(M)}$ and $\gamma\colon W \to V$ be linear mappings. Then*

$$\forall x \in W\colon\ Ax \leq \boldsymbol{o} \implies \gamma x \preceq 0$$

if and only if

$$\exists \boldsymbol{u} \in V^M,\ \boldsymbol{u} \succeq \boldsymbol{o}\colon\ \iota\boldsymbol{u}^{\mathrm{T}}A = \gamma\,.$$

Indeed, although the "if" part of Hypothesis 1 is trivial, the "only if" part does not hold in general.

Counterexample 1. For simplicity, let us consider $F = \mathbb{R}$, the field of the real numbers, and $V = \mathbb{R}^1$, the one-dimensional real axis. Let $W = c_{00} = \mathbb{R}^{(\mathbb{N})}$ be the functional space of all sequences $\boldsymbol{x} = (x_i)_{i=1}^{\infty}$ of real numbers with only a finite number of non-zero entries.

Let $M = \mathbb{N} \cup \{\omega\} = \{1, 2, 3, \dots\} \cup \{\omega\}$ be the set of all finite natural numbers with a transfinite element. Let us consider the forms $\alpha_i\boldsymbol{x} = x_i$ for $i = 1, 2, 3, \dots$, and $\alpha_\omega\boldsymbol{x} = \sum_{i=1}^{\infty} -x_i$, putting $\gamma\boldsymbol{x} = \sum_{i=1}^{\infty} -ix_i$ for an $\boldsymbol{x} = (x_i)_{i=1}^{\infty} \in W$. In a less formal way, we can represent α_1, α_2, α_3, ..., and α_ω with γ as row vectors:

$$\begin{array}{rcrrrrcl} \alpha_1 & = (& 1 & 0 & 0 & 0 & \dots &)\,, \\ \alpha_2 & = (& 0 & 1 & 0 & 0 & \dots &)\,, \\ \alpha_3 & = (& 0 & 0 & 1 & 0 & \dots &)\,, \\ \alpha_4 & = (& 0 & 0 & 0 & 1 & \dots &)\,, \\ \multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ \alpha_\omega & = (& -1 & -1 & -1 & -1 & \dots &)\,, \\ \hline \gamma & = (& -1 & -2 & -3 & -4 & \dots &)\,. \end{array}$$

Thanks to the choice of the space $W = c_{00}$, only a finite number of the linear forms α_i is non-zero at a point $\boldsymbol{x} \in W$, and the form α_ω with the mapping γ are well defined because only a finite number of the terms is non-zero in the sums.

Now, choose an $\boldsymbol{x} = (x_i)_{i=1}^{\infty} \in W = c_{00}$. If $\alpha_i\boldsymbol{x} = x_i \leq 0$ for $i = 1$, 2, 3, ..., and $\alpha_\omega\boldsymbol{x} \leq 0$, i.e. $\sum_{i=1}^{\infty} x_i \geq 0$, then $\boldsymbol{x} = \boldsymbol{o}$, hence $\gamma\boldsymbol{x} = 0$, so $\gamma\boldsymbol{x} \preceq 0$. However, there exist no non-negative numbers u_1, u_2, u_3, ..., and u_ω such that $\gamma = u_\omega\alpha_\omega + \sum_{i=1}^{\infty} u_i\alpha_i$.

The counterexample motivates us to introduce a certain constraint qualification: we shall exclude the case described in Counterexample 1.

(*Remark.* It turns out that Counterexample 1 describes the only basic situation that can be found to counter the statement of Hypothesis 1.)

7

Definition 1 (F-linear independence)**.** Let $A\colon W \to F^{(M)}$ be a linear mapping so that we have an indexed collection $\{\alpha_i\}_{i\in M}$ of linear forms such that, for any $x \in W$, the set $\{\, i \in M : \alpha_i x \neq 0 \,\}$ is finite. Now, let $M^* \subseteq M$ be any subset of the index set M. We say that the subcollection $\{\alpha_i\}_{i\in M^*}$ is *F-linearly independent* iff

$$\forall \boldsymbol{\lambda}_{M^*} \in F^{M^*}\colon\ \iota\boldsymbol{\lambda}_{M^*}^T A_{M^*} = \sum_{i\in M^*} \iota\lambda_i\alpha_i = o \implies \boldsymbol{\lambda}_{M^*} = \boldsymbol{o}\,.$$

Definition 2 (Constraint Qualification (CQ))**.** Let $A\colon W \to F^{(M)}$ be a linear mapping. We say that the linear mapping A satisfies the constraint qualification (CQ) iff, for any subset $M^* \subseteq M$ such that the subcollection $\{\alpha_i\}_{i\in M^*}$ is F-linearly independent, and for any infinite subset $M^- \subseteq M^*$, there exists a point $x \in W$ such that

$$Ax \le \boldsymbol{o} \qquad \text{and} \qquad A_{M^-}x \neq \boldsymbol{o}\,.$$

The latter condition means that $\alpha_i x \neq 0$, hence $\alpha_i x < 0$, for at least one $i \in M^-$.

