Laplacian spectrum of weakly quasi-threshold graphs*

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Abstract. In this paper we study the class of weakly quasi-threshold graphs that are obtained from a vertex by recursively applying the operations (i) adding a new isolated vertex, (ii) adding a new vertex and making it adjacent to all old vertices, (iii) disjoint union of two old graphs, and (iv) adding a new vertex and making it adjacent to all neighbours of an old vertex. This class contains the class of quasi-threshold graphs. We show that weakly quasi-threshold graphs are precisely the comparability graphs of a forest consisting of rooted trees with each vertex of a tree being replaced by an independent set. We also supply a quadratic time algorithm in the the size of the vertex set for recognizing such a graph. We completely determine the Laplacian spectrum of weakly quasi-threshold graphs. It turns out that weakly quasi-threshold graphs are Laplacian integral. As a corollary we obtain a closed formula for the number of spanning trees in such graphs. A conjecture of Grone and Merris asserts that the spectrum of the Laplacian of any graph is majorized by the conjugate of the degree sequence of the graph. We show that the conjecture holds for cographs.

Key words. quasi-threshold graphs, weakly quasi-threshold graphs, cographs, Laplacian Matrix, degree sequence.

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The study of graph spectra is an important topic in Algebraic Graph Theory (See Cvetkovic *et al* [4]). Although spectral graph theory mainly involves the study of the eigenvalues of the adjacency matrix, the Laplacian eigenvalues have also been receiving considerable attention in recent years. Since the eigenvalues of the adjacency matrix, or of the Laplacian, are invariant under a relabeling of the vertices of the graph, these eigenval-

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ues, or suitable functions of them, occur naturally in the search for a graph invariant under isomorphism. The question of determining whether two nonisomorphic graphs can have the same spectrum can be thought of as a discrete analogue of the well-known problem "Can we hear the shape of a drum" by Marc Kac; see Merris [11] for a readable account of this analogy. Incidentally, for a regular graph, the eigenvalues of the adjacency matrix and the Laplacian are closely related, and the class of regular graphs is the one which is intensively studied in spectral graph theory. Laplacian eigenvalues are also related to the Wiener index of a graph. In particular, for a tree, the Wiener index is a function of the Laplacian eigenvalues (see Merris [11]).

Our motivation for the present work was to use the connection between the Laplacian eigenvalues and the number of spanning trees to provide a closed formula for the number of spanning trees in certain classes of graphs. Since Laplacian eigenvalues behave well under complementation and disjoint union, graph classes defined using these two operations provide a natural setting for the study of Laplacian eigenvalues.

Throughout the article only simple graphs are considered. A *rooted tree* is a directed graph obtained from a tree by assigning each edge a direction so that there exists a special vertex r, called the *root*, and there is a unique directed path from r to each vertex. If u is a vertex in a rooted tree $T = (V, E)$ then a vertex v such that $uv \in E$ is called a *child* of u. A vertex v is called a *descendant (ancestor)* of u if there is a directed u-v (v-u) path in T. The set of children, descendants and ancestors of a vertex u will be denoted by $\mathrm{child}(u)$, $\mathrm{des}(u)$ and $\mathrm{anc}(u)$, respectively. By F_u we shall denote the *family* of u which consists of u and its descendants. Thus $F_r = V$. The *comparability graph* of a rooted tree $T = (V, E)$ is the undirected graph $G = (V, E')$, where $uv \in E'$ if and only if v is a descendant of u or u is a descendant of v. For any undirected graph G, $\deg_G(v)$ will denote the degree of the vertex v in G.

The following is well-known.

Theorem 0.1. [15] *A graph G is the comparability graph of a rooted tree if and only if G is P_4-free and C_4-free, i.e., G does not have an induced subgraph isomorphic to the path on 4 vertices or to the cycle on 4 vertices.*

A graph G which is P_4-free is called a *cograph*. A cograph which is C_4-free is called a *quasi-threshold graph*. A quasi-threshold graph which is $2K_2$-free is called a *threshold graph*.

Graphs obtained from a vertex by recursively applying the following operations: (i) adding a new vertex, (ii) adding a new vertex that is adjacent to all old vertices, and (iii) disjoint union of two graphs are precisely the quasi-threshold graphs (see Yan *et al.* [16]).

In this article, we introduce the class $\mathcal{C}$ of *weakly quasi-threshold graphs* which are obtained from an empty graph by recursively applying the following operations:

(1) adding a new isolated vertex,
(2) adding a new vertex and making it adjacent to all old vertices,
(3) disjoint union of two old graphs, and
(4) adding a new vertex and making it adjacent to all neighbours of an old vertex,

where the *neighbourhood* of a vertex $v \in V$ is defined as the set $N(v) = \{u \in V : uv \in E\}$. Observe that the class $\mathcal{C}$ obviously contains the quasi-threshold graphs.

We now outline the contents of the paper. In Section 2, we show that the weakly quasi-threshold graphs are precisely the comparability graph of a forest consisting of rooted trees with each vertex of the tree replaced by an independent set. We also supply a quadratic time algorithm in the the size of the vertex set for recognizing such a graph. We also prove a characterization of weakly quasi-threshold graphs.

In Section 3, we define the Laplacian matrix $L(G)$ of a graph G and completely determine the eigenalues of $L(G)$ for each $G \in \mathcal{C}$. It follows that such graphs are Laplacian integral. That is, we show that the eigenvalues of $L(G)$ for each $G \in \mathcal{C}$ is a nonnegative integer. As a corollary, we get a closed formula for the number of spanning trees in such graphs. This formula when restricted to quasi-threshold graphs improves a result of Nikolopoulos and Papadopoulos [14].

For a sequence of positive integers $d_1 \geq d_2 \geq \ldots \geq d_n$, the *conjugate sequence* of $d = (d_1, d_2, \ldots, d_n)$ is the sequence $d^* = (d_1^*, d_2^*, \ldots, d_n^*)$ given by $d_i^* = |\{j : d_j \geq i\}|$. A conjecture of Grone and Merris [6] asserts that the spectrum of the Laplacian of any graph is majorized by the conjugate of the degree sequence of the graph. In Section 4, we show that the conjecture holds for cographs.

In this paper, most of the graph theoretic definitions and notations are explained. If any notation or definition is missed, we refer the reader to the book of Harary [7] for the same.

1. Weakly quasi-threshold graphs

Let T be a rooted tree with each vertex v_i (for $1 \leq i \leq n$) replaced by a set $Q_i = \{v_{i1}, \cdots, v_{in_i}\},\ n_i \geq 1$. Here the sets $Q_1, \ldots, Q_n$ are pairwise disjoint. We shall refer to such a tree as a *rooted tree with independent vertex sets as vertices*. The notion of the comparability graph can be extended to such trees.

