Max-type Recursive Distributional Equations

by

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M.A. (University of California, Berkeley) 2000

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Statistics

in the

GRADUATE DIVISION
of the
UNIVERSITY OF CALIFORNIA, BERKELEY

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Spring 2003

The dissertation of Antar Bandyopadhyay is approved:

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Abstract

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In certain problems in a variety of applied probability settings (from probabilistic analysis of algorithms to mean-field statistical physics models), the central requirement is to solve a fixed point equation of the form $X \stackrel{d}{=} g\left(\left(\xi_i, X_i\right), i \geq 1\right)$, where $\left(\xi_i\right)_{i \geq 1}$ and $g\left(\cdot\right)$ are given and $\left(X_i\right)_{i \geq 1}$ are independent copies of X with unknown distribution. We call such an equation a *recursive distributional equation*. Exploiting the natural recursive structure one can associate a tree-indexed process with every solution, and such a process is called a *recursive tree process*. This process in some sense is a solution of an infinite system of recursive distributional equations.

The dissertation is devoted to the study of such fixed point equations and the associated recursive tree process when the given function $g\left(\cdot\right)$ is essentially a "maximum" or a "minimum" function. Such equations arise in the context of optimization problems and branching random walks. The present work mainly concentrates on the theoretical question of *endogeny* : the tree-indexed process being measurable with respect to the given i.i.d *innovations* $(\xi_{\mathbf{i}})$. We outline some basic general theory which is natural from a statistical physics point of view and then specialize to study some concrete examples arising from various different contexts.

Professor David J. Aldous
Dissertation Committee Chair

To Baba and Ma

Contents

Acknowledgments

I am most grateful to my thesis advisor Professor David J. Aldous whose encouragement and enthusiasm have been the central motivating factor for my academic life at Berkeley. He introduced me to various exciting research projects including this dissertation topic, provided me with his brilliant ideas and extremely valuable advise, and helped me in every respect. Undoubtedly I learned the most from discussing with him. His generosity and support will always be in my memory.

I am also very grateful to Professor Jim W. Pitman for his continuous encouragement and valuable advise. Many thanks to him for providing me with several important references which were very helpful for this dissertation work. I would also like to thank him for giving the course on "Combinatorial Stochastic Processes" in Spring 2002, which really helped me to learn and understand a lot of probability theory. I am very grateful to Professor Steven N. Evans for being such helpful person in every respect. My sincere thanks to him for reviewing the first draft of this dissertation very carefully and for pointing out some typing errors as well as a mistake in the proof of Lemma 13 which was corrected in the later version. I am also very grateful to Professor Alistair Sinclair for being such helpful outside member on my oral and dissertation committee.

Professor Yuval Peres deserves many thanks for his great spirit, enthusiasm and continuous encouragement. Many thanks to Professor Ani Adhikari for her support and for the good advise. Special thanks to Professor John Rice for his support and help, and also for giving me the opportunity to teach an upper division course on probability theory at Berkeley, which was undoubtedly a great experience.

The help, kindness and love I have received from all the staff members of the Statistics Department are unforgettable. Without their help it would have been impossible to finish this dissertation work. Special thanks to Angie for being such caring and immensely efficient graduate secretary. Many thanks to Judy, Pat, James, Jane, Norma and Sarah for their assistance and help.

It would be incomplete if I do not acknowledge the support and encouragement I have always received from the faculty of Indian Statistical Institute, Calcutta. Professors B. V. Rao, T. Krishnan, A. K. Roy, K. S. Vijayan, Arup Bose and S. M. Bendre deserve many thanks for their kindness and advice. I owe as much to them as what I owe to the faculty of UC, Berkeley.

It would have been impossible for me to adjust in this new country if I didn't have my friends, I would like to thank all of them for their friendship and love. Special thanks to Rajarshi and Rahul for being such great room-mates and for all the support I have received from them. My special thanks to Nilim for his great spirit, kindness and being such a good friend. Many thanks to Rahul, Rajarshi, Nilim and Gaurav, to Rajarshi Datta, Kiran and Lakshmi, and to Anindita and Rumana for all the cheerful time we have spent together watching movies, discussing world issues, playing Monopoly, dining out and on road trips. Special thanks to Anindita and Rumana for all the wonderful food they cooked for me. My special thanks to Rakhee for her great spirit and for the encouragement I received from her. Special thanks to all my friends from Indian Statistical Institute, especially to Krishanu, Saikat, Arnab, Sanjay and Tathagata for their friendship and for all the brilliant discussions we had over phone.

My fellow graduate students, especially Noam, Lea, Von-Bing, Dave, Jonathan, Apratim, Aiyou, Manjunath, Young, Ingileif, Alan, Katherina, Pawel, Jon, David, Vivian, Gabor and Xiaoyue made my graduate student life at Berkeley memorable, thank you all for everything. Special thanks to Noam and Lea for all the discussions we had on the corridors of Evans Hall. Special thanks to Xiaoyue for her help with computer programming, especially with S-plus. Many thanks to Von-Bing for providing the thesis format which really helped in typing this dissertation.

Finally I would like to express my gratitude to my wonderful family – to my father and mother and to my brothers. My parents taught me to think rationally and it was their love for learning which gave me confidence to pursue a research career. It is only because of their omnipresent support and love I have been able to come up to this stage without any obstacle. My special thanks to my twin brother Abakash who has always been a great friend and always supported me at every stage of my life. I am immensely grateful to my elder brother Akash who always encouraged me to study mathematics. His support and help has been a decisive factor in my life.

Berkeley, California *Antar Bandyopadhyay*
May 15, 2003

Chapter 1

Introduction

Let $\mathcal{P}(S)$ be the set of all probability measures on a *Polish* space S; in most of our examples, S will either be a discrete set, or the set of real numbers, or a subset of it. Suppose that we are given a joint distribution for a family of random variables say $(\xi_i, i \leq 1)$, and a S-valued function $g(\cdot)$ with some appropriate domain (we give a precise definition in Section 2.1 of Chapter 2).Then we can define a map $T : \mathcal{P}(S) \to \mathcal{P}(S)$ as follows.

$$T(\mu) \stackrel{d}{=} g\left(\left(\xi_i, X_i\right), i \geq 1\right), \tag{1.1}$$

where $(X_i)_{i \geq 1}$ are i.i.d samples from the distribution μ and are independent of the given family $(\xi_i)_{i \geq 1}$. Within this general framework one can ask about the *existence* and *uniqueness* of fixed points of the operator T, that is distributions μ such that

$$T(\mu) \stackrel{d}{=} \mu. \tag{1.2}$$

We will rewrite the fixed point equation (1.2) in terms of the random variables as

$$X \stackrel{d}{=} g\left(\left(\xi_i, X_i\right), i \geq 1\right) \quad \text{on} \quad S, \tag{1.3}$$

where we will implicitly assume the independence assumption. We call such an equation a *recursive distributional equation* (RDE). RDEs have arisen in a variety of settings, for example, in the study of the Galton-Watson branching process and related trees (see Section 1.1), characterization of probability distributions [19], probabilistic analysis of random algorithms with suitable recursive structures, in particular the study of Quicksort algorithm

[26, 27, 16] and the study of the `find` algorithm [12], in the study of branching random walk [9, 10], and in various statistical physics models on trees [4, 5], to name a few.

Perhaps the most well studied case is the case when $S = \mathbb{R}^+$ and g is of the form

$$g\left(\left(\xi_i, X_i\right), i \geq 1\right) = \sum_{i=0}^{N} \xi_i X_i, \tag{1.4}$$

where $N \leq \infty$ is a constant or a given random variable taking values in $\mathbb{Z}^+ = \{1, 2, \ldots; \infty\}$. Durrett and Liggett [14] studied this case extensively for non-random fixed N. The extension to random N have been developed by Liu [21, 22].

Our main interest is to study a subclass of RDEs, those involving "*max-type*" functions g. Such RDEs have arisen in the study of branching random walks [9], statistical physics models on trees, like *frozen percolation* process on trees [4], and also in the probabilistic study of combinatorial optimization and local weak convergence method [3].

This thesis mainly focuses on the study of some "max-type" RDEs. On the theoretical side we develop some general theory which is rather natural from a statistical physics point of view but may be less apparent from the algorithms viewpoint. We formalize RDEs in terms of tree-indexed processes which we call *recursive tree processes*(RTPs). In particular we define the endogeny property, that in a RTP the variables $X_{\mathbf{i}}$ are measurable functions of the basic driving tree-indexed innovation process $(\xi_{\mathbf{i}})$, and hence no external randomness is needed. We also set out some general facts to study the tail of a RTP. The rest of the work mainly devoted to carefully studying some particular examples of RDEs arising from various different contexts. The main interest has been to show existence, uniqueness and if possible to check the endogenous property of the associated RTP.

1.1 A Motivating Example

Consider a sub-critical/critical Galton-Watson branching process with progeny distribution N. To exclude the trivial cases we assume that $\mathbf{P}(N = 1) \neq 1$ and $\mathbf{P}(N = 0) \neq 1$. A classical result [8] shows that the branching process starting from one individual goes extinct a.s. and hence the random family tree of the individual is a.s. finite. So the random variable

$$H := \min\left\{ d \,\middle|\, \text{no individual in generation } d \right\} \tag{1.5}$$

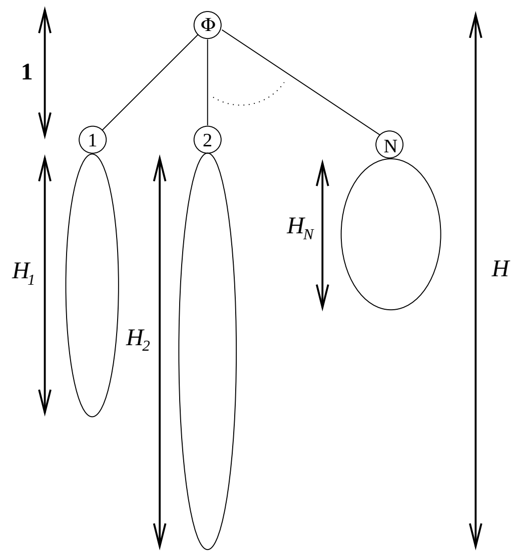


Figure 1.1: Recursion for H in a (sub)-critical Galton-Watson branching process

is well defined. Naturally $H = 1+$ height of the family tree.

Thus H satisfies the following simple RDE

$$H \stackrel{d}{=} 1 + \max\left(H_1, H_2, \ldots, H_N\right) \quad \text{on} \quad S = \{1, 2, \ldots\}, \tag{1.6}$$

where $H_1, H_2, \ldots$ are the i.i.d. copies of H which are heights of the family trees of the children of the first individual. Note that here we define maximum over an empty set (like $N = 0$ in (1.6)) as zero. The RDE (1.6) is a natural prototype for "max-type" RDEs and one of our main motivating examples. So although the following result is implicit in the classical literature [8, 6] we provide a proof which is a simple consequence of the above RDE.