(*Remark.* The constraint qualification (CQ) presented in Definition 2 is *weaker* than that originally presented in the paper [BARTL, D. Farkas' Lemma, Gale's Theorem, and Linear Programming: the Infinite Case in an Algebraic Way. *Global Journal of Mathematical Sciences (GJMS)*, 1 (2012) 18–23. ISSN 2164-3709]. The old (CQ) requested that, whenever $M^- \subseteq M^* \subseteq M$ with M^- infinite and $\{\alpha_i\}_{i\in M^*}$ being F-linearly independent, there exists a point $x \in W$ such that $Ax \le \boldsymbol{o}$ with $A_{M^-}x \neq \boldsymbol{o}$ and $A_{M^*\setminus M^-}x = \boldsymbol{o}$. The last condition ($A_{M^*\setminus M^-}x = \boldsymbol{o}$) is unnecessary.)

Assuming the constraint qualification (CQ), the "only if" part of Farkas' Lemma (Hypothesis 1) becomes to hold true (see the next subsection).

Now, we shall be concerned with an infinite version of Gale's Theorem. Let us consider an infinite system $Ax \le \boldsymbol{b}$ with the linear mapping $A\colon W \to F^{(M)}$. Thus, it might seem plausible that we should have $\boldsymbol{b} \in F^{(M)}$. Then, however, the system $Ax \le \boldsymbol{b}$ would not be interesting: we would have a finite system $\alpha_i x \le b_i$ for $i \in M$ with $b_i \neq 0$ and the remaining, possibly infinite, part $\alpha_i x \le 0$ for $i \in M$ with $b_i = 0$. Therefore, we shall consider the more general case when $\boldsymbol{b} \in F^M$. Formally, a column vector $\boldsymbol{b} = (b_i)_{i\in M} \in F^M$ of scalars of the field F is a sequence or mapping

$$\begin{aligned} \boldsymbol{b}&\colon M \longrightarrow F\,,\\ \boldsymbol{b}&\colon\ i \longmapsto b_i\,. \end{aligned}$$

Naturally, we have to assume that the column vector $\boldsymbol{b}$ comprises only a finite number of negative entries. (Otherwise, the system $Ax \le \boldsymbol{b}$ could not have a solution as only a finite number of entries of the left-hand column can be non-zero.) In order that the sum $\iota\boldsymbol{\lambda}^T\boldsymbol{b} = \sum_{i\in M}\iota\lambda_i b_i$ is well defined, we shall require that only a finite number of the terms is non-zero, i.e. the set $\{\, i \in M : \lambda_i \neq 0 \,\wedge\, b_i \neq 0 \,\}$ is finite. Thus, we conjecture that the following version of Gale's Theorem could hold:

Hypothesis 2 (An infinite version of Gale's Theorem)**.** *Let W be a vector space over a linearly ordered (commutative or skew) field F, let M be an index* *set, let $A\colon W \to F^{(M)}$ be a linear mapping, and let $\boldsymbol{b} \in F^M$ be a column vector. Under the assumption* $\operatorname{Fin}\{\, i \in M : b_i < 0 \,\}$, *it holds that*

$$\nexists x \in W\colon \ \ Ax \leq \boldsymbol{b}$$

if and only if

$$\exists \boldsymbol{\lambda} \in F^M,\ \boldsymbol{\lambda} \geq \boldsymbol{o},\ \operatorname{Fin}\{\, i \in M : \lambda_i \neq 0 \ \wedge\ b_i \neq 0 \,\}\colon \ \iota\boldsymbol{\lambda}^{\mathrm{T}} A = o \ \wedge\ \iota\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b} < 0\,.$$

Again, while the "if" part of Hypothesis 2 is obvious, its "only if" part does not hold in general.