Definition. Let T be a rooted tree with independent vertex sets $Q_1, \ldots, Q_n$ as vertices, where $Q_i = \{v_{ik} : k = 1, \ldots, n_i\},\ n_i \geq 1$. Then the *comparability graph* of T is the graph $G = (V, E)$ where $V = \bigcup\limits_{i=1}^{n} Q_i$ and $v_{ik}v_{js} \in E$ if Q_i is either an ancestor or a descendant of Q_j in T. The comparability graph of a forest of such trees is the disjoint union of the comparability graphs of the trees.

The following is an easy observation.

Proposition **1.1** *A graph G is weakly quasi-threshold if and only if each connected component of G is weakly quasi-threshold.*

Proof. To prove the necessary part let G be a weakly quasi-threshold graph. Then by definition G is obtained by applying a sequence of operations (1)–(4) recursively starting with an empty graph. Let H be a connected component of G. Consider the subsequence of operations involving vertices of H. Since none of these operations involve a vertex outside of H, it follows that H is created by this subsequence of operations. Hence H is also weakly quasi-threshold.

Converse follows easily by repeated application of operation (3). ■

Let $G = (V, E)$ be a graph. Then recall that the *neighbourhood* of a vertex $v \in V$ is defined as the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighbourhood* of a vertex v is $N[v] = N(v) \cup \{v\}$. Thus $\deg_G(v)$ equals $|N(v)|$, the cardinality of the set $N(v)$. For an independent vertex set Q_i of a rooted tree T, child(Q_i), anc(Q_i) and des(Q_i) will respectively denote the children, ancestors and descendants of Q_i.

Let $G = (V, E)$ be a graph. Define a relation on V as follows: Let $u, v \in V$. Then $u \sim v$ if $N(u) = N(v)$. We understand here that any two isolated vertices are related by $\sim$. Observe that $\sim$ is an equivalence relation and the equivalence classes are independent vertex sets in G.

Example **1.2** In Figure 1, the picture on the left is an example of a rooted tree T with independent vertex sets as vertices. The graph G on the right is the comparability graph of T. For this tree, child$(Q_1) = \{Q_2, Q_3\}$, anc$(Q_1) = \emptyset$ and des$(Q_1) = \{Q_2, Q_3, Q_4, Q_5\}$. Notice that if we consider a new tree T' by replacing the independent vertex sets $Q_4 = \{v_{41}, v_{42}\}$ and $Q_5 = \{v_{51}\}$ with a single independent vertex set $Q'_4 = \{v_{41}, v_{42}, v_{51}\}$, then T' also has the same comparability graph. Consider the graph G. Under the equivalence relation defined above, $\{v_{11}, v_{12}\}$, $\{v_{21}, v_{22}, v_{23}\}$, $\{v_{31}, v_{32}\}$, and $\{v_{41}, v_{42}, v_{51}\}$ are the equivalence classes.

Theorem 1.3. *Let G be a weakly quasi-threshold graph with $Q_1, \ldots, Q_r$ as the equivalence classes under the relation $\sim$. Then G is the comparability graph of a forest F of rooted trees with independent vertex sets $Q_1, \ldots, Q_r$ as vertices.*

Proof. We proceed by induction on the number of vertices n of G. The assertion trivially holds for $n = 1$. Suppose that the assertion holds for all weakly quasi-threshold graphs with $n \leq k$ vertices. Let G be a weakly quasi-threshold graph on $n = k + 1$ vertices. Our proof depends on the last operation used in the creation of G and we proceed by cases.

Case 1: Suppose operation (1) is applied on a weakly quasi-threshold graph G' to obtain G. Let v be the isolated vertex added to G' and let F' be the forest associated to G'. We have two subcases.

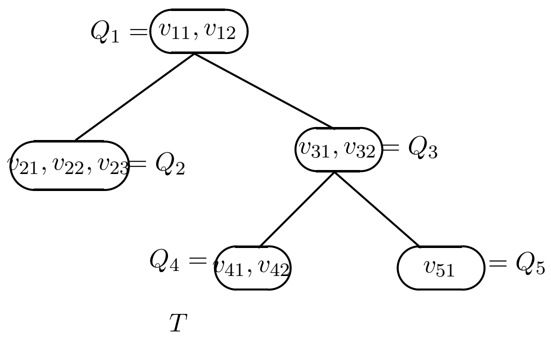

Fig. 1. A rooted tree with independent vertex sets as vertices and its comparability graph.

Case 1.1: Forest F' has an isolated independent vertex set Q'_i. Consider the forest F obtained from F' by replacing the vertex set Q'_i with $Q'_i \cup \{v\}$. By induction hypothesis G' is the comparability graph of F' whose vertices are the equivalence classes of vertices of G'. It follows that G is the comparability graph of F whose vertices are the equivalence classes of vertices of G.

Case 1.2: Forest F' has no isolated vertex. Consider the forest F that is obtained from F' by introducing a new isolated vertex $Q_v = \{v\}$. It can be seen that G is indeed the comparability graph of F whose vertices are the equivalence classes of vertices of G.

Case 2: Suppose operation (2) is applied on a weakly quasi-threshold graph G' to obtain G. Let v be the new vertex added which is made adjacent to all vertices of G'. In this case it can be seen that the equivalence classes of vertices of G can be obtained from the equivalence classes of vertices of G' by adding a new class $\{v\}$. Let F' be the forest associated to G' and put $Q_v = \{v\}$. Consider the forest F with $V(F) = V(F') \cup \{Q_v\}$ and $E(F) = E(F') \cup \{Q_vQ_{u_1}, \ldots, Q_vQ_{u_s}\}$, where Q_{u_i}'s are the roots of trees in F'. We can see that G is the comparability graph of F whose vertices are the equivalence classes of vertices of G.

Case 3: Suppose operation (3) is applied on two graphs G_1 and G_2 to obtain G. By Proposition 1.1, G_1 and G_2 are weakly quasi-threshold. Using induction hypothesis, let F_1 and F_2 be the forests corresponding to the graphs G_1 and G_2, respectively. Let F be the forest obtained by taking the disjoint union of F_1 and F_2, if F_1 or F_2 does not have an isolated vertex. If F_1 has an isolated vertex Q_1 and F_2 has an isolated vertex Q_2, then consider the forest $F_2 - Q_2$ which is obtained by deleting the vertex Q_2 from F_2. Take F to be the forest obtained from the disjoint union of F_1 and $F_2 - Q_2$ and replacing the independent vertex set Q_1 by $Q_1 \cup Q_2$. It can be verified that G is the comparability graph of F whose vertices are the equivalence classes of vertices of G.