Proposition 1 *Let N be a non-negative integer valued random variable with $\mathbf{E}\left[N\right] \leq 1$ and assume that $0 < \mathbf{P}\left(N = 0\right) < 1$. Let ϕ be the probability generating function of N. Then the RDE (1.6) has unique solution given by $\mathbf{P}\left(H \leq n\right) = \phi_n(0)$, for $n \geq 1$, where ϕ_n is the n-fold composition of ϕ.*

Proof : Let H be a solution of the RDE (1.6). Write $F(x) = \mathbf{P}\left(H \leq x\right)$, $x \in \mathbb{R}$ as the distribution function of H. Clearly $F(1) = \mathbf{P}(N = 0) = \phi(0)$. Fix $n \geq 2$ then

$$\begin{aligned} F(n) &= \mathbf{P}\left(\max\left(H_1, H_2, \ldots, H_N\right) \leq n-1\right) \\ &= \sum_{k=0}^{\infty} \mathbf{P}\left(N = k\right) \mathbf{P}\left(\max\left(H_1, H_2, \ldots, H_k\right) \leq n-1\right) \\ &= \sum_{k=0}^{\infty} \mathbf{P}\left(N = k\right) \left(F(n-1)\right)^k \\ &= \phi\left(F(n-1)\right) \end{aligned}$$

The rest follows by induction. □

It is interesting to note that the distribution of H is completely characterized by the distribution of N. In fact we will show later (using Lemma 8) that H is indeed a measurable function of the family tree.

1.2 Outline

In Chapter 2 we develop some basic general theory which formalizes RDEs in terms of *recursive tree processes* (RTPs). In particular we define endogeny property, bivariate uniqueness of 1st and 2nd kind and also study the tail of the RTPs. Chapters 3, 4 and 5 provides interesting examples of some "max-type" RDEs. In Chapter 3 we specialize to RDEs where the function g is of the form $g\left(\eta, \left(\xi_i, X_i\right), i \geq 1\right) = \eta + \max_{i \geq 1} \xi_i X_i$ and discuss related examples. Chapter 4 is devoted to study a particular RDE we call the Logistic RDE, which appears in Aldous' proof of $\zeta(2)$-limit for the mean-field random assignment problem [5]. Chapter 5 focuses on another example of a "max-type" RDE which has arisen in the study of the *frozen-percolation* process on infinite binary tree [4]. Unfortunately for this RDE we do not have a conclusive result about the endogenous property of the RTP but, we provide numerical results which suggest that the RTP is not endogenous. Chapters 2 and 3 represent parts of a joint work with David J. Aldous [2].

Chapter 2

Basic General Theory

In this chapter we outline some basic general theory to study RDEs. Exploiting the recursive nature one can associate with a RDE a much richer probabilistic model which we call *recursive tree process* (RTP) (see Section 2.4 for a formal definition). This treatment is some what similar to the general theory of Markov random field [17] but our emphasis is quite different.

2.1 The General Setting

Here we record a careful setup for RDEs. Let $(S, \mathcal{S})$ be a measurable space, and $\mathcal{P}(S)$ the set of probability measures on $(S, \mathcal{S})$. Let $(\Theta, \mathfrak{T})$ be another measurable space. Construct

$$\Theta^* := \Theta \times \bigcup_{0 \leq m \leq \infty} S^m \tag{2.1}$$

where the union is a disjoint union, S^m is product space, interpreting S^∞ as the usual infinite product space and S^0 as a singleton set, which we will write as $\{\Delta\}$. Let $g : \Theta^* \to S$ be a measurable function. Let ν be a probability measure on $\Theta \times \bar{\mathbb{Z}}^+$, where $\bar{\mathbb{Z}}^+ := \{0, 1, 2, \ldots; \infty\}$. These objects can now be used to define an operator $T : \mathcal{P}(S) \to \mathcal{P}(S)$ as follows. We write $\leq^* N$ to mean $\leq N$ for $N < \infty$ and to mean $< \infty$ for $N = \infty$.

Definition 2 For $\mu \in \mathcal{P}(S)$, $T(\mu)$ is the distribution of $g(\xi, X_i, 1 \leq i \leq^* N)$, where

1. $(X_i, i \geq 1)$ are independent with distribution μ;
2. (ξ, N) has distribution ν;
3. the families in 1 and 2 are independent.

Equation (1.1) fits this setting by writing $\xi = (\xi_i)_{i\geq 1}$. In most examples there is a sequence $(\xi_i)_{i\geq 1}$, but for theoretical discussion we regard such a sequence as a single random element ξ.

In examples where $\mathbf{P}\left(N = \infty\right) > 0$ a complication often arise. It may be that $g(\cdot)$ is not well-defined on all of $\Theta \times S^{\infty}$ while $g\left(\xi, X_i, 1 \leq i \leq^* N\right)$ is well-defined almost surely for $\left(X_i\right)_{i\geq 0}$ i.i.d with distribution in a restricted class of probability measures on S. For such examples and also for other cases where it is natural to restrict attention to distributions satisfying some conditions (like moment conditions), we allow the more general setting where we are given a subset $\mathcal{P} \subseteq \mathcal{P}(S)$ with $g\left(\xi, X_i, 1 \leq i \leq^* N\right)$ is well-defined almost surely for $\left(X_i\right)_{i\geq 0}$ i.i.d with distribution in $\mathcal{P}$. Now T is well-defined as a map

$$T : \mathcal{P} \mapsto \mathcal{P}(S) \tag{2.2}$$

2.2 Basic Monotonicity and Contraction Lemmas

In this section we describe two very basic standard tools for studying maps $T : \mathcal{P} \to \mathcal{P}(S)$ which do not depend on the map arising in the particular way of Definition 2.

First suppose $S \subseteq \mathbb{R}^+$ is an interval of the form $[0, x_0]$ for some $x_0 < \infty$, or $S = [0, \infty)$. Let $\preceq$ be the standard *stochastic partial order* on $\mathcal{P}(S)$. We will say T is *monotone* if

$$\mu_1 \preceq \mu_2 \Rightarrow T(\mu_1) \preceq T(\mu_2).$$

Note that, writing δ_0 for the probability measure degenerate at 0, if T is monotone then the sequence of iterates $T^n(\delta_0)$ is increasing, and thus the weak limit

$$T^n(\delta_0) \ \stackrel{d}{\to} \ \mu_\star$$

exists for some probability measure $\mu_\star$ on the compactified interval $[0, \infty]$. The following lemma is obvious and we omit the proof of it.

Lemma 3 (Monotonicity Lemma) *Let S be an interval as above. Suppose T is monotone. If $\mu_\star$ gives non-zero measure to $\{\infty\}$ then T has no fixed point on $\mathcal{P}(S)$. If $\mu_\star$ gives zero measure to $\{\infty\}$, and if T is continuous with respect to increasing limits, that is, $\mu_n \uparrow \mu_\infty$ implies $T(\mu_n) \uparrow T(\mu_\infty)$, then $\mu_\star$ is a fixed point of T, and $\mu_\star \preceq \mu$, for any other fixed point μ.*

This obvious result parallels the notion of *lower invariant measure* in attractive interacting particle systems [20].

Now considering the case of general S, the Banach contraction mapping principle specializes to yet another obvious result.

Lemma 4 (Contraction Method) *Let $\mathcal{P}$ be a subset of $\mathcal{P}(S)$ such that T maps $\mathcal{P}$ into $\mathcal{P}$. Suppose there is a complete metric d on $\mathcal{P}$ such that T is a (strict) contraction, that is*

$$\sup_{\mu_1 \neq \mu_2 \in \mathcal{P}} \frac{d(T(\mu_1), T(\mu_2))}{d(\mu_1, \mu_2)} < 1.$$

Then T has a unique fixed point μ in $\mathcal{P}$, whose domain of attraction is all of $\mathcal{P}$.

2.3 Recursive Tree Framework

Consider again the setup from Section 2.1. Rather than considering only the induced map T, one can make a richer structure by interpreting

$$X = g(\xi, X_i, 1 \leq i \leq^* N)$$

as a relationship between random variables. In brief we regard X as a value associated with a "parent" which is determined by the values X_i at N "children" and by some "random noise" ξ associated with the parent. One can then extend to grandchildren, great grandchildren and so on in the obvious way. We write out the details carefully in rest of this section

Let $\mathbb{T} = (\mathcal{V}, \mathcal{E})$ be the canonical infinite tree with vertex set $\mathcal{V} := \cup_{m \geq 0} \mathbb{N}^m$ (where $\mathbb{N}^0 := \{\emptyset\}$), and the edge set $\mathcal{E} := \{e = (\mathbf{i}, \mathbf{i}j) \,|\, \mathbf{i}\mathcal{V}, j \in \mathbb{N}\}$. We consider $\emptyset$ as the root of the tree, and will write $\emptyset j = j \ \forall \ j \in \mathbb{N}$. A typical vertex of the tree, say $\mathbf{i} = i_1 i_2 \cdots i_d$ denotes a d^{th} generation individual, which is the i_d^{th} child of its parent $i_1 i_2 \cdots i_{d-1}$. We will write gen($\mathbf{i}$)

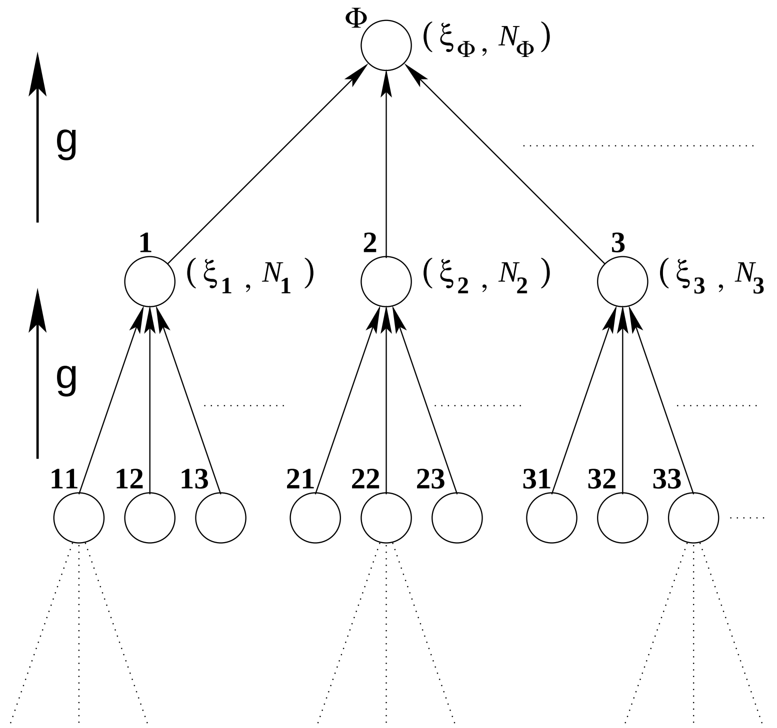


Figure 2.1: Recursive tree framework

for the generation of the vertex $\mathbf{i}$. Of course the generation of the root is defined to be 0. Given the distribution ν on $\Theta \times \bar{\mathbb{Z}}^+$ from Section 2.1, for each $\mathbf{i} \in \mathcal{V}$ let $(\xi_{\mathbf{i}}, N_{\mathbf{i}})$ have distribution ν, independent as $\mathbf{i}$ varies. Recall also the function g from Section 2.1.

Definition 5 A triplet of the form $\left(\mathbb{T}, (\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}, g\right)$ is called a *recursive tree framework* (RTF). The infinite tree $\mathbb{T}$ is called the *skeleton* and the i.i.d random variables $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ are called *innovation process* of the RTF (see Figure 2.1).