Counterexample 2. Take $F = \mathbb{R}$, the field of the real numbers, with $W = c_{00} = \mathbb{R}^{(\mathbb{N})}$, the functional space of all sequences $\boldsymbol{x} = (x_i)_{i=1}^{\infty}$ of real numbers with only a finite number of non-zero entries. Consider the system

$$\begin{array}{rcl} -x_1 & \leq & -1\,, \\ -x_2 + x_1 & \leq & \frac{1}{2}\,, \\ -x_3 + x_2 \qquad & \leq & \frac{1}{4}\,, \\ -x_4 + x_3 \qquad\qquad & \leq & \frac{1}{8}\,, \\ -x_5 + x_4 \qquad\qquad\qquad & \leq & \frac{1}{16}\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \end{array}$$

Obviously, the system has no solution in the space $W = c_{00}$. However, no finite linear combination of the left-hand sides yields the zero linear form on W: all the left hand-sides have to be summed up; then, however, the sum of the right-hand sides is zero, not negative.

3.3 The main results

Let V be a linearly ordered vector space over a linearly ordered (commutative or skew) field F, let W be a vector space over the field F, let M be an index set, let $A\colon W \to F^{(M)}$ be a linear mapping satisfying the constraint qualification (CQ), let $\boldsymbol{b} \in F^M$ be a column vector with $\operatorname{Fin}\{\, i \in M : b_i < 0 \,\}$, and let $\gamma\colon W \to V$ be a linear mapping. Then, the following three results hold true:

Lemma 4 (Farkas' Lemma)**.** *If*

$$\forall x \in W\colon \ \ Ax \leq \boldsymbol{o} \ \implies\ \gamma x \preceq 0\,, \tag{5}$$

then

$$\exists \boldsymbol{u} \in V^M,\ \boldsymbol{u} \succeq \boldsymbol{o}\colon \ \iota\boldsymbol{u}^{\mathrm{T}} A = \gamma\,. \tag{6}$$

Theorem 5 (Gale's Theorem)**.** *If*

$$\nexists x \in W\colon \ \ Ax \leq \boldsymbol{b}\,, \tag{7}$$

then

$$\exists \boldsymbol{\lambda} \in F^{(M)},\ \boldsymbol{\lambda} \geq \boldsymbol{o}\colon \ \iota\boldsymbol{\lambda}^{\mathrm{T}} A = o \ \wedge\ \iota\boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{b} < 0\,. \tag{8}$$

Theorem 6 (Duality Theorem). *Consider the following primal and dual problem of linear programming:*

$$(\mathrm{P}) \quad \begin{array}{ll} \text{maximize} & \gamma x \\ \text{s.t.} & Ax \le \boldsymbol{b}\,, \end{array} \qquad\qquad (\mathrm{D}) \quad \begin{array}{ll} \text{minimize} & \iota \boldsymbol{u}^T \boldsymbol{b} \\ \text{s.t.} & \iota \boldsymbol{u}^T A = \gamma\,, \\ & \boldsymbol{u} \succeq \boldsymbol{o}\,, \\ \multicolumn{2}{l}{\mathrm{Fin}\{\, i \in M : u_i \neq 0 \,\wedge\, b_i \neq 0 \,\}\,,} \end{array}$$

where $x \in W$ and $\boldsymbol{u} \in V^M$ are variable. Then:

I. *If $x^* \in W$ is an optimal solution to the primal problem* (P)*, then there is an optimal solution $\boldsymbol{u}^* \in V^M$ to the dual problem* (D) *with $\gamma x^* = \iota \boldsymbol{u}^{*T} \boldsymbol{b}$.*

II. *If $\boldsymbol{u}^* \in V^M$ is an optimal solution to the dual problem* (D) *and the vector space V is non-trivial, then there is an optimal solution $x^* \in W$ to the primal problem* (P) *with $\gamma x^* = \iota \boldsymbol{u}^{*T} \boldsymbol{b}$.*

Gale's Theorem 5 is surprising: if the system $Ax \le \boldsymbol{b}$ has no solution, then, by (8), some finite subsystem of it has no solution. The condition $\mathrm{Fin}\{\, i \in M : \lambda_i \neq 0 \,\wedge\, b_i \neq 0 \,\}$ is not necessary in Gale's Theorem 5 (though conjectured in Hypothesis 2), but its variant is essential in the dual problem (D) in Duality Theorem 6. Let us observe that, if the set M is finite, e.g. $M = \{1, \ldots, m\}$, then the constraint qualification (CQ) is naturally satisfied – there is no infinite subset $M^- \subseteq M^* \subseteq M$. Thus, Farkas' Lemma 4, Gale's Theorem 5, and Duality Theorem 6 generalizes Farkas' Lemma 1, Gale's Theorem 2, and Duality Theorem 3, respectively.

The proofs of the main results are long. The author is really sorry that he has not published them yet, regrettably...