Case 4: Suppose operation (4) is applied on a weakly quasi-threshold graph G' to obtain G. Let v be the new vertex added which is made adjacent to the neighbors $N(u)$

of a vertex u in G'. Let F' be the forest associated to G' and Q'_u be an independent vertex set of F' containing u. Then the forest F corresponding to G can be obtained from F' by just replacing the independent vertex set Q'_u with $Q_u = Q'_u \cup \{v\}$. We see that the vertices of F are precisely the equivalence classes of vertices of G.

Therefore, in each case, we have obtained a rooted forest F such that the comparability graph of F is G and the vertices of F are precisely the equivalence classes of vertices of G. ∎

REMARK **1.4** Notice that if Q is an equivalence class in a weakly quasi-threshold graph G and G' is obtained from G by adding new vertices applying (1)–(4) sequentially, then there is an equivalence class Q' for G' such that $Q \subseteq Q'$.

Theorem 1.5. *Let G be the comparability graph of a forest F of rooted trees with independent sets $Q_1, \ldots, Q_r$ as vertices. Then G is weakly quasi-threshold.*

Proof. Note that in view of Proposition 1.1, it is enough to prove this for a rooted tree with independent sets $Q_1, \ldots, Q_r$ as vertices. We employ induction on r. For $r = 1$ the assertion is trivial. Assume that the assertion holds for $r \leq k$ and let T be a rooted tree with independent sets $Q_1, \ldots, Q_{k+1}$ as vertices with $Q_1 = \{v_{11}, \ldots, v_{1n_1}\}$ as the root. Consider the forest $F = T - Q_1$. It consists of rooted trees with independent sets as vertices. By induction hypothesis the comparability graph of each of these rooted trees is weakly quasi-threshold. By Proposition 1.1, their disjoint union H is weakly quasi-threshold. Thus H can be obtained from the empty graph by recursively applying a sequence of operations (1)–(4). We extend that sequence as described below.

- Apply operation (2) with new vertex v_{11}.
- For $i = 2, \ldots, n_1$, apply operation (4) with new vertex v_{1i} and old vertex v_{11}.

So the comparability graph of T is weakly quasi-threshold and the proof is complete. ∎

Theorem 1.6. *Let G be a graph and let $Q_1, \ldots, Q_k$ be the equivalence classes under $\sim$. Suppose that $|Q_1| > 1$ and let $v \in Q_1$ be a vertex. Then G is weakly quasi-threshold if and only if $G - v$ is weakly quasi-threshold.*

Proof. Let G be weakly quasi-threshold. By Theorem 1.3, there is a forest F of rooted trees with the independent sets $Q_1, \ldots, Q_k$ as vertices such that G is the comparability graph of F. It is clear that $G - v$ is the comparability graph of F', where F' is obtained from F by replacing Q_1 with $Q_1 \setminus \{v\}$. Hence $G - v$ is weakly quasi-threshold, by Theorem 1.5.

Conversely, suppose that $G - v$ is weakly quasi-threshold. Note that $Q_1 \setminus \{v\}, Q_2, \ldots, Q_k$ are the equivalence classes of vertices of $G - v$ under the relation $\sim$. By Theorem 1.3, there is a forest F' of rooted trees with the independent sets $Q_1 \setminus \{v\}, Q_2, \ldots, Q_k$ as vertices

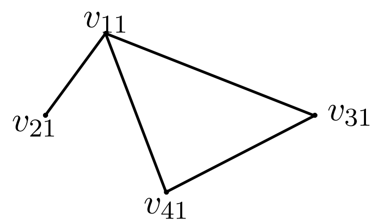

Fig. 2. A subgraph of representatives of the graph in Figure 1.

such that $G - v$ is the comparability graph of F'. Obtain F from F' by replacing $Q_1 \setminus \{v\}$ by Q_1. Then G is the comparability graph of F and hence it is weakly quasi-threshold. ■

REMARK **1.7** An extension of Theorem 1.6 is the following. *Suppose that G is a weakly quasi-threshold graph and v is any vertex. Then $G - v$ is weakly quasi-threshold.* To see this, in view of Theorem 1.6, let $Q_1 = \{v\}, Q_2, \cdots Q_k$ be the equivalence classes and F be the forest of rooted trees with the independent sets $Q_1, \ldots, Q_k$ as vertices such that G is the comparability graph of F. If Q_1 has no ancestors or no predecessors, it means v is an isolated vertex and hence it follows that $G - v$ is the comparability graph of $F - Q_1$. For the other case, let Q_2 be the nearest ancestor of Q_1 (parent) and $Q_3, \cdots, Q_r$ be the children of Q_1. Then consider the forest F' obtained from $F - Q_1$ by adding edges Q_2Q_i, for $i = 3, \cdots, r$. It is easy to see that the comparability graph of F' is $G - v$. *Thus an induced subgraph of a weakly quasi-threshold graph is a weakly quasi-threshold graph.* That is, the class of weakly quasi-threshold graphs is *hereditary.*

Definition. Let G be a graph with $Q_1, \ldots, Q_k$ as the equivalence classes under the relation $\sim$. For each $i = 1, \ldots, k$ choose a vertex $u_i \in Q_i$. We call the subgraph $\tilde{G}$ of G induced by $u_1, \ldots, u_k$ as a *subgraph of representatives* of G.

For example, consider the graph G in Figure 1. A subgraph of representatives of G is shown in Figure 2.

REMARK **1.8** Let G be a weakly quasi-threshold graph. Using Theorem 1.3, any subgraph of representatives of G is a quasi-threshold graph.

COROLLARY **1.9** *Let G be a graph. Then G is weakly quasi-threshold if and only if a subgraph of representatives is quasi-threshold.*

Proof. Follows immediately from Theorem 1.6 and Remark 1.8. ■

In view of the algorithm given in Yan et al. (see [16]) and Corollary 1.9 we can give an algorithm for recognizing weakly quasi-threshold graphs. This algorithm has running time $O(n^2)$ where n is the number of vertices.

Algorithm WQT: Test whether a graph is weakly quasi-threshold.

Input: A graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$.