Remark : Associated with any RTF there is a Galton-Watson branching process tree $\mathcal{T}$ rooted at $\emptyset$ which is embedded in the skeleton tree $\mathbb{T}$, and has the offspring distribution given by the marginal of ν on $\bar{\mathbb{Z}}^+$. Indeed the $\bar{\mathbb{Z}}^+$-valued random variables $\mathbb{N}_{\mathbf{i}}$ defines a Galton-Watson branching forest on $\mathcal{V}$ with the required offspring distribution, set $\mathcal{T}$ as the connected component containing the root $\emptyset$. In most of the applications this branching

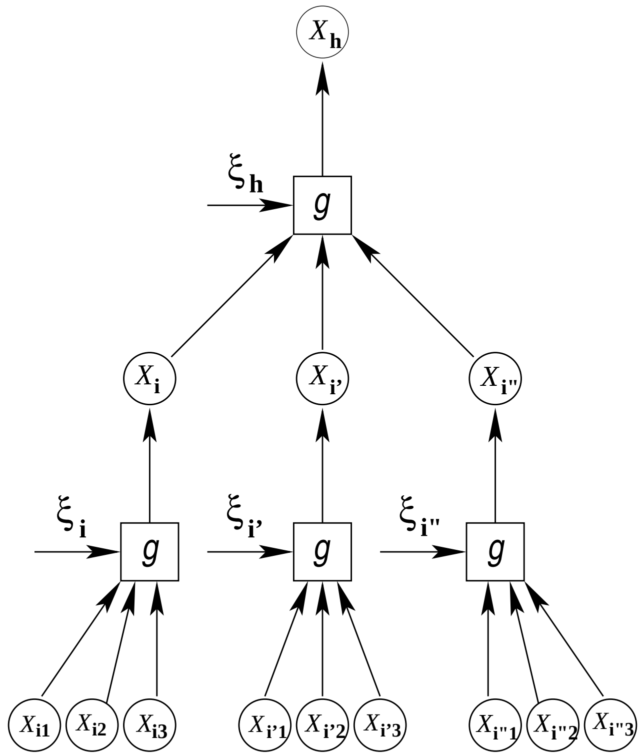


Figure 2.2: Recursive tree process

process tree $\mathcal{T}$ is the most important part of the skeleton.

2.4 Recursive Tree Process

In the setting of a RTF suppose that, jointly with the random objects above, we can construct S-valued random variables $X_{\mathbf{i}}$ such that for each $\mathbf{i}$

$$X_{\mathbf{i}} = g(\xi_{\mathbf{i}}, X_{\mathbf{i}j}, 1 \leq j \leq^* N_{\mathbf{i}}) \text{ a.s.} \tag{2.3}$$

and such that, conditional on the values of $\{\xi_{\mathbf{i}}, N_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^m, m \leq d-1\}$, the random variables $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^d\}$ are i.i.d. with some distribution μ_d. Call this structure (a RTF jointly with the $X_{\mathbf{i}}$'s) a *recursive tree process* (RTP). If the random variables $X_{\mathbf{i}}$ are defined only for vertices $\mathbf{i}$ of $\text{gen}(\mathbf{i}) \leq d$ then call it a RTP of depth d. See Figure 2.2.

Now an RTF has an induced map $T : \mathcal{P}(S) \to \mathcal{P}(S)$ as in Definition 2. In the extended case 2.2 we need to assume that T maps $\mathcal{P}$ into $\mathcal{P}$. Note that the relationship between a RTF and a RTP mirrors the relationship between a Markov transition kernel and a Markov chain. Fix a RTF, given d and an arbitrary distribution μ^0 on S, there is a RTP of depth d in which the generation-d vertices are defined to have distribution $\mu_d = \mu^0$. Then the distributions $\mu_d, \mu_{d-1}, \mu_{d-2}, \ldots, \mu_0$ at decreasing generations $d, d-1, d-2, \ldots, 0$ of the tree are just the successive iterates $\mu^0, T(\mu^0), T^2(\mu^0), \ldots, T^d(\mu^0)$ of the map T.

One should take a moment to distinguish RTPs from other structures involving tree-indexed random variables. For instance, a *branching Markov chain* can also be represented as a family $(X_{\mathbf{i}})$. But its essential property is that, conditional on the value $X_{\mathbf{i}}$ at a parent $\mathbf{i}$, the values $(X_{\mathbf{i}1}, X_{\mathbf{i}2}, \ldots)$ at the children $\mathbf{i}1$, $\mathbf{i}2, \ldots$ are i.i.d.. A RTP in general does not have this property. Conceptually, in branching processes one thinks of the "arrow of time" as pointing away from the root, whereas in a RTF the arrow points toward the root.

Calling a RTP *invariant* if the marginal distributions of $X_{\mathbf{i}}$ are identical at all depths, then we have the following obvious analog of Markov chain stationarity.

Lemma 6 *Consider a RTF. A distribution μ is a fixed point of the induced map T if and only if there is an invariant RTP with marginal distributions μ.*

Proof : First of all if there is an invariant RTP with marginal μ then naturally the induced map T has a fixed point.

Conversely, suppose that the induced map T of the given RTF has a fixed point μ. So for any $d \geq 0$ we can construct an invariant RTP of depth d with marginal μ. The existence of an invariant RTP follows from Kolmogorov's consistency theorem. $\square$

An invariant RTP could be regarded as a particular case of a *Markov random field*, but the special "directed tree" structure of RTFs makes them worth distinguishing from general Markov random fields.

2.5 Endogeny and Bivariate Uniqueness Property

Now imagine (2.3) as a system of equations for "unknowns" $X_{\mathbf{i}}$ in terms of "known data" $(\xi_{\mathbf{i}}, N_{\mathbf{i}})$. It is natural to ask if there is a unique solution which depends only on the data.

We formalize this as the following *endogenous* property. Write

$$\mathcal{G}_{\mathbb{T}} = \sigma(\xi_{\mathbf{i}}, N_{\mathbf{i}}, \mathbf{i} \in \mathcal{V}). \tag{2.4}$$

Definition 7 An invariant RTP is called *endogenous* if

$$X_{\emptyset} \text{ is } \mathcal{G}_{\mathbb{T}}\text{-measurable.}$$

We here note that by considering the embedded Galton-Watson branching process tree $\mathcal{T}$ one can rephrase the endogeny property by saying that

$$X_{\emptyset} \text{ is } \mathcal{G}\text{-measurable,}$$

where

$$\mathcal{G} := \sigma\left(\xi_{\mathbf{i}}, N_{\mathbf{i}}, \mathbf{i} \text{ is a vertex of } \mathcal{T}\right).$$

In this work we will mainly use this formulation.

It is intuitively clear that when the embedded Galton-Watson tree $\mathcal{T}$ is a.s. finite then there will be an unique invariant RTP and it will be endogenous. The following lemma proves this fact.

Lemma 8 *Consider a RTF such that the Galton-Watson branching process tree embedded in the skeleton is a.s. finite, then the associated RTP has an unique solution and it is endogenous.*

Proof : Let $\mathfrak{I}$ be the set of all finite rooted trees with each vertex having a weight from Θ. Using similar nomenlature as done above we will write a vertex as $\mathbf{i}$ and its weight $w_{\mathbf{i}}$ and the number of children it has as $n_{\mathbf{i}}$. Define the function $h : \mathfrak{I} \to S$ as follows.

- At each leaf $\mathbf{i}$ define $x_{\mathbf{i}} = g\left(w_{\mathbf{i}}, \varDelta\right)$;
- for an internal vertex $\mathbf{i}$ recursively define $x_{\mathbf{i}} = g\left(w_{\mathbf{i}}, \left(x_{\mathbf{i}j}, 1 \le j \le n_{\mathbf{i}}\right)\right)$;

h is then the value $x_{\emptyset}$. Observe that h is well defined since each tree in $\mathfrak{I}$ is finite. Now consider a RTF with the Galton-Watson tree embedded in the skeleton a.s. finite. Let $\mathcal{T}$ be the a.s. finite Galton-Watson branching process tree rooted at $\emptyset$. Let $\mathcal{T}_{\mathbf{i}}$ denote the tree rooted at the vertex $\mathbf{i}$ for $\mathbf{i}$ a vertex of $\mathcal{T}$. Define $X_{\mathbf{i}} = h(\mathcal{T}_{\mathbf{i}})$. Clearly $(X_{\mathbf{i}})$ is an invariant RTP, proving that the associated RDE has a solution. Further this solution is endogenous from definition.

Now assume that the associated RDE has a solution and let $(Y_{\mathbf{i}})$ be the corresponding invariant RTP. Let $\mathcal{T}$ be the associated Galton-Watson branching process tree rooted at $\emptyset$ which is a.s. finite by assumption. Define $X_{\mathbf{i}}$ for $\mathbf{i}$ a vertex of $\mathcal{T}$ as above. Let $\mathbf{i}$ be a leaf of the tree $\mathcal{T}$ from definition $Y_{\mathbf{i}} = g(\xi_{\mathbf{i}}, \Delta)$ but so is $X_{\mathbf{i}}$, and hence $Y_{\mathbf{i}} = X_{\mathbf{i}}$ for all $\mathbf{i}$. This proves the uniqueness of the solution. $\square$

When $\mathcal{T}$ is infinite the "boundary behavior" may cause uniqueness and/or endogeny to fail. The following trivial example shows that one can not tell whether or not the endogenous property holds just by looking at T, even when the T has a unique fixed point.

Example 1 *Take $S = \mathbb{R}$ and define $T : \mathcal{P}(S) \to \mathcal{P}(S)$ by $T(\mu) = Normal(0,1)$ for all μ. So $Normal(0,1)$ is the unique fixed point of T. We will construct two RTPs with the same induced map T such that in one case it will be endogenous while in the other case it will fail to be endogenous.*

For the first case take (ξ, N) with $N \equiv 1$ and $\xi \sim \text{Normal}(0,1)$, and $g(t, x_1) = t$. Clearly the induced map is T. Observe that for the associated RTP we have $X_\emptyset = \xi_\emptyset$ and so the endogenous property holds.

For the other case take the same pair (ξ, N) and let $g(t, x_1) = \text{sign}(x_1) \times t$. Once again it is easy to see that the induced map is T. But notice that $g(\xi, -X_1) = -g(\xi, X_1)$ and also $X_1 \stackrel{d}{=} -X_1$ when $X_1 \sim \text{Normal}(0,1)$. So we can change the value of the root variable $X_\emptyset$ by changing X_1 to $-X_1$ and not changing any of the values of the innovation process. So naturally the RTP is not endogenous. More formally one can show that in this example the bivariate uniqueness fails and hence as discussed in the following sections the invariant RTP is not endogenous.

Theorem 12 will show that the endogenous property is equivalent to a certain *bivariate uniqueness* property, which is explained next.