3.4 An application in semi-infinite linear programming

Now, the above results are quite general and abstract. It is an interesting question whether they can be applied to some problems of infinite linear programming whose solution is already known (e.g., Anderson & Nash, 1987), perhaps establishing a new approach to solving those problems. In this section, we show that they can be applied to problems of semi-infinite linear programming (Goberna & López, 1998).

Let a row vector $\boldsymbol{c}^T \in \mathbb{R}^{1\times n}$ be given. We also have an index set T, row vectors $\boldsymbol{a}_t^T \in \mathbb{R}^{1\times n}$ and numbers $b_t \in \mathbb{R}$ for $t \in T$. Consider the primal problem of semi-infinite linear programming (Goberna & López, 1998, Section 1.1, p. 3), where $\boldsymbol{x} \in \mathbb{R}^n$ is variable:

$$(\mathrm{P}_{\mathrm{LSIP}}) \qquad \begin{array}{lll} \inf & \boldsymbol{c}^T \boldsymbol{x} & \\ \text{s.t.} & \boldsymbol{a}_t^T \boldsymbol{x} \ge b_t & \quad \text{for } t \in T\,. \end{array}$$

Write the variables $\boldsymbol{x}$ as a difference $\boldsymbol{x} = \boldsymbol{x}^+ - \boldsymbol{x}^-$ of two non-negative magnitudes $\boldsymbol{x}^+, \boldsymbol{x}^- \ge \boldsymbol{o}$, and introduce new non-negative variables $z_t \ge 0$ for $t \in T$. Then, we can write problem $(\mathrm{P}_{\mathrm{LSIP}})$ as follows:

$$\begin{array}{lll} \min & \boldsymbol{c}^T \boldsymbol{x}^+ - \boldsymbol{c}^T \boldsymbol{x}^- & \\ \text{s.t.} & \boldsymbol{a}_t^T \boldsymbol{x}^+ - \boldsymbol{a}_t^T \boldsymbol{x}^- - z_t = b_t & \quad \text{for} \quad t \in T\,, \\ & \boldsymbol{x}^+,\, \boldsymbol{x}^- \ge \boldsymbol{o}, \quad z_t \ge 0 & \quad \text{for} \quad t \in T\,. \end{array} \tag{9}$$

This problem is of the form (D). Formulate the corresponding primal problem (P):

$$\begin{aligned}
\max \quad & \sum_{t\in T} \lambda_t b_t \\
\text{s.t.} \quad & \sum_{t\in T} \lambda_t \boldsymbol{a}_t^T \le \boldsymbol{c}^T , \\
& -\sum_{t\in T} \lambda_t \boldsymbol{a}_t^T \le -\boldsymbol{c}^T , \\
& -\lambda_t \le 0 \qquad \text{for } t \in T ,
\end{aligned} \tag{10}$$

where $\lambda_t \in \mathbb{R}$ are variable. We must have $\boldsymbol{\lambda} \in \mathbb{R}^{(T)}$ in order to satisfy the condition that the mapping "$A\colon W \to F^{(M)}$" on the left-hand side of the constraints attains only finitely many non-zero values at any point. So we can write the problem in the form

$$\begin{aligned}
(\mathrm{D_{LSIP}}) \qquad \sup \quad & \sum_{t\in T} \lambda_t b_t \\
\text{s.t.} \quad & \sum_{t\in T} \lambda_t \boldsymbol{a}_t^T = \boldsymbol{c}^T , \\
& \lambda_t \ge 0 \qquad \text{for } t \in T , \\
& \mathrm{Fin}\{\, t \in T : \lambda_i \ne 0 \,\} ,
\end{aligned}$$

which is the *Haar Dual* (Goberna & López, 1998, Section 2.2, p. 49) of the primal problem ($\mathrm{P_{LSIP}}$) of the semi-infinite linear programming.

Now, it is a question whether the mapping "$A\colon W \to F^{(M)}$" on the left-hand side of the constraints of the "primal problem (P)" (10), satisfies the constraint qualification (CQ). Applying Definition 2 to problem (10), we obtain that, for any infinite set $T^- \subseteq T$, there must exist a non-negative point $\boldsymbol{\lambda} \in \mathbb{R}^{(T)}$ such that $\sum_{t\in T} \lambda_t \boldsymbol{a}_t^T = \boldsymbol{o}^T$ with $\lambda_i > 0$ for at least one $i \in T^-$. Then Duality Theorem 6 applies to problems ($\mathrm{P_{LSIP}}$) and ($\mathrm{D_{LSIP}}$).

3.5 Further research

Problems of the subsequent form are often considered in infinite programming (Anderson & Nash, 1987, Section 3.3, pp. 38–40).