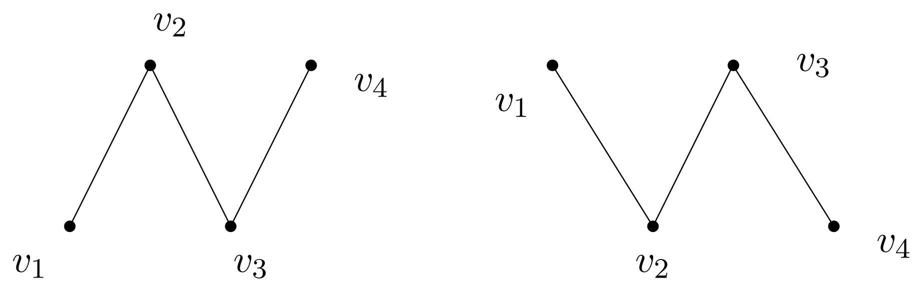

Fig. 3. Possible configurations of v_1, v_2, v_3, and v_4

Output: If G is weakly quasi-threshold, the output is a rooted forest F whose comparability graph is G, otherwise output is "no".
Method: Each $N(v_i)$ may be viewed as a 0-1 string of length at most n. We apply radix sort to sort them in $O(n^2)$ time. With another n^2 comparisons we get the equivalence classes $Q_1, \ldots, Q_k$ of vertices of G under the relation $\sim$. To generate the subgraph G_1 of representatives we require k^2 time. Now we use the algorithm of [16] to test whether G_1 is quasi-threshold or not. If it is not then we conclude that G is not weakly quasi-threshold. Otherwise we get a forest, say F_1, such that G_1 is the comparability graph of F_1. Let F be obtained from F_1 by replacing each vertex u_i in F_1 by the corresponding Q_i. Then the comparability graph of F is G.

We end this section, by proving the following equivalent condition for a graph G to be weakly quasi-threshold.

Theorem 1.10. *Let G be a connected graph. Then the following are equivalent.*

(a) G is a weakly quasi-threshold.
(b) G is a cograph and there is no induced $C_4 = [v_1, v_2, v_3, v_4]$ whenever $N(v_1) \neq N(v_3)$ and $N(v_2) \neq N(v_4)$.

Proof. To prove (a) implies (b), let G be weakly quasi-threshold and $Q_1, \ldots, Q_k$ be the equivalence classes of vertices of G under $\sim$. By Theorem 1.3, G is the comparability graph of a rooted tree T with independent vertex sets $Q_1, \ldots, Q_k$.

Suppose that G has an induced $P_4 = [v_1, v_2, v_3, v_4]$. It follows that these four points satisfy the following conditions in T.

(i) Either v_2 is an ancestor of v_1, v_3 or v_2 is a descendant of v_1, v_3.
(ii) Either v_3 is an ancestor of v_2, v_4 or v_3 is a descendant of v_2, v_4.

Hence we see that *either* (i) v_2 is an ancestor of v_1, v_3 and v_3 is a descendant of v_4, *or* (ii) v_2 is a descendant of v_1, v_3 and v_3 is an ancestor of v_4, holds. (See Figure 3.)

In case of the configuration on the left, if v_2 and v_4 belong to different equivalence classes, considering the fact that the root is an ancestor of each vertex, we get that T

cannot be a tree. Hence, v_2 and v_4 belong to the same equivalence class and so the edge v_1v_4 must be present in G. This results in a contradiction to our assumption that G has an induced subgraph isomorphic to $P_4 = [v_1, v_2, v_3, v_4]$.

In case of the configuration on the right, we get a similar contradiction. Thus G cannot have an induced P_4.

Now suppose that G has an induced $C_4 = [v_1, v_2, v_3, v_4]$. Proceeding in a similar way as above, we see that in the case of the configuration on the left, v_2 and v_4 are in the same equivalence class, that is, $N(v_2) = N(v_4)$. Similarly, in case of the configuration on the right, $N(v_1) = N(v_3)$.

To show (b) implies (a), let G be a connected graph satisfying (b) and $Q_1, \ldots, Q_k$ be the equivalence classes of vertices of G under $\sim$. It follows that a subgraph of representatives of G does not have an induced P_4 and does not have an induced C_4. Hence the subgraph of representatives is quasi-threshold. By Corollary 1.9, G is weakly quasi-threshold. ■

2. Laplacian Spectrum of weakly quasi-threshold graphs

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Let $D(G) = \mathrm{diag}(\deg_G(v_1), \deg_G(v_2), \ldots, \deg_G(v_n))$ be the diagonal matrix of vertex degrees. The *Laplacian matrix* is defined by $L(G) = D(G) - A(G)$, where $A(G)$ is the $(0,1)$-adjacency matrix of G. It is well-known that $L(G)$ is a positive semidefinite matrix and 0 is an eigenvalue of $L(G)$ with the vector of all 1's as the corresponding eigenvector. Also, 0 is an eigenvalue of $L(G)$ with multiplicity one if and only if G is connected. To know some interesting facts about Laplacian matrices and its eigenvalues we refer the reader to [11,13]. The set consisting of all eigenvalues of $L(G)$ is called the *Laplacian spectrum* of G and a graph G is called *Laplacian integral* if all the eigenvalues of $L(G)$ are integers. The readers can refer to [9,12] for a few papers that study the Laplacian integrability of certain graphs.

Let G be a connected weakly quasi-threshold graph. Then G is the comparability graph of a rooted tree T on independent vertex sets $Q_1, Q_2, \ldots, Q_k$ with Q_1 as the root vertex set. For $1 \leq i \leq k$, let $n_i = |Q_i|$, $c_i = |\mathrm{child}(Q_i)|$ and $d_i = |\mathrm{des}(Q_i)|$. For the sake of clarity, we assume that $n_i > 1$ for each non-pendant vertex set Q_i, where we recall that a vertex v of a tree T is called a *pendant* vertex if $\deg_T(v) = 1$. Let us assume that the number of non-pendant independent vertex sets in G is ℓ.

Let $u_1, u_2, \ldots, u_k$ be the vertex representatives of the vertex sets $Q_1, Q_2, \ldots, Q_k$, respectively. Consider the graph $\tilde{G}$, the subgraph of representatives of G, and the corresponding rooted tree $\tilde{T}$ such that $\tilde{G}$ is the comparability graph of $\tilde{T}$. In the rooted tree $\tilde{T}$, let m_i for $2 \leq i \leq k$, denote the distance of the vertex u_i from the root vertex u_1. Then, there exist vertices u_{i_j}'s such that $u_1 = u_{i_1}, u_{i_2}, \ldots, u_{i_{m_i}}, u_{i_{m_i+1}} = u_i$ is the path from the vertex u_1

to the vertex u_i As the vertices $u_1 = u_{i_1}, u_{i_2}, \ldots, u_{i_{m_i+1}} = u_i$ are respectively, the vertex representatives of the independent vertex sets $Q_1 = Q_{i_1}, Q_{i_2}, \ldots, Q_{i_{m_i+1}} = Q_i$, we use the corresponding n_{i_j}'s to define $\tilde{n}_i = \sum_{j=1}^{m_i+1} n_{i_j}$. We also define $\widehat{n}_i = \sum_{j=1}^{i} n_j$. Note that $\widehat{n}_i$'s are defined to take care of the order of the sub-matrices that appear in the Laplacian matrix.