2.5.1 Bivariate Uniqueness Property of 1st Kind

In the general setting of a RTF we have the induced map $T : \mathcal{P} \to \mathcal{P}(S)$. Now consider a bivariate version, let $\mathcal{P}^{(2)}$ be the space of probability measures on $S^2 = S \times S$, with marginals in $\mathcal{P}$. We can now define a map $T^{(2)} : \mathcal{P}^{(2)} \to \mathcal{P}\left(S^2\right)$ as follows

Definition 9 For a probability $\mu^{(2)} \in \mathcal{P}^{(2)}$, $T^{(2)}\left(\mu^{(2)}\right)$ is the joint distribution of

$$\begin{pmatrix} g\left(\xi, X_i^{(1)}, 1 \le i \le^* N\right) \\ g\left(\xi, X_i^{(2)}, 1 \le i \le^* N\right) \end{pmatrix}$$

where we assume

1. $\left(X_i^{(1)}, X_i^{(2)}\right)_{i \ge 1}$ are independent with joint distribution $\mu^{(2)}$ on S^2;
2. (ξ, N) has distribution ν;
3. the families of random variables in 1 and 2 are independent.

The point is that we use the *same realization(data)* of (ξ, N) in both components. Immediately from the definitions we conclude the following

Lemma 10 *(a) If μ is a fixed point for T then the associated* diagonal measure $\mu^{\nearrow}$ *is a fixed point for $T^{(2)}$, where*

$$\mu^{\nearrow} = \mathrm{dist}(X, X) \textit{ for } \mu = \mathrm{dist}(X).$$

(b) If $\mu^{(2)}$ is a fixed point for $T^{(2)}$ then each marginal distribution is a fixed point for T.

So if μ is a fixed point for T then $\mu^{\nearrow}$ is a fixed point for $T^{(2)}$ and there may or may not be other fixed points of $T^{(2)}$ with marginals μ.

Definition 11 An invariant RTP with marginal μ has the *bivariate uniqueness property of 1st kind* if $\mu^{\nearrow}$ is the unique fixed point of $T^{(2)}$ with marginal μ.

The following example shows that even if T has a unique fixed-point say μ, $T^{(2)}$ may have several fixed-points with marginal μ.

Example 2 *Take* $S = \{0, 1\}$ *and let* ξ *have Bernoulli*(q) *distribution for* $0 < q < 1$*. Fix* $N = 1$*. Consider the RDE*

$$X \stackrel{d}{=} X_1 + \xi \ (mod\ 2). \tag{2.5}$$

Here T *maps a Bernoulli*(p) *to Bernoulli*$(p(1-q) + q(1-p))$*, so Bernoulli*$(1/2)$ *is the unique fixed point. We will show that any distribution on* S^2 *with marginal Bernoulli*$(1/2)$ *is a fixed-point for* $T^{(2)}$*.*

Let (X_1, Y_1) be a pair with some bivariate distribution on S^2 with marginal Bernoulli$(1/2)$. Put $\theta := \mathbf{P}(X_1 = 1, Y_1 = 1)$, then observe that $\theta = \mathbf{P}(X_1 = 0, Y_1 = 0)$ also. Let ξ be independent of (X_1, Y_1) and has Bernoulli(q) distribution, then

$$\mathbf{P}(X_1 + \xi = 1, Y_1 + \xi = 1) = \theta \times q + \theta \times (1-q) = \theta.$$

So $\text{dist}\,(X_1, Y_1)$ is a fixed-point for $T^{(2)}$.

Further we note that it is possible that T has several solutions some of which are endogenous and some are not. One such non-trivial example is considered in Section 3.3 of Chapter 3

2.5.2 The First Equivalence Theorem

Here we state a version of the general result linking endogeny and bivariate uniqueness. This result is similar to results about Gibbs measures and Markov random fields [17].

Theorem 12 *Suppose* S *is a* Polish *space. Consider an invariant RTP with marginal distribution* μ*.*

(a) If the endogenous property holds then the bivariate uniqueness property of 1st kind holds.

(b) Conversely, suppose the bivariate uniqueness property of 1st kind holds. If also $T^{(2)}$ *is continuous with respect to weak convergence on the set of bivariate distributions with marginals* μ*, then the endogenous property holds.*

(c) Further, the endogenous property holds if and only if ${T^{(2)}}^n(\mu \otimes \mu) \stackrel{d}{\to} \mu^{\nearrow}$*, where* $\mu \otimes \mu$ *is product measure.*

Remark : In part (b) we need the technical condition of continuity of $T^{(2)}$ for writing the proof, though we do not have an example where the equivalence in part (b) fails without the condition. Also it is worth noting that for part (c) we do not need to assume continuity of $T^{(2)}$. The part (c) can be used non-rigorously to investigate endogeny via numerical or Monte Carlo methods (see Section 5.3 of Chapter 5).

2.5.3 Intuitive Picture for the Equivalence Theorem

Intuitively we can think of the root variable $X_\emptyset$ of an invariant RTP as a function (*output*) of the innovation process of the associated RTF and possibly some other *external* variables which are given as *inputs* at the boundary at infinity. Then endogeny simply means that the effect of the boundary values have no influence on the output $X_\emptyset$. If that is the case then giving independent inputs at the boundary at infinity should still give the same output at the root (see Figure 2.3). The bivariate uniqueness of 1st kind should then hold by breaking up the skeleton tree $\mathcal{T}$ in subtrees rooted at the children of the root $\emptyset$. The proof of the above equivalence theorem is essentially a formalization of this intuitive idea.

2.5.4 Proof of the First Equivalence Theorem

(a) Let ν be a fixed point of $T^{(2)}$ with marginals μ. Consider a bivariate RTP $\left(\left(X_{\mathbf{i}}^{(1)}, X_{\mathbf{i}}^{(2)}\right), \mathbf{i} \in \mathcal{T}\right)$ with $\nu = \mathrm{dist}\left(X_\emptyset^{(1)}, X_\emptyset^{(2)}\right)$. Define $\mathcal{G}_n := \sigma\left((\xi_{\mathbf{i}}, N_{\mathbf{i}}), \mathrm{gen}(\mathbf{i}) \leq n\right)$. Observe that $\mathcal{G}_n \uparrow \mathcal{G}$.

Fix $\Lambda : S \to \mathbb{R}$ a bounded continuous function. Notice that from the construction of the bivariate RTP,

$$\left(X_\emptyset^{(1)}; (\xi_{\mathbf{i}}, N_{\mathbf{i}}),\ \mathrm{gen}(\mathbf{i}) \leq n\right) \stackrel{d}{=} \left(X_\emptyset^{(2)}; (\xi_{\mathbf{i}}, N_{\mathbf{i}}),\ \mathrm{gen}(\mathbf{i}) \leq n\right).$$

So

$$\mathbf{E}\left[\Lambda(X_\emptyset^{(1)}) \middle| \mathcal{G}_n\right] = \mathbf{E}\left[\Lambda(X_\emptyset^{(2)}) \middle| \mathcal{G}_n\right] \text{ a.s.} \tag{2.6}$$

Now by martingale convergence

$$\mathbf{E}\left[\Lambda(X_\emptyset^{(1)}) \middle| \mathcal{G}_n\right] \xrightarrow{\text{a.s.}} \mathbf{E}\left[\Lambda(X_\emptyset^{(1)}) \middle| \mathcal{G}\right] \stackrel{\text{a.s.}}{=} \Lambda(X_\emptyset^{(1)}) \tag{2.7}$$

the last equality because of the *endogenous* assumption for the univariate RTP. Similarly,

$$\mathbf{E}\left[\Lambda(X_\emptyset^{(2)}) \middle| \mathcal{G}\right] \stackrel{\text{a.s.}}{=} \Lambda(X_\emptyset^{(2)}).$$

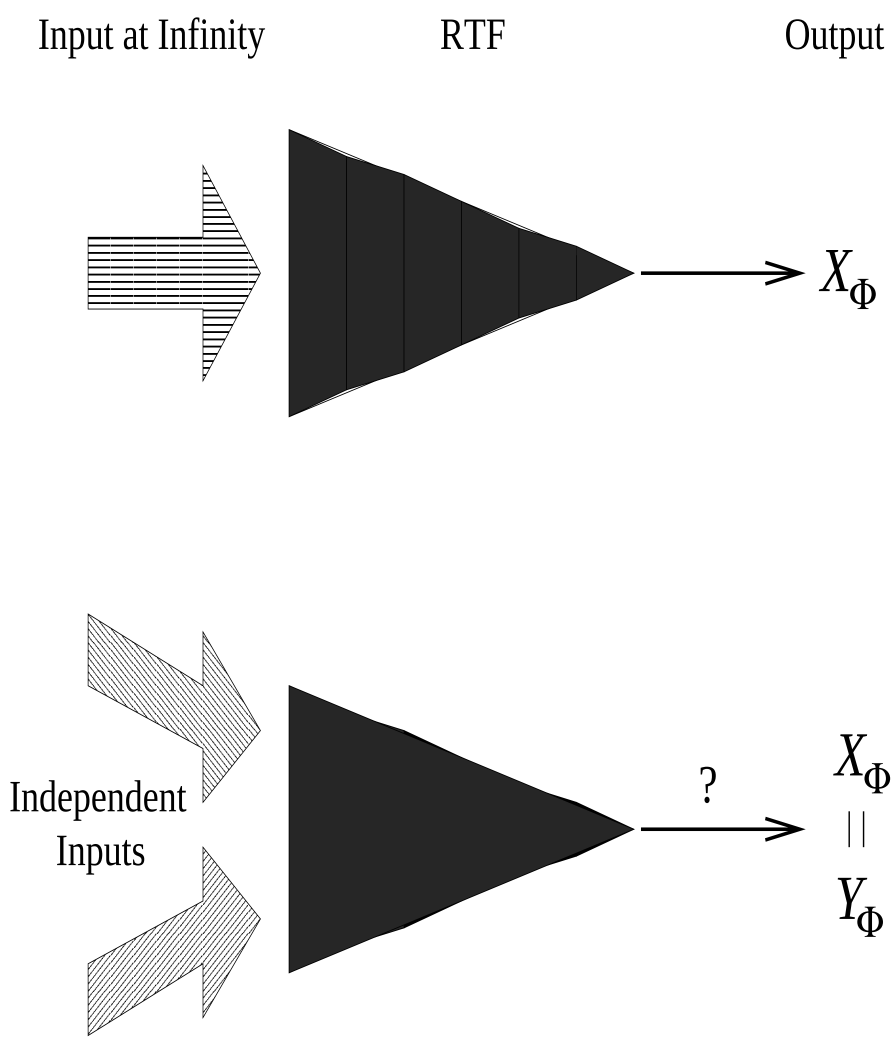


Figure 2.3: Intuitive picture for bivariate uniqueness of 1st kind

Thus by (2.6) we see that $\Lambda(X_\emptyset^{(1)}) = \Lambda(X_\emptyset^{(2)})$ a.s.. Since this is true for every bounded continuous Λ we deduce $X_\emptyset^{(1)} = X_\emptyset^{(2)}$ a.s., proving bivariate uniqueness of 1st kind.