Let X be a real vector space and let Y be a locally convex topological vector space over the field $\mathbb{R}$ of the real numbers. Let a linear mapping $A\colon X \to Y$, a point $b \in Y$, and a linear functional $c\colon X \to \mathbb{R}$ be given. Let a non-negative cone $P \subseteq X$ be given in the space X. Write $x \ge 0$ iff $x \in P$.

The algebraic dual of the space X is to be denoted by $X^\#$, the topological dual of the space Y is to be denoted by Y^*. For a $y \in Y^*$, we have $\iota by = yb$, the value of the functional y at b. For a $y \in Y^*$, we also have $\iota Ay = yA$, the composition of the mapping A and y. And, for $c, d \in X^\#$, write $d \le c$ iff $cx \le dx$ for each $x \in P$. (So that $\iota Ay \le c$ means $yAx \le cx$ for all $x \in P$.)

Consider the next primal and dual problem (Anderson & Nash, 1987, Section 3.3, pp. 38–40), where $x \in X$ and $y \in Y^*$ are variable:

$$\begin{aligned}
(\mathrm{EP}) \quad \inf \quad & cx \\
\text{s.t.} \quad & Ax = b , \\
& x \ge 0 ,
\end{aligned}
\qquad\qquad
\begin{aligned}
(\mathrm{EP}^*) \quad \sup \quad & \iota by \\
\text{s.t.} \quad & \iota Ay \le c .
\end{aligned}$$

The purpose is to prove the Duality Theorem of the following form (Anderson & Nash, 1987, Theorem 3.3, p. 41): *Problem* (EP) *is subconsistent and has a finite subvalue M if and only if problem* (EP*) *is consistent and has a finite value M.* That is a certain least upper bound (infimum) is equal to the greatest lower bound (supremum) of problem (EP*).

Duality Theorem 6 for problems (P) a (D) is of different nature. Firstly, we assume that there *exists* an optimal solution, i.e. the maximum or minimum is *attained.* The mentioned Duality Theorem of Anderson & Nash (1987) uses suprema and infima only – optimal solutions (points) may not exist, i.e. we can only get arbitrarily close to the optimal value. (The existence of true optimal solutions of (EP) and (EP*) is discussed in Anderson & Nash (1987, Chapter 3) too.) Secondly, the mapping A of problems (P) and (D) must satisfy the constraint qualification (CQ).

Notice the space of the values of the objective functions of problems (EP) and (EP*) is the space $\mathbb{R}^1$, the real line. It holds the property it is complete, i.e. each non-empty above (or below) bounded set has a supremum (or infimum). It is known that this is the *only* linearly ordered vector space of that property. (If G is a linearly ordered group which is complete in the above sense, then it is isomorphic with the additive group of the field $\mathbb{R}$. Hölder Theorem.)

A general linearly ordered vector space V is that of the values of the objective functions of problems (P) and (D), which we consider. It follows the existence of suprema and infima is not guaranteed there. That fact does not matter in Duality Theorem 6 for problems (P) and (D) because we assume the *existence* of optimal solutions, hence, we assume that the optimal values are *attained* there.

It is a question if it is possible to weaken the Duality Theorem for problems (P) and (D) somehow even in our very general algebraic setting (the space W and the linearly ordered space V over F). For example, we may consider just to show that "there is no duality gap" in the following way:

Let $V^{\downarrow} = \big\{\, u \in V : \exists x \in W\colon\; Ax \leq \boldsymbol{b} \,\wedge\, u \preceq \gamma(x) \,\big\}$ be the set of all vectors $v \in V$ that are bounded from above by some objective value of problem (P). Analogously, let $V^{\uparrow} = \big\{\, v \in V : \exists \boldsymbol{u} \in V^M\colon\; \iota\boldsymbol{u}^T A = \gamma \,\wedge\, \boldsymbol{u} \succeq \boldsymbol{o} \,\wedge\, \mathrm{Fin}\{\, i \in M : u_i \neq 0 \,\wedge\, b_i \neq 0 \,\} \,\wedge\, v \succeq \iota\boldsymbol{u}^T\boldsymbol{b} \,\big\}$ je be the set of all vectors $v \in V$ that are bounded from below by some objective value of problem (D).

Then, it holds (?) that $u \preceq v$ for all $u \in V^{\downarrow}$ and for all $v \in V^{\uparrow}$ and that $V^{\downarrow} \cup V^{\uparrow} = V$. (An analogy of the Dedekind Cut.)

In this way, we can express essentially the same as the equation "sup = inf" does while we do not need the concept of the supremum and infimum, which may not exist in the space V either.

The question whether such a result can be established is a motivation of our further research.

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