In this section, we write $\mathbf{1}_n$ to denote an n-tuple of all 1's, $\mathbf{e}_i$ to denote the vector which has 1 at the i^{th} position and zero elsewhere, $J_{m,n}$ is a matrix of all 1's of order $m \times n$. With the notations defined above, we state our main theorem of this section, which gives the exact information about the eigenvalues of any weakly quasi-threshold graph. In particular, we show that the eigenvalues of weakly quasi-threshold graphs are integers.

Theorem 2.1. *Let G be a connected weakly quasi-threshold graph on n vertices. Let G be the comparability graph of a rooted tree T with $Q_1, Q_2, \ldots, Q_k$ as the independent vertex sets with $n_i = |Q_i|$ for $1 \le i \le k$. Suppose Q_1 is the root of T. Let $L(G)$ be the Laplacian matrix of the graph G. If u_i is the representative of the independent vertex set Q_i, $1 \le i \le k$, then the non-zero eigenvalues of $L(G)$ are the following:*

1. $\deg_G(u_i)$, *repeated $n_i - 1$ times for each i, $1 \le i \le k$,*
2. $\deg_G(u_i) + n_i$, *repeated exactly once for each non-pendant vertex set Q_i, and*
3. $\tilde{n}_i$, *repeated $c_i - 1 = |child(Q_i)| - 1$ times for each non-pendant vertex set Q_i.*

Proof. As mentioned above, 0 is an eigenvalue of $L(G)$ with corresponding eigenvector $\mathbf{1}_n$. As the graph G is connected, there will be exactly $n-1$ non-zero eigenvalues of $L(G)$. Now note that $n = \sum_{i=1}^{k} n_i$ and therefore the total number of non-zero eigenvalues of the form $\deg_G(u_i)$, counted with multiplicity is $n-k$ (listed in the first part). Also, observe that the total number of eigenvalues in the second and the third list add up to $k-1$ (for each non-pendant independent vertex set Q_i with $c_i = |\text{child}(Q_i)|$, the number of eigenvalues to the total sum is $1 + (c_i - 1) = c_i$ and there are $k-1$ vertex independent sets excluding Q_1).

We now describe the Laplacian matrix of a weakly quasi-threshold graph G. It is easy to observe that the Laplacian matrix $L(G)$ is a block matrix, with blocks of sizes $n_1, n_2, \ldots, n_k$. For $1 \le i, j \le k$, the (i,i)-block is $\deg_G(u_i) I_{n_i}$ and the (i,j)-block is the zero matrix of order $n_i \times n_j$ if (u_i, u_j) is not an edge in G and is the matrix $-J_{n_i,n_j}$ if (u_i, u_j) is an edge in G.

The proof is based on giving the eigenspaces for each of the eigenvalues mentioned in the statement of the theorem. Let $\mathbf{y}$ be an eigenvector for an eigenvalue λ. In the list below, we write $\mathbf{y}$ in the block form, with $\mathbf{y}(Q_i)$ representing the i^{th} block corresponding to the vertex independent set Q_i. Also, recall that $|Q_i| = n_i$ for $1 \le i \le k$.

eigenvalue	eigenvector
$\deg_G(u_i)$	$\mathbf{e}_{\widehat{n_{i-1}}+1} - \mathbf{e}_{\widehat{n_{i-1}}+2}, \ldots, \mathbf{e}_{\widehat{n_{i-1}}+1} - \mathbf{e}_{\widehat{n_i}}$ $\quad (n_i - 1$ in number).
$\deg_G(u_i) + n_i$	Let $\operatorname{des}(Q_i) = \{Q_{i_1}, Q_{i_2}, \ldots, Q_{i_{d_i}}\}$. Then the non-zero blocks of the eigenvector $\mathbf{y}$ are: $\mathbf{y}(Q_i) = -\Big(\sum_{j=1}^{d_i} n_{i_j}\Big)\mathbf{1}_{n_i},\ \ \mathbf{y}(Q_{i_j}) = n_i \mathbf{1}_{n_{i_j}},\ 1 \le j \le d_i.$
$\tilde{n}_i$	Let $\operatorname{child}(Q_i) = \{Q_{i_1}, Q_{i_2}, \ldots, Q_{i_{c_i}}\}$ with $F_{i_1}, \ldots, F_{i_{c_i}}$ as there respective families. Then the non-zero blocks of the eigenvector $\mathbf{y}_{i_j},\ j \neq 1$ are: $\mathbf{y}_{i_j}(v) = \begin{cases} -\lvert F_{i_j}\rvert & \text{if } v \in F_{i_1} \\ \lvert F_{i_1}\rvert & \text{if } v \in F_{i_j} \\ 0 & \text{otherwise} \end{cases} \quad (c_i - 1 \text{ in number}).$

Observe that all the vectors $\mathbf{y}$ given above are orthogonal to the vector $\mathbf{1}_n$. We now prove that the vectors $\mathbf{y}$'s are indeed eigenvectors corresponding to the given eigenvalues. As the vertex independent set Q_1 is the root, we have

$$L(G) = \left[\begin{array}{c|c|c|c|c} \deg_G(u_1) I_{n_1} & -J_{n_1,n_2} & -J_{n_1,n_3} & \cdots & -J_{n_1,n_k} \\ \hline -J_{n_2,n_1} & \deg_G(u_2) I_{n_2} & \star & \cdots & \star \\ \hline \star & \star & \deg_G(u_3) I_{n_3} & \cdots & \star \\ \hline \vdots & \star & \star & \ddots & \vdots \\ \hline -J_{n_k,n_1} & \star & \star & \cdots & \deg_G(u_k) I_{n_k} \end{array}\right] \tag{2.1}$$

where the i^{th} block corresponds to the vertex independent set Q_i and the entry $\star$ indicates the presence of either the submatrix $-J_{n_i,n_j}$ or $\mathbf{0}_{n_i,n_j}$ for appropriate choices of n_i and n_j. As there are three types of eigenvalues, we need to consider them separately. We give the proof of the first two types. The proof for the third type is similar to the proof of the second type and hence we omit it. From (2.1), we note that we need to consider the matrix product $J_{n_i,n_j}\mathbf{1}_{n_j}$ which equals $n_j\mathbf{1}_{n_i}$.