(b) To prove the converse, again fix $\Lambda : S \to \mathbb{R}$ bounded continuous. Let $(X_{\mathbf{i}})$ be the invariant RTP with marginal μ. Again by martingale convergence

$$\mathbf{E}\left[\Lambda(X_\emptyset)\middle|\mathcal{G}_n\right] \xrightarrow[\mathcal{L}_2]{\text{a.s.}} \mathbf{E}\left[\Lambda(X_\emptyset)\middle|\mathcal{G}\right]. \tag{2.8}$$

Independently of $(X_{\mathbf{i}}, \xi_{\mathbf{i}}, N_{\mathbf{i}},\ \mathbf{i} \in \mathbb{T})$, construct random variables $(V_{\mathbf{i}}, \mathbf{i} \in \mathbb{T})$ which are i.i.d with distribution μ. For $n \geq 1$, define $Y_{\mathbf{i}}^n := V_{\mathbf{i}}$ if $\mathrm{gen}(\mathbf{i}) = n$, and then recursively define $Y_{\mathbf{i}}^n$ for $\mathrm{gen}(\mathbf{i}) < n$ by (2.3) to get an invariant RTP $(Y_{\mathbf{i}}^n)$ of depth n. Observe that $X_\emptyset \stackrel{d}{=} Y_\emptyset^n$. Further given $\mathcal{G}_n$, the variables $X_\emptyset$ and $Y_\emptyset^n$ are conditionally independent and identically distributed given $\mathcal{G}_n$. Now let

$$\sigma_n^2(\Lambda) := \left\|\mathbf{E}\left[\Lambda(X_\emptyset)\middle|\mathcal{G}_n\right] - \Lambda(X_\emptyset)\right\|_2^2. \tag{2.9}$$

We calculate

$$\begin{aligned}\sigma_n^2(\Lambda) &= \mathbf{E}\left[\left(\Lambda(X_\emptyset) - \mathbf{E}\left[\Lambda(X_\emptyset)\middle|\mathcal{G}_n\right]\right)^2\right] \\ &= \mathbf{E}\left[\mathbf{Var}\left(\Lambda(X_\emptyset)\middle|\mathcal{G}_n\right)\right] \\ &= \tfrac{1}{2}\mathbf{E}\left[\left(\Lambda(X_\emptyset) - \Lambda(Y_\emptyset^n)\right)^2\right].\end{aligned} \tag{2.10}$$

The last equality uses the conditional form of the fact that for any random variable U one has $\mathbf{Var}(U) = \frac{1}{2}\mathbf{E}\left[(U_1 - U_2)^2\right]$, where U_1, U_2 are i.i.d copies of U.

Now suppose we show that

$$(X_\emptyset, Y_\emptyset^n) \stackrel{d}{\to} (X^\star, Y^\star), \text{ say} \tag{2.11}$$

for some limit. From the construction,

$$\begin{bmatrix} X_\emptyset \\ Y_\emptyset^{n+1} \end{bmatrix} \stackrel{d}{=} T^{(2)}\left(\begin{bmatrix} X_\emptyset \\ Y_\emptyset^n \end{bmatrix}\right),$$

and then the weak continuity assumption on $T^{(2)}$ implies

$$\begin{bmatrix} X^\star \\ Y^\star \end{bmatrix} \stackrel{d}{=} T^{(2)}\left(\begin{bmatrix} X^\star \\ Y^\star \end{bmatrix}\right).$$

Also by construction we have $X_\emptyset \stackrel{d}{=} Y_\emptyset^n \stackrel{d}{=} \mu$ for all $n \geq 1$, and hence $X^\star \stackrel{d}{=} Y^\star \stackrel{d}{=} \mu$. The bivariate uniqueness of 1st kind assumption now implies $X^\star = Y^\star$ a.s. Since Λ is a bounded continuous function, (2.11) implies $\Lambda(X_\emptyset) - \Lambda(Y_\emptyset^n) \to 0$ a.s. and so using (2.10) we see that $\sigma_n^2(\Lambda) \longrightarrow 0$. Hence from (2.9) and (2.8) we conclude that $\Lambda(X_\emptyset)$ is $\mathcal{G}$-measurable. This is true for every bounded continuous Λ, proving that $X_\emptyset$ is $\mathcal{G}$-measurable, as required.

Now all remains is to show that a limit (2.11) exists. Fix $f : S \to \mathbb{R}$ and $h : S \to \mathbb{R}$, two bounded continuous functions. Again by martingale convergence

$$\mathbf{E}\left[f(X_\emptyset) \middle| \mathcal{G}_n \right] \xrightarrow[\mathcal{L}_1]{\text{a.s.}} \mathbf{E}\left[f(X_\emptyset) \middle| \mathcal{G} \right],$$

and similarly for h. So

$$\begin{aligned} \mathbf{E}\left[f(X_\emptyset) h(Y_\emptyset^n) \right] &= \mathbf{E}\left[\mathbf{E}\left[f(X_\emptyset) h(Y_\emptyset^n) \middle| \mathcal{G}_n \right] \right] \\ &= \mathbf{E}\left[\mathbf{E}\left[f(X_\emptyset) \middle| \mathcal{G}_n \right] \mathbf{E}\left[h(X_\emptyset) \middle| \mathcal{G}_n \right] \right], \end{aligned}$$

the last equality because of conditional on $\mathcal{G}_n$ $X_\emptyset$ and $Y_\emptyset^n$ independent and identically distributed as of $X_\emptyset$ given $\mathcal{G}_n$. Letting $n \to \infty$ we get

$$\mathbf{E}\left[f(X_\emptyset) h(Y_\emptyset^n) \right] \longrightarrow \mathbf{E}\left[\mathbf{E}\left[f(X_\emptyset) \middle| \mathcal{G} \right] \mathbf{E}\left[h(X_\emptyset) \middle| \mathcal{G} \right] \right]. \tag{2.12}$$

Moreover note that $X_\emptyset \stackrel{d}{=} Y_\emptyset^n \stackrel{d}{=} \mu$ and so the sequence of bivariate distributions $(X_\emptyset, Y_\emptyset^n)$ is tight. Tightness, together with convergence (2.12) for all bounded continuous f and h, implies weak convergence of $(X_\emptyset, Y_\emptyset^n)$.

(c) First assume that $T^{(2)^n}(\mu \otimes \mu) \stackrel{d}{\to} \mu^\nearrow$, then with the same construction as in part (b) we get that

$$\left(X_\emptyset, Y_\emptyset^n \right) \stackrel{d}{\to} \left(X_\emptyset, X_\emptyset \right).$$

Further recall that Λ is bounded continuous, thus using (2.10), (2.9) and (2.8) we conclude that $\Lambda(X_\emptyset)$ is $\mathcal{G}$-measurable. Since it is true for any bounded continuous function Λ, thus $X_\emptyset$ is $\mathcal{G}$-measurable. So the RTP is endogenous.

Conversely, suppose that the RTP with marginal μ is endogenous. Let Λ_1 and Λ_2 be two bounded continuous functions. Note that the variables $\left(X_\emptyset, Y_\emptyset^n \right)$, as defined in part (b) have joint distribution as $T^{(2)^n}(\mu \otimes \mu)$. Further, given $\mathcal{G}_n$, they are conditionally independent and have same conditional law as of $X_\emptyset$ given $\mathcal{G}_n$. So

$$\begin{aligned}
\mathbf{E}\left[\Lambda_1(X_\emptyset)\,\Lambda_2(Y_\emptyset^n)\right] &= \mathbf{E}\left[\mathbf{E}\left[\Lambda_1(X_\emptyset)\middle|\mathcal{G}_n\right]\,\mathbf{E}\left[\Lambda_2(X_\emptyset)\middle|\mathcal{G}_n\right]\right] \\
&\to \mathbf{E}\left[\mathbf{E}\left[\Lambda_1(X_\emptyset)\middle|\mathcal{G}\right]\,\mathbf{E}\left[\Lambda_2(X_\emptyset)\middle|\mathcal{G}\right]\right] \\
&= \mathbf{E}\left[\Lambda_1(X_\emptyset)\,\Lambda_2(X_\emptyset)\right].
\end{aligned}$$

The convergence is by martingale convergence, and the last equality is by endogeny. So from definition we get

$$T^{(2)^n}(\mu\otimes\mu) \stackrel{d}{=} \left(X_\emptyset, Y_\emptyset^n\right) \stackrel{d}{\to} \left(X_\emptyset, X_\emptyset\right) \stackrel{d}{=} \mu^{\nearrow}.$$

□

2.6 Tail of a RTP

Consider a RTF such that the induced map T has a fixed-point $\mu \in \mathcal{P}$ and let $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ be an invariant RTP with marginal distribution μ. Let $\mathcal{H}$ be the tail sigma algebra of the RTP defined as

$$\mathcal{H} = \bigcap_{n\geq 0} \mathcal{H}_n, \tag{2.13}$$

where $\mathcal{H}_n := \sigma\left(X_{\mathbf{i}}, \mathrm{gen}(\mathbf{i}) \geq n\right)$. Intuitively it is clear that if the invariant RTP is endogenous then the tail $\mathcal{H}$ is trivial. Formally, let $\mathcal{G}_n := \sigma\left(\left(\xi_{\mathbf{i}}, N_{\mathbf{i}}\right), \mathrm{gen}(\mathbf{i}) \leq n\right)$. From definition we have $\mathcal{H}_n \downarrow \mathcal{H}$ and $\mathcal{G}_n \uparrow \mathcal{G}$. Also for each $n \geq 0$, $\mathcal{G}_n$ is independent of $\mathcal{H}_{n+1}$. So clearly $\mathcal{G}$ is independent of $\mathcal{H}$. Hence if the RTP is endogenous then $X_\emptyset$ is $\mathcal{G}$-measurable, so it is independent of $\mathcal{H}$. The following lemma proves the rest.

Lemma 13 *$X_\emptyset$ is independent of $\mathcal{H}$ if and only if $\mathcal{H}$ is trivial.*

Proof : If the tail $\mathcal{H}$ is trivial then naturally $X_\emptyset$ is independent of it. For proving the converse we will need the following standard measure theoretic fact whose proof is easy using Dynkin's π-λ Theorem (see [13]) and is omitted here.

Lemma 14 *Suppose $(\Omega, \mathcal{I}, \mathbf{P})$ be a probability space and let $\mathcal{F}^*, \mathcal{G}^*$ and $\mathcal{H}^*$ be three sub-σ-algebras such that $\mathcal{F}^*$ is independent of $\mathcal{H}^*$; $\mathcal{G}^*$ is independent of $\mathcal{H}^*$; and $\mathcal{F}^*$ and $\mathcal{G}^*$ are independent given $\mathcal{H}^*$. Then $\sigma\left(\mathcal{F}^* \cup \mathcal{G}^*\right)$ is independent of $\mathcal{H}^*$.*

To complete the proof of the Lemma 13 we denote $\mathcal{F}_n^0 := \sigma\left(X_{\mathbf{i}}, \mathrm{gen}(\mathbf{i}) = n\right)$ and $\mathcal{F}_n := \sigma\left(X_{\mathbf{i}}, \mathrm{gen}(\mathbf{i}) \leq n\right)$. From assumption $X_{\mathbf{i}}$ is independent of $\mathcal{H}$ for all $\mathbf{i} \in \mathcal{V}$. Fix $n \geq 1$ and let $\mathbf{i} \neq \mathbf{i}'$ be two vertices at generation n. From the definition of RTP $X_{\mathbf{i}}$ and $X_{\mathbf{i}'}$ are independent, moreover they are independent given $\mathcal{H}_{n+k}$ for any $k \geq 1$. Letting $k \to \infty$ we conclude that $X_{\mathbf{i}}$ and $X_{\mathbf{i}'}$ are independent given $\mathcal{H}$. Thus by Lemma 14 we get that $(X_{\mathbf{i}}, X_{\mathbf{i}'})$ is independent of $\mathcal{H}$, and hence by induction $\mathcal{F}_n^0$ is independent of $\mathcal{H}$.