Type 1: The vectors $\mathbf{y}$ for $1 \le i \le k$ are of the form $\mathbf{e}_{\widehat{n_{i-1}}+1} - \mathbf{e}_{\widehat{n_{i-1}}+s}$ for $2 \le s \le n_i$. That is, the blocks $\mathbf{y}(Q_j)$ are given by

$$\mathbf{y}(Q_j) = \begin{cases} \mathbf{e}_1 - \mathbf{e}_s & \text{if } j = i, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Note that $L(G)\mathbf{y} = \deg_G(u_i)\mathbf{y}$ as

$$I_{n_i}(\mathbf{e}_1 - \mathbf{e}_s) = \mathbf{e}_1 - \mathbf{e}_s, \quad -J_{n_j,n_i}(\mathbf{e}_1 - \mathbf{e}_s) = \mathbf{0}, \quad \text{and} \quad \mathbf{0}(\mathbf{e}_1 - \mathbf{e}_s) = \mathbf{0}.$$

Type 2: The vectors $\mathbf{y}$ for $1 \le i \le k$ are of the form

$$\mathbf{y}(Q_i) = -\Big(\sum_{j=1}^{d_i} n_{i_j}\Big)\mathbf{1}_{n_i},\ \ \mathbf{y}(Q_{i_j}) = n_i\mathbf{1}_{n_{i_j}},\ 1 \le j \le d_i$$

where $n_{i_j} = |Q_{i_j}|$ for $1 \le j \le d_i$ and $\mathrm{des}(Q_i) = \{Q_{i_1}, Q_{i_2}, \ldots, Q_{i_{d_i}}\}$. The i^{th} block of $L(G)\mathbf{y}$ is given by

$$\begin{aligned}(L(G)\mathbf{y})_i &= \deg_G(u_i) I_{n_i} \Big(-\sum_{j=1}^{d_i} n_{i_j}\Big)\mathbf{1}_{n_i} + \sum_{j=1}^{d_i} -J_{n_i,n_{i_j}} \cdot n_i \mathbf{1}_{n_{i_j}} \\ &= -\Big(\deg_G(u_i)\sum_{j=1}^{d_i} n_{i_j}\mathbf{1}_{n_i} + n_i \sum_{j=1}^{d_i} n_{i_j}\mathbf{1}_{n_i}\Big) \\ &= (\deg_G(u_i) + n_i)(-\sum_{j=1}^{d_i} n_{i_j}\mathbf{1}_{n_i}) = (\deg_G(u_i) + n_i)\mathbf{y}(Q_i).\end{aligned}$$

We now compute the j^{th} block of $L(G)\mathbf{y}$. For doing so, note the following:
1. If the vertex independent set $Q_s \in \mathrm{des}(u_{i_j})$ for some $j,\ 1 \le j \le d_i$, then $Q_s \in \mathrm{des}(u_i)$,
2. $\deg_G(u_i) = |\mathrm{anc}(u_i)| + |\mathrm{des}(u_i)|$ and
3. $|\mathrm{anc}(u_{i_j})| = |\mathrm{anc}(u_i)| + n_i$ for $1 \le j \le d_i$.

Hence, for $1 \le j \le d_i$ the j^{th} block of $L(G)\mathbf{y}$ equals

$$\begin{aligned}&-J_{n_{i_j},n_i}\Big(-\sum_{j=1}^{d_i} n_{i_j}\Big)\mathbf{1}_{n_i} + \deg_G(u_{i_j}) I_{n_{i_j}} \cdot n_i \mathbf{1}_{n_{i_j}} + \sum_{Q_s \in \mathrm{des}(u_{i_j})} -J_{n_{i_j},n_s} \cdot n_i \mathbf{1}_{n_s} \\ &= n_i(\sum_{j=1}^{d_i} n_{i_j})\mathbf{1}_{n_{i_j}} + n_i \deg_G(u_{i_j})\mathbf{1}_{n_{i_j}} - n_i|\mathrm{des}(u_{i_j})|\mathbf{1}_{n_{i_j}} \\ &= (\sum_{j=1}^{d_i} n_{i_j} + \deg_G(u_{i_j}) - |\mathrm{des}(u_{i_j})|) \cdot n_i \mathbf{1}_{n_{i_j}} \\ &= (\deg_G(u_i) - |\mathrm{anc}(u_i)| + |\mathrm{anc}(u_{i_j})| + |\mathrm{des}(u_{i_j})| - |\mathrm{des}(u_{i_j})|) \cdot n_i \mathbf{1}_{n_{i_j}} \\ &= (\deg_G(u_i) - |\mathrm{anc}(u_i)| + |\mathrm{anc}(u_i)| + n_i) \cdot n_i \mathbf{1}_{n_{i_j}} \\ &= (\deg_G(u_i) + n_i)\mathbf{y}(Q_{i_j}).\end{aligned}$$

Therefore, in this case $L(G)\mathbf{y} = (\deg_G(u_i) + n_i)\mathbf{y}$. A similar reasoning gives the proof for the third case. ■

Note that the above theorem can be easily generalized to a disconnected weakly quasi-threshold graph. As an immediate corollary of Theorem 2.1, we obtain all the eigenvalues of the Laplacian matrix of a quasi-threshold graph. The proof of the Corollary is omitted as it follows from observing that for quasi-threshold graphs, $n_i = 1$ and $\tilde{n}_i = m_i + 1$ for all $i,\ 1 \le i \le k$.

Corollary **2.2** *Let G be a connected quasi-threshold graph. Suppose G is the comparability graph of a rooted tree T with vertices $u_1, u_2, \ldots, u_k$. Then the nonzero eigenvalues of G are*

1. $\deg_G(u_i) + 1$, *repeated exactly once for each non-pendant vertex u_i, and*
2. $m_i + 1$, *repeated $c_i - 1$ times for each non-pendant vertex u_i.*

We are now ready to determine a formula for the number of spanning trees for weakly quasi-threshold graphs. In particular, this formula when restricted to quasi-threshold graphs improves Theorem 4.1 of Nikolopoulos and Papadopoulos [14] as their formula is recursive in nature. To do so, we start with some definitions and known results related with the spanning tree of a graph.

For a graph G on n vertices, a *spanning tree* of G is a connected, acyclic, spanning subgraph of G. For any subgraph H of a complete graph K_N on N vertices, $K_N - H$ denotes the graph obtained from K_N by removing the edges of H. Observe that if H is also a subgraph on N vertices, then $K_N - H$ coincides with $\bar{H}$, the *complement* of H. In [14], Nikolopoulos and Papadopoulos give a recursive formula for computing the number of spanning trees in K_N-complements of quasi-threshold graphs. The computation of the number of spanning trees of a graph is a well-studied problem. As an application of our result on eigenvalues of quasi-threshold and weakly quasi-threshold graphs, we give a closed form formula for obtaining the number of spanning trees for such graphs and also for their K_N-complements. We now state a well-known lemma which helps us in computing the spanning trees of all the graphs mentioned above.

LEMMA **2.3** [8] *Let G be a connected graph on n vertices and let $L(G)$ be its Laplacian matrix. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L(G)$. Then the number of spanning trees of G equals the product* $\frac{\prod_{i=2}^{n} \lambda_i}{n}$.

Now, using Theorem 2.1, Corollary 2.2 and Lemma 2.3, we have the following.

Theorem 2.4. *Let G be a connected weakly quasi-threshold graph on n vertices. Suppose G is the comparability graph of a rooted tree T with $Q_1, Q_2, \ldots, Q_k$ as the independent vertex sets and $u_1, u_2, \ldots, u_k$ as their respective representatives. Also, let Q_1 be the root vertex of T. Then the number of spanning trees of G equals*

$$\prod_{i=1}^{k}(\deg_G(u_i))^{n_i-1}\prod_{i=1}^{\ell}(\deg_G(u_i)+n_i)\prod_{i=1}^{\ell}(\tilde{n}_i)^{c_i-1},$$

where $u_1, u_2, \ldots, u_\ell$ correspond to the non-pendant vertices.

In particular, for a connected quasi-threshold graph G, the number of spanning trees equals

$$\prod_{i=1}^{\ell}(\deg_G(u_i)+1)\prod_{i=1}^{\ell}(m_i+1)^{c_i-1}.$$

Recall that $L(K_N)$, the Laplacian matrix of K_N, has 0 as an eigenvalue with eigenvector $\mathbf{1}_N$, and N as an eigenvalue with eigenspace equal to the orthogonal complement of $\mathbf{1}_N$. That is, N is an eigenvalue with multiplicity $N-1$ and any vector orthogonal to $\mathbf{1}_N$ is an eigenvector corresponding to N. Since the eigenvectors of a weakly quasi-threshold graph corresponding to the non-zero eigenvalues are orthogonal to $\mathbf{1}_N$, we can easily compute all the eigenvalues of K_N-complements of weakly quasi-threshold graphs. The above observations, along with Theorem 2.1 and Corollary 2.2, give us the following theorem.

Theorem 2.5. *Let G be a connected weakly quasi-threshold graph on n vertices with $Q_1, Q_2, \ldots, Q_k$ as the independent vertex sets in the rooted tree T and $u_1, u_2, \ldots, u_k$ as their respective representatives. Then the number of spanning trees of $K_N - G$ equals*

$$N^{N-1-n} \prod_{i=1}^{k} (N - \deg_G(u_i))^{n_i - 1} \prod_{i=1}^{\ell} (N - \deg_G(u_i) - n_i) \prod_{i=1}^{\ell} (N - \tilde{n}_i)^{c_i - 1},$$

where $u_1, u_2, \ldots, u_\ell$ correspond to the non-pendant vertices.

In particular, for a connected quasi-threshold graph G, the number of spanning trees of $K_N - G$ equals

$$N^{N-1-n} \prod_{i=1}^{\ell} (N - \deg_G(u_i) - 1) \prod_{i=1}^{\ell} (N - m_i - 1)^{c_i - 1}.$$

Theorem 2.5 gives a closed formula for the number of spanning trees of weakly quasi-threshold graphs. This formula when restricted to quasi-threshold graphs improves Theorem 4.1 of Nikolopoulos and Papadopoulos [14].

Example **2.6** *1. Consider the rooted tree T and its weakly quasi-threshold graph G given in Figure 1. The nonzero eigenvalues of G are* $8, 2, 2, 5, 4, 4, 10, 7, 2$. *Hence, the number of spanning trees of $K_N - G$ is*

$$N^{N-11}(N-2)^3(N-4)^2(N-5)(N-7)(N-8)(N-10).$$

2. Consider c-split graphs [14]. These graphs are quasi-threshold and can be obtained as comparability graphs of rooted trees whose description is given below:
Let T be a rooted tree on $p = k + s$ vertices with each of the vertices $v_1, v_2, \ldots, v_{k-1}$ having exactly one child and let the number of children of the vertex v_k be s (see Figure 4). Suppose G is the comparability graph of T. Then G is a quasi-threshold graph with 0 *as an eigenvalue repeated exactly once, p as an eigenvalue repeated k times, and k as an eigenvalue repeated $s - 1$ times. Hence, the number of spanning trees of the graph $K_N - G$ is given by*

$$N^{N-1-p}(N-k)^{s-1}(N-p)^k.$$

Note that this formula coincides with the corresponding formula in [14].

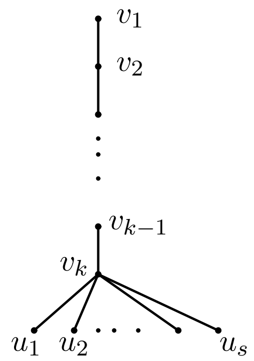

Fig. 4. a c-split quasi-threshold graph

3. Grone-Merris conjecture holds for cographs

In the third section, we have shown that weakly quasi-threshold graphs are Laplacian integral. We now state a conjecture of Grone and Merris (see [6]) and prove a result (Theorem 3.1) that may help in proving the conjecture for quasi-threshold graphs. To do so, we assume that all the graphs in this section will be simple and undirected.

For a graph G on n vertices, let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ with $d_i \geq d_{i+1}$ for $1 \leq i \leq n-1$ be the degree sequence of G and let $\mathbf{d}^* = (d_1^*, d_2^*, \ldots, d_n^*)$, where $d_i^* = |\{j : d_j \geq i\}|$ be the conjugate degree sequence of G. Observe that the condition that G is a simple graph on n vertices, implies that $d_n^*(G) = 0$. Also, let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ be the Laplacian spectrum of G. Before stating the conjecture, we need the following definition.

Let $\mathbf{s} = \{s_1, s_2, \ldots, s_n\}$ and $\mathbf{t} = \{t_1, t_2, \ldots, t_n\}$ be two non-increasing sequences of real numbers. Then the sequence $\mathbf{s}$ is said to *majorize* the sequence $\mathbf{t}$, denoted $\mathbf{s} \succ \mathbf{t}$ if

$$s_1 + s_2 + \cdots + s_i \geq t_1 + t_2 + \cdots + t_i, \quad \text{for } 1 \leq i \leq n-1 \text{ and } \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} t_i.$$

Since the Laplacian matrix is symmetric and positive semidefinite, its spectrum majorizes its main diagonal [10, p. 218]. Hence, $\Lambda \succ \mathbf{d}$. Grone and Merris (see [6]) noted that $\mathbf{d}^* \succ \mathbf{d}$ and made the following conjecture:
Let G be a connected graph. Then $\mathbf{d}^* \succ \Lambda$.

This conjecture has been studied by several researchers and a few partial results are known (for example, see [1,12]). We now prove that this conjecture holds true for all cographs. To begin with, we prove two lemmas.