Now $\mathcal{G}_n$ is independent of $\mathcal{H}$ from definition. Further $\mathcal{G}_n$ is independent of $\mathcal{F}_{n+1}^0$ given $\mathcal{H}_{n+k}$ for any $k \geq 1$. Once again letting $k \to \infty$ we conclude that $\mathcal{G}_n$ and $\mathcal{F}_{n+1}^0$ are independent given $\mathcal{H}$. So again using Lemma 14 it follows that $\sigma\left(\mathcal{G}_n \cup \mathcal{F}_{n+1}^0\right)$ is independent of $\mathcal{H}$. But $\mathcal{F}_n \subseteq \sigma\left(\mathcal{G}_n \cup \mathcal{F}_{n+1}^0\right)$ so $\mathcal{F}_n$ is independent of $\mathcal{H}$. But $\mathcal{F}_n \uparrow \mathcal{H}_0$ and hence $\mathcal{H}$ is independent of $\mathcal{H}_0 \supseteq \mathcal{H}$. This proves that $\mathcal{H}$ is trivial. $\square$

So one way to conclude that the RTP is not endogenous will be to prove that the tail $\mathcal{H}$ is non-trivial. The following trivial example shows that the converse is not necessarily true.

Example 3 *Consider the RDE in Example 2. We have seen that $T^{(2)}$ has several fixed-points with marginal Bernoulli$(1/2)$. So the bivariate uniqueness of 1st kind fails, hence the associated RTP is not endogenous. Now we will show that the tail of the RTP is trivial.*

For that we will use the part (b) of Theorem 18 of the next section. Let (X, Y) be S^2-valued random pair with some distribution such that the marginals are both Bernoulli$(1/2)$. Let $\theta = \mathbf{P}\left(X = 1, Y = 1\right) = \mathbf{P}\left(X = 0, Y = 0\right)$. Suppose further that the distribution of (X, Y) satisfies the following bivariate RDE

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} X_1 + \xi \\ Y_1 + \eta \end{pmatrix} \pmod{2},$$

where (X_1, Y_1) is a copy of (X, Y) and independent of (ξ, η) which are i.i.d. Bernoulli(q). So we get the following equation for θ

$$\theta = q^2\theta + (1-q)^2\theta + 2q(1-q)(1/2 - \theta). \tag{2.14}$$

The only solution of (2.14) is $\theta = 1/4$, that is X and Y are independent. So using part (b) of Theorem 18 we conclude that the RTP has a trivial tail.

In the same spirit of Section 2.5.3 we can intuitively argue that in order to check whether $X_\emptyset$ is independent of $\mathcal{H}$ (and so by the lemma above $\mathcal{H}$ is trivial) we can think of having same input at "infinity" in two RTFs with independent and identical innovations, and getting two outputs say $X_\emptyset$ and $Y_\emptyset$. Intuitively if $X_\emptyset$ and $Y_\emptyset$ are independent then there is no influence of the boundary at infinity, that is, $X_\emptyset$ should be independent of $\mathcal{H}$. See Figure 2.4. Theorem 18 tries to make this idea formal in a similar way as done in the case of Theorem 12

2.6.1 Bivariate Uniqueness Property of 2nd Kind

In the general setting of a RTF consider the induced map $T : \mathcal{P} \to \mathcal{P}(S)$. Now we will consider another bivariate version of it. With the same notation from Section 2.5.1, write $\mathcal{P}^{(2)}$ for the space of probability measures on $S^2 = S \times S$, with marginals in $\mathcal{P}$. We can now define a map $T \otimes T : \mathcal{P}^{(2)} \to \mathcal{P}\left(S^2\right)$ as follows

Definition 15 For a probability $\mu^{(2)} \in \mathcal{P}^{(2)}$, $(T \otimes T)\left(\mu^{(2)}\right)$ is the joint distribution of

$$\left(\begin{array}{c} g\left(\xi, X_i^{(1)}, 1 \leq i \leq^* N\right) \\ g\left(\eta, X_i^{(2)}, 1 \leq i \leq^* M\right) \end{array}\right)$$

where we assume

1. $\left(X_i^{(1)}, X_i^{(2)}\right)_{i \geq 1}$ are independent with joint distribution $\mu^{(2)}$ on S^2;
2. (ξ, N) and (η, M) are i.i.d ν;
3. the families of random variables in 1 and 2 are independent.

Unlike in the case of $T^{(2)}$ as defined in Section 2.5.1, the point is that we use *independent (data)realizations* of in the two components. Once again immediately from the definition we conclude the following

Lemma 16 *(a) If μ is a fixed point for T then the associated* product measure $\mu \otimes \mu$ *is a fixed point for $T \otimes T$,*

(b) If $\mu^{(2)}$ is a fixed point for $T \otimes T$ then each marginal distribution is a fixed point for T.

So if μ is a fixed point for T then $\mu \otimes \mu$ is a fixed point for $T \otimes T$ and there may or may not be other fixed points of $T \otimes T$ with marginal μ.

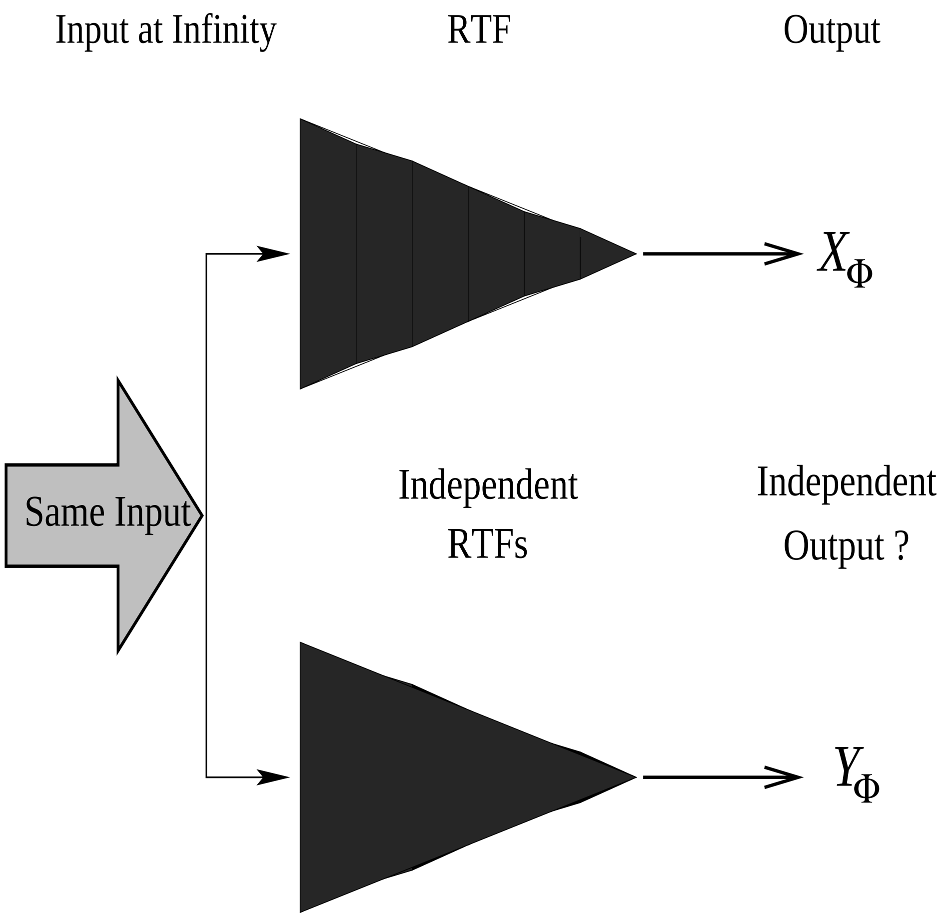


Figure 2.4: Intuitive picture for bivariate uniqueness of 2nd kind

Definition 17 An invariant RTP with marginal μ has the *bivariate uniqueness property of 2nd kind* if $\mu \otimes \mu$ is the unique fixed point of $T \otimes T$ with marginal μ.

2.6.2 The Second Equivalence Theorem

The following result is a general result linking tail triviality and bivariate uniqueness property of 2nd kind. This result and its proof is basically the formalization of the intuitive picture described above. The proof of this theorem parallels the proof of the first equivalence theorem (Theorem 12).

Theorem 18 *Suppose S is a* Polish *space. Consider an invariant RTP with marginal distribution μ.*

(a) If the endogenous property holds then the bivariate uniqueness property of 2nd kind holds.

(b) Suppose the bivariate uniqueness property of 2nd kind holds. If also $T \otimes T$ is continuous with respect to weak convergence on the set of bivariate distributions with marginals μ, then the tail of the RTP is trivial.

(c) Further, the RTP has a trivial tail if and only if $(T \otimes T)^n \left(\mu^{\nearrow}\right) \stackrel{d}{\to} \mu \otimes \mu$.

Remark : This theorem is not really an *equivalence* of the two properties, namely tail triviality of the RTP and the bivariate uniqueness of the 2nd kind. The part (a) assumes endogeny which is much stronger than tail triviality, though because of part (c) it is natural to believe that part (a) is true if we just assume tail triviality. Unfortunately we do not have a proof of this neither do we have a counter-example. The main usefulness of this theorem is to show non-endogeny by showing bivariate uniqueness of 2nd kind does not hold, which in some examples can possibly be simpler. Once again part (c) can be used for non-rigorous investigation using numerical or Monte Carlo methods.

2.6.3 Proof of the Second Equivalence Theorem

(a) Let λ be a fixed point of $T \otimes T$ with marginals μ. Consider two RTFs with independent and identical innovation processes given by $((\xi_{\mathbf{i}}, N_{\mathbf{i}}), \mathbf{i} \in \mathcal{V})$ and $((\eta_{\mathbf{i}}, M_{\mathbf{i}}), \mathbf{i} \in \mathcal{V})$. Exactly the same way as done in Lemma 6 we can now construct a bivariate RTP $((X_{\mathbf{i}}^{(1)}, X_{\mathbf{i}}^{(2)}), \mathbf{i} \in \mathcal{V})$ with $\lambda = \text{dist}(X_{\emptyset}^{(1)}, X_{\emptyset}^{(2)})$. Notice that $\left(X_{\mathbf{i}}^{(1)}\right)_{\mathbf{i}\in\mathcal{V}}$ and $\left(X_{\mathbf{i}}^{(2)}\right)_{\mathbf{i}\in\mathcal{V}}$ are two (*univariate*) RTPs with marginal μ.

Let $\mathcal{G}_{\mathbb{T}} := \sigma\left((\xi_{\mathbf{i}}, N_{\mathbf{i}}), \mathbf{i} \in \mathcal{V}\right)$ be as before and $\mathcal{G}_{\mathbb{T}}^{\star} := \sigma\left((\eta_{\mathbf{i}}, M_{\mathbf{i}}), \mathbf{i} \in \mathcal{V}\right)$. Trivially $\mathcal{G}_{\mathbb{T}}$ and $\mathcal{G}_{\mathbb{T}}^{\star}$ are independent.

Since we assume that invariant RTP with marginal μ has the endogenous property, thus $X_{\emptyset}^{(1)}$ is $\mathcal{G}_{\mathbb{T}}$-measurable and $X_{\emptyset}^{(2)}$ is $\mathcal{G}_{\mathbb{T}}^{\star}$-measurable. Hence $X_{\emptyset}^{(1)}$ is independent of $X_{\emptyset}^{(2)}$, so $\lambda = \mu \otimes \mu$.

(b) Let $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ be the invariant RTP with marginal μ. Define $\mathcal{H}_n := \sigma\left(X_{\mathbf{i}}, \text{gen}(\mathbf{i}) \geq n\right)$. Observe that $\mathcal{H}_n \downarrow \mathcal{H}$. Now fix $\Lambda : S \to \mathbb{R}$ a bounded continuous function. So by reverse martingale convergence

$$\mathbf{E}\left[\Lambda(X_{\emptyset})\middle|\mathcal{H}_n\right] \xrightarrow[\mathcal{L}_2]{\text{a.s.}} \mathbf{E}\left[\Lambda(X_{\emptyset})\middle|\mathcal{H}\right]. \tag{2.15}$$

Let $(\eta_{\mathbf{i}}, M_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ be independent innovations which are independent of $(X_{\mathbf{i}}, \xi_{\mathbf{i}}, N_{\mathbf{i}},\ \mathbf{i} \in \mathcal{V})$. For $n \geq 1$, define $Y_{\mathbf{i}}^n := X_{\mathbf{i}}$ if $\text{gen}(\mathbf{i}) = n$, and then recursively define $Y_{\mathbf{i}}^n$ for $\text{gen}(\mathbf{i}) < n$ using (2.3) but replacing $\xi_{\mathbf{i}}$ by $\eta_{\mathbf{i}}$ and $N_{\mathbf{i}}$ by $M_{\mathbf{i}}$ to get an invariant RTP $(Y_{\mathbf{i}}^n)$ of depth n. Observe that $X_{\emptyset} \stackrel{d}{=} Y_{\emptyset}^n$. Further given $\mathcal{H}_n$, the variables $X_{\emptyset}$ and $Y_{\emptyset}^n$ are conditionally independent and identically distributed. Now let

$$\bar{\sigma}_n^2(\Lambda) := \left\|\mathbf{E}\left[\Lambda(X_{\emptyset})\middle|\mathcal{H}_n\right] - \mathbf{E}\left[\Lambda(X_{\emptyset})\right]\right\|_2^2. \tag{2.16}$$

We calculate

$$\begin{aligned}
\bar{\sigma}_n^2(\Lambda) &= \mathbf{E}\left[\left(\mathbf{E}\left[\Lambda(X_{\emptyset})\middle|\mathcal{H}_n\right] - \mathbf{E}\left[\Lambda(X_{\emptyset})\right]\right)^2\right] \\
&= \mathbf{Var}\left(\mathbf{E}\left[\Lambda(X_{\emptyset})\middle|\mathcal{H}_n\right]\right) \\
&= \mathbf{Var}\left(\Lambda(X_{\emptyset})\right) - \mathbf{E}\left[\mathbf{Var}\left(\Lambda(X_{\emptyset})\middle|\mathcal{H}_n\right)\right] \\
&= \mathbf{Var}\left(\Lambda(X_{\emptyset})\right) - \tfrac{1}{2}\mathbf{E}\left[\left(\Lambda(X_{\emptyset}) - \Lambda(Y_{\emptyset}^n)\right)^2\right].
\end{aligned} \tag{2.17}$$

The last equality follows from similar reason given for (2.10).

Now suppose we show that

$$(X_{\emptyset}, Y_{\emptyset}^n) \xrightarrow{d} (X^{\star}, Y^{\star}) \tag{2.18}$$

for some limit $(X^\star, Y^\star)$. From the construction,

$$\begin{bmatrix} X_\emptyset \\ Y_\emptyset^{n+1} \end{bmatrix} \stackrel{d}{=} (T \otimes T)\left(\begin{bmatrix} X_\emptyset \\ Y_\emptyset^{n} \end{bmatrix}\right),$$

and then the weak continuity assumption on $T \otimes T$ implies

$$\begin{bmatrix} X^\star \\ Y^\star \end{bmatrix} \stackrel{d}{=} (T \otimes T)\left(\begin{bmatrix} X^\star \\ Y^\star \end{bmatrix}\right).$$

Also by construction we have $X_\emptyset \stackrel{d}{=} Y_\emptyset^n \stackrel{d}{=} \mu$ for all $n \geq 1$, and hence $X^\star \stackrel{d}{=} Y^\star \stackrel{d}{=} \mu$. The bivariate uniqueness of 2nd kind assumption now implies $X^\star$ and $Y^\star$ are independent. Since Λ is a bounded continuous function, (2.18) implies

$$\mathbf{E}\left[\left(\Lambda(X_\emptyset) - \Lambda(Y_\emptyset^n)\right)^2\right] \to \mathbf{E}\left[\left(\Lambda(X^\star) - \Lambda(Y^\star)\right)^2\right] = 2\mathbf{Var}\left(\Lambda(X_\emptyset)\right) \tag{2.19}$$

and so using (2.17) we see that $\bar{\sigma}_n^2(\Lambda) \longrightarrow 0$. Hence from (2.16) and (2.15) we conclude that $\Lambda(X_\emptyset)$ is independent of $\mathcal{H}$. This is true for every bounded continuous Λ, proving that $X_\emptyset$ is independent of $\mathcal{H}$, so from Lemma 13 it follows that $\mathcal{H}$ is trivial.

Now all remains is to show that limit (2.18) exists. Fix $f : S \to \mathbb{R}$ and $h : S \to \mathbb{R}$, two bounded continuous functions. Again by reverse martingale convergence

$$\mathbf{E}\left[f(X_\emptyset)\middle|\mathcal{H}_n\right] \xrightarrow[\mathcal{L}_1]{\text{a.s.}} \mathbf{E}\left[f(X_\emptyset)\middle|\mathcal{H}\right],$$

and similarly for h. So

$$\begin{aligned} \mathbf{E}\left[f(X_\emptyset)h(Y_\emptyset^n)\right] &= \mathbf{E}\left[\mathbf{E}\left[f(X_\emptyset)h(Y_\emptyset^n)\middle|\mathcal{H}_n\right]\right] \\ &= \mathbf{E}\left[\mathbf{E}\left[f(X_\emptyset)\middle|\mathcal{H}_n\right]\mathbf{E}\left[h(X_\emptyset)\middle|\mathcal{H}_n\right]\right], \end{aligned}$$

the last equality because of conditional on $\mathcal{H}_n$ $X_\emptyset$ and $Y_\emptyset^n$ are independent and identically distributed. Letting $n \to \infty$ we get

$$\mathbf{E}\left[f(X_\emptyset)h(Y_\emptyset^n)\right] \longrightarrow \mathbf{E}\left[\mathbf{E}\left[f(X_\emptyset)\middle|\mathcal{G}\right]\mathbf{E}\left[h(X_\emptyset)\middle|\mathcal{G}\right]\right]. \tag{2.20}$$

Moreover note that $X_\emptyset \stackrel{d}{=} Y_\emptyset^n \stackrel{d}{=} \mu$ and so the sequence of bivariate distributions $(X_\emptyset, Y_\emptyset^n)$ is tight. Tightness, together with convergence (2.20) for all bounded continuous f and h, implies weak convergence of $(X_\emptyset, Y_\emptyset^n)$.

(c) First assume that $(T \otimes T)^n \left(\mu^{\nearrow}\right) \stackrel{d}{\rightarrow} \mu \otimes \mu$, then with the same construction as done in part (b) we get that

$$\left(X_{\emptyset}, Y_{\emptyset}^n\right) \stackrel{d}{\rightarrow} \left(X^{\star}, Y^{\star}\right),$$

where $X^{\star}$ and $Y^{\star}$ are independent copies of $X_{\emptyset}$. Further recall that Λ is bounded continuous, thus using (2.17), (2.16) and (2.15) we conclude that $\Lambda(X_{\emptyset})$ is independent of $\mathcal{H}$. Since it is true for any bounded continuous function Λ, thus $X_{\emptyset}$ is independent of $\mathcal{H}$. Thus again by Lemma 13 the RTP has trivial tail.

Conversely, suppose that the invariant RTP with marginal μ has trivial tail. Let Λ_1 and Λ_2 ne two bounded continuous functions. Note that the variables $\left(X_{\emptyset}, Y_{\emptyset}^n\right)$, as defined in part (b) has joint distribution $(T \otimes T)^n \left(\mu^{\nearrow}\right)$. Further, given $\mathcal{H}_n$, they are conditionally independent and have same conditional law as of $X_{\emptyset}$ given $\mathcal{H}_n$. So

$$\begin{aligned}
\mathbf{E}\left[\Lambda_1(X_{\emptyset})\,\Lambda_2(Y_{\emptyset}^n)\right] &= \mathbf{E}\left[\mathbf{E}\left[\Lambda_1(X_{\emptyset})\middle|\mathcal{H}_n\right]\,\mathbf{E}\left[\Lambda_2(X_{\emptyset})\middle|\mathcal{H}_n\right]\right] \\
&\rightarrow \mathbf{E}\left[\mathbf{E}\left[\Lambda_1(X_{\emptyset})\middle|\mathcal{H}\right]\,\mathbf{E}\left[\Lambda_2(X_{\emptyset})\middle|\mathcal{H}\right]\right] \\
&= \mathbf{E}\left[\Lambda_1(X_{\emptyset})\right]\mathbf{E}\left[\Lambda_2(X_{\emptyset})\right].
\end{aligned}$$

The convergence is by reverse martingale convergence, and the last equality is by tail triviality and Lemma 13. So from definition we get

$$(T \otimes T)^n \left(\mu^{\nearrow}\right) \stackrel{d}{=} \left(X_{\emptyset}, Y_{\emptyset}^n\right) \stackrel{d}{\rightarrow} \mu \otimes \mu.$$

$\square$

Chapter 3

Discounted Tree Sums

3.1 Background and Motivation

Athreya [7] studied the following RDE which he called "discounted branching random walk"

$$X \stackrel{d}{=} \eta + c \max\left(X_1, X_2, \ldots, X_N\right) \quad \text{on} \quad S = \mathbb{R}^+, \tag{3.1}$$

where $(X_i)_{i \geq 0}$ are i.i.d. copies of X and are independent of (η, N), where $\eta \geq 0$, N is a non-negative integer valued random variable independent of η, and $0 < c < 1$ is a fixed constant ("discounting factor"). He considered the super-critical case, that is when $\mathbf{E}\left[N\right] > 1$. It is easy to see that the above RDE translates to the following integral equation

$$F(x) = \int_0^\infty \phi\left(F\left(\frac{x-t}{c}\right)\right) dH(t), \;\; x \geq 0, \tag{3.2}$$

where F and H are the distribution functions of X and η, and ϕ is the probability generating function of N. In particular he studied the equation (3.2) for $N \equiv 2$ and $\eta \sim$ Exponential(1). In this Chapter we will study the following generalization of the RDE (3.1)

$$X \stackrel{d}{=} \eta + \max_{1 \leq i \leq^* N} \xi_i X_i \quad \text{on} \quad S = \mathbb{R}^+ \tag{3.3}$$

where $(X)_{i \geq 1}$ are i.i.d. copies of X and $\left(\eta, \left(\xi_i, 1 \leq i \leq^* N\right), N\right)$ has some given distribution. We will only study the case where η has all moments finite and $0 \leq \xi_i < 1$. Assuming that (3.3) has a solution there is a natural interpretation of the associated RTP. Suppose at time $t = 0$ we start with one individual who lives for a random $\eta_\emptyset$ amount of time, after which it dies and gives birth to $N_\emptyset$ number of children, whose life-spans get *discounted* by a random factor ξ_i for the i^{th}-child. The children then evolves in a similar manner independent of each others. If $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ be the RTP then $X_{\mathbf{i}}$ is the *time of survival of the family started by the individual* $\mathbf{i}$. Durrett and Limic [15] proves the existence of such a process when $\eta \sim$ Exponential(1) and $(\xi_i)_{i\geq 1}$ are points of a Poisson point process of rate 1 on $(0,\infty)$ and they are independent. We will study this example in Section 3.3.