Theorem 3.1. *Let G be a quasi-threshold graph on n vertices. Let $d = (d_1, d_2, \ldots, d_n)$ be the degree sequence of G and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the sequence of eigenvalues of $L(G)$. Then for $1 \leq i \leq f(d)$, $\lambda_i = d_i + 1$, where $f(d) = |\{j : d_j \geq j\}|$ is known as the Durfee number of d.*

Proof. Suppose G is the comparability graph of a rooted tree T on vertices $u_1, u_2, \ldots, u_n$ with u_1 as the root and $\deg_G(u_i) = d_i,\ 1 \le i \le n$. By Corollary 2.2, the eigenvalues of G are

1. $d_i + 1$ with multiplicity 1, for each non-pendant vertex u_i of T, and
2. $m_i + 1$ with multiplicity $c_i - 1$, for each non-pendant vertex u_i of T.

Suppose $f(d) = k$. Thus $d_i \ge k$ for $1 \le i \le k$ and $d_{k+1} < k+1$. On the contrary suppose there exists an

$$i \in \{1, 2, \ldots, k\} \text{ such that } \lambda_i \notin \{d_1+1, d_2+1, \ldots, d_k+1\}. \tag{3.2}$$

Thus $\lambda_i = m_j + 1$ for some non-pendant vertex u_j. That is, $\lambda_i \ge d_k + 1 \ge k+1$. This means that the non-pendant vertex u_j is at a height/distance of at least k from the root u_1. But in that case, there are at least k vertices $u_{j_1}, u_{j_2}, \ldots, u_{j_k}$ that lie on the path from u_1 to u_j. And all these vertices u_{j_r}'s are non-pendant vertices with $\deg(u_{j_r}) \ge m_j$ for all $r,\ 1 \le r \le k$. This implies that we have already obtained k eigenvalues $\deg(u_{j_1})+1, \deg(u_{j_2})+1, \ldots, \deg(u_{j_k})+1$ all of which are greater than or equal to m_j+1. This contradicts (3.2). Thus the result follows. ■

LEMMA **3.2** *Let H be a graph on n vertices and let G be its complement graph. Let $\Lambda(H)$ (resp. $\Lambda(G)$) and $\mathbf{d}^*(H)$ (resp. $\mathbf{d}^*(G)$) be the spectral sequence and the conjugate degree sequence, respectively of the graph H (resp. G). Then $\mathbf{d}^*(G) \succ \Lambda(G)$ whenever $\mathbf{d}^*(H) \succ \Lambda(H)$.*

Proof. Suppose $\mathbf{d}^*(H) \succ \Lambda(H)$. So, for $1 \le i \le n-1$
$d_1^*(H) + d_2^*(H) + \cdots + d_i^*(H) \ge \lambda_1(H) + \lambda_2(H) + \cdots + \lambda_i(H)$ and $\sum\limits_{i=1}^{n} \lambda_i(H) = \sum\limits_{i=1}^{n} d_i^*(H)$.
Or equivalently, for $0 \le p \le n-1$,

$$\sum_{k=n-p}^{n} d_k^*(H) \le \sum_{k=n-p}^{n} \lambda_k(H), \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i(H) = \sum_{i=1}^{n} d_i^*(H). \tag{3.3}$$

As G is the complement of H, $L(G) + L(H) = nI - J$, where J is the matrix of all 1's. Hence, it follows that $\lambda_i(G) = n - \lambda_{n-i}(H)$ for $1 \le i \le n-1$. Also, $d_i(G) = n - 1 - d_i(H)$ for $1 \le i \le n$ as $G = \bar{H}$. Therefore, for $1 \le k \le n-1$,

$$\begin{aligned} d_k^*(G) = |\{i : d_i(G) \ge k\}| = {} & |\{i : n-1-d_i(H) \ge k\}| \\ = |\{i : d_i(H) \le n-k-1\}| = {} & n - |\{i : d_i(H) > n-k-1\}| \\ = n - |\{i : d_i(H) \ge n-k\}| & \\ = n - d_{n-k}^*(H). & \end{aligned}$$

As $d_n^*(G) = d_n^*(H) = 0$ and $\lambda_n(G) = \lambda_n(H) = 0$, for $1 \le p \le n-1$, we get

$$\sum_{k=1}^{p} d_k^*(G) = \sum_{k=1}^{p} (n - d_{n-k}^*(H)) = np - \sum_{k=n-p}^{n-1} d_k^*(H) = np - \sum_{k=n-p}^{n} d_k^*(H)$$

$$\geq np - \sum_{k=n-p}^{n} \lambda_k(H) = np - \sum_{k=n-p}^{n-1} \lambda_k(H)$$
$$= \sum_{k=n-p}^{n-1} (n - \lambda_k(H)) = \sum_{k=1}^{p} \lambda_k(G).$$

It can easily be verified that $\sum_{k=1}^{n} d_k^*(G) = \sum_{k=1}^{n} \lambda_k(G)$. Hence, we have shown that $\mathbf{d}^*(G) \succ \Lambda(G)$ whenever $\mathbf{d}^*(H) \succ \Lambda(H)$. ∎

By Lemma 3.2, we see that if G is the complement graph of a graph H, then $d^*(G) \succ \Lambda(G)$ whenever $\mathbf{d}^*(H) \succ \Lambda(H)$. Also, it is known that if G is the disjoint union of two graphs G_1 and G_2 then $\mathbf{d}^*(G) \succ \Lambda(G)$ whenever both $\mathbf{d}^*(G_1) \succ \Lambda(G_1)$ and $\mathbf{d}^*(G_2) \succ \Lambda(G_2)$ hold true (see [10, p. 22]). It can be easily seen that the conjecture holds for the graph K_1 (the graph having a single vertex). Therefore, the Grone and Merris conjecture will hold true for any graph that is obtained from a single vertex by recursively applying the graph operations “complementation” and “disjoint union”.

It is well-known that such graphs are precisely the cographs defined after Theorem 0.1 (see [2] for several equivalent conditions for a graph to be a cograph). In fact, the term cograph is derived from “complement-reducible” graph. To be exact, a cograph may equivalently be defined as a simple graph defined by the following criteria (see [2])

1. K_1 is a cograph,
2. If G is a cograph, then so is its complement graph, and
3. If G_1 and G_2 are cographs, then so is their graph union $G_1 \cup G_2$.

Hence, using Lemma 3.2 and the above observation, we get the following theorem.

Theorem 3.3. *Let G be a cograph on n vertices. Let Λ and $\mathbf{d}^*$ be the spectral sequence and the conjugate degree sequence, respectively of the graph G. Then $\mathbf{d}^* \succ \Lambda$.*

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