3.2 Contraction Argument

Let T be the associated operator then it is easy to see that T is monotone and hence by Lemma 3 the RDE (3.3) will have a solution if an only if $T^n(\delta_0)$ is tight. The following theorem which is a generalization of Rachev and Rüschendorf's result (see the comments on equation (9.1.18) of [25]) which only deals with non-random fixed N, gives a sufficient condition for existence of a solution.

Theorem 19 *Suppose $0 \leq \xi_i < 1$ for all $i \geq 1$ and $\eta \geq 0$ has all moments finite. For $p \geq 1$ write $c(p) := \sum_{i=1}^{\infty} \mathbf{E}\left[\xi_i^p\right] \leq \infty$. Suppose $c(p_0) < \infty$ for some $p_0 \geq 1$, then*

(a) There exists $p \geq p_0$ such that $c(p) < 1$ and then T is a strict contraction on $\mathcal{F}_p$, the space of all probability measures on $[0,\infty)$ with finite p^{th} moments under the standard Mallows p-metric.

(b) There is a solution $\mu_\star$ of the RDE (3.3) with all moments finite such that $T^n(\mu) \xrightarrow{d} \mu_\star$ for all $\mu \in \mathcal{F}_p$. In particular $T^n(\delta_0) \xrightarrow{d} \mu_\star$.

(c) The RTP with marginal $\mu_\star$ is endogenous.

Proof : (a) By assumption $c(p_0) < \infty$ for some $p_0 \geq 1$. Since $0 \leq \xi_i < 1$ for all $i \geq 1$, we clearly have $c(p) \downarrow 0$ as $p \uparrow \infty$. So choose and fix $p \geq p_0$ such that $c(p) < 1$. Recall

$$\mathcal{F}_p := \left\{ \mu \,\middle|\, \mu \text{ is a probability on } \mathbb{R}^+ \text{ and } \mu \text{ has finite } p^{\text{th}} - \text{moment} \right\}. \qquad (3.4)$$

We will first check that $T(\mathcal{F}_p) \subseteq \mathcal{F}_p$. Let $\mu \in \mathcal{F}_p$ and let $(X)_{i\geq 1}$ be i.i.d samples from μ which are independent of $\left((\xi_i)_{i\geq 1}, N; \eta\right)$. Define $[\mu]_p$ as the p^{th}-moment of μ. Observe that

$$\begin{aligned}
\mathbf{E}\left[\left(\max_{1\le i\le^* N}(\xi_i X_i)\right)^p\right] &= \mathbf{E}\left[\max_{1\le i\le^* N}(\xi_i^p X_i^p)\right] \\
&\le \mathbf{E}\left[\sum_{1\le i\le^* N}\xi_i^p X_i^p\right] \\
&= \mathbf{E}\left[\sum_{i=1}^{\infty}\xi_i^p X_i^p I(N\ge i)\right] \\
&\le \mathbf{E}\left[\sum_{i=1}^{\infty}\xi_i^p X_i^p\right] \\
&= \sum_{i=1}^{\infty}\mathbf{E}\left[\xi_i^p\right]\mathbf{E}\left[X_i^p\right] \\
&= [\mu]_p \times c(p) < \infty.
\end{aligned}$$

Further we have assumed that $\mathbf{E}\left[\eta^p\right] < \infty$, thus using (3.3) we conclude that T maps $\mathcal{F}_p$ to itself.

Let d_p be the Mallows metric on $\mathcal{F}_p$ defined as

$$d_p(\mu,\nu) := \inf\left\{ \left(\mathbf{E}\left[|Z-W|^p\right]\right)^{1/p} \,\middle|\, Z\sim\mu \text{ and } W\sim\nu \right\}. \tag{3.5}$$

Fix $\mu,\nu\in\mathcal{F}_p$. By standard coupling argument construct i.i.d samples $((X_i,Y_i))_{i\ge 1}$ such that

- They are independent of $\left((\xi_i)_{i\ge 1}, N; \eta\right)$;
- $X_i\sim\mu$ and $Y_i\sim\nu$ for all $i\ge 1$;
- $d_p(\mu,\nu) = \left(\mathbf{E}\left[|X_i - Y_i|^p\right]\right)^{1/p}$.

Put $Z = \eta + \max_{1 \leq i \leq^* N} (\xi_i X_i)$ and $W = \eta + \max_{1 \leq i \leq^* N} (\xi_i Y_i)$. Notice that from definition $Z \sim T(\mu)$ and $W \sim T(\nu)$.

$$
\begin{aligned}
\left(d_p\left(T(\mu), T(\nu)\right)\right)^p & \leq \mathbf{E}\left[|Z - W|^p\right] \\
& = \mathbf{E}\left[\left|\max_{1 \leq i \leq^* N} \xi_i X_i - \max_{1 \leq i \leq^* N} \xi_i Y_i\right|^p\right] \\
& \leq \mathbf{E}\left[\max_{1 \leq i \leq^* N} |\xi_i X_i - \xi_i Y_i|^p\right] \\
& \leq \mathbf{E}\left[\sum_{i=1}^{\infty} |\xi_i X_i - \xi_i Y_i|^p\right] \\
& = \sum_{i=1}^{\infty} \mathbf{E}\left[\xi_i^p\right] \left(d_p\left(\mu, \nu\right)\right)^p \\
& = c(p) \times \left(d_p\left(\mu, \nu\right)\right)^p
\end{aligned}
$$

So T is a strict contraction map on $\mathcal{F}_p$.

(b) Since $\mathcal{F}_p$ under the metric d_p is a complete metric space so by Banach contraction mapping principle it follows that there is a unique $\mu_\star \in \mathcal{F}_p$ such that $T^n(\mu_0) \xrightarrow{d} \mu_\star$ for all $\mu \in \mathcal{F}_p$ and $T(\mu_\star) = \mu_\star$. So $\mu_\star$ is a solution of the RDE (3.3). In particular $T^n(\delta_0) \xrightarrow{d} \mu_\star$. Since $c(p) \downarrow 0$ as $p \uparrow \infty$ so repeating same argument for larger and larger p it follows that $\mu_\star$ has all moments finite.

(c) From part (b) we know that $T^n(\delta_0) \xrightarrow{d} \mu_\star$. But easy recursion shows that $T^n(\delta_0)$ is the distribution of the following random variable

$$
D_n := \max_{\mathbf{i} = (\emptyset = i_0, i_1, \ldots, i_{n-1})} \sum_{d=0}^{n-1} \eta_{(i_0, i_1, \ldots, i_d)} \prod_{j=1}^{d} \xi_{(i_0, i_1, \ldots, i_j)} \tag{3.6}
$$

where $(\xi_{\mathbf{i}}, N_{\mathbf{i}}; \eta_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ is the i.i.d innovation process of the associated RTF. Note that here each $\xi_{\mathbf{i}}$ is a infinite string of random variables written as $(\xi_{\mathbf{i}1}, \xi_{\mathbf{i}2}, \ldots)$ which are the respective discounting factors for the 1st, 2nd, $\ldots$ child of $\mathbf{i}$. Now let $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ be the RTP with marginal $\mu_\star$ then from construction

$$
D_1 \leq D_2 \leq \cdots \leq D_n \leq D_{n+1} \leq \cdots \leq X_\emptyset .
$$

The last inequality follows because $X_{\mathbf{i}} \geq 0$ a.s. for all $\mathbf{i}$ and the right hand side of (3.3) is pointwise increasing in X-variables. But $D_n \xrightarrow{d} \mu_\star = \text{dist}(X_\emptyset)$ and hence $X_\emptyset = \lim_{n \to \infty} D_n$. Thus the RTP is endogenous. $\square$

So the theorem tells us that under suitable condition (like $c(p) < 1$) the RDE (3.3) has a solution with all moments finite which is unique among the class of measures in $\mathcal{F}_p$. As we will see in the next section that this is unfortunately not the complete answer. It is possible that other solutions might exist with none or some moments finite. Athreya [7] observes the same in the particular case (3.1) with a slightly more general sufficient condition. Under our assumption Athreya's condition is satisfied giving the same solution $\mu_\star$ with all moments finite.

3.3 A Concrete Example

Durrett and Limic [15] studied a particular Poisson process arising from a species competition model which they comment that can be thought of as the following branching Markov process (the authors write "David Aldous has pointed out that time reversal of our process together with a transformation of $(0,1)$ to $(0,\infty)$ by $x \to -\log(1-x)$ is an interesting Markov chain ...") taking values in the space of countable subsets of $(0,\infty)$. At time $t = 0$ start with one individual at position 0. Each individual at position x at time t lives for an independent Exponential(e^x) amount of time after which it dies and instantaneously gives birth to infinite number of children to be placed at positions $(x + \xi_i)_{i \geq 1}$ where $(\xi_i)_{i \geq 1}$ are points of an independent Poisson rate 1 process on $(0,\infty)$. Durrett and Limic [15] showed that for each $\lambda < 1$ the Poisson rate λ process on $(0,\infty)$ is a stationary law for for the above branching Markov process. It is easy to see that the time of extinction of the process when started with only one particle at position 0 satisfies the following RDE

$$X \stackrel{d}{=} \eta + \max_{i \geq 1} e^{-\xi_i} X_i \quad \text{on} \quad S = \mathbb{R}^+, \tag{3.7}$$

where (X_i) are i.i.d copies of X which are independent of $\Big((\xi_i)_{i \geq 1} ; \eta \Big)$, $\eta \sim$ Exponential(1) and is independent of $(\xi_i)_{i \geq 1}$. This is a special case of the RDEs (3.3) discussed in the previous sections.

Now for this example we first compute the quantity $c(p)$ for $p \geq 1$ defined in Theorem 19.

$$\begin{aligned} c(1) &= \sum_{i=1}^{\infty} \mathbf{E}\left[e^{-p\,\xi_i}\right] \\ &= \sum_{i=1}^{\infty} \left(\mathbf{E}\left[e^{-p\,\xi_1}\right]\right)^i \\ &= \sum_{i=1}^{\infty} \left(\tfrac{1}{1+p}\right)^i = \frac{1}{p} < \infty \end{aligned} \tag{3.8}$$

Thus from Theorem 19 we get that there is a solution with all moments finite which is unique among all solutions with finite $(1+\varepsilon)^{\text{th}}$ moments for any $\varepsilon > 0$. Further this solution is also endogenous.

We now consider the *homogeneous* equation, that is with $\eta \equiv 0$

$$X \stackrel{d}{=} \max_{i \geq 1} e^{-\xi_i} X_i \quad \text{on} \quad S = \mathbb{R}^+. \tag{3.9}$$

Naturally δ_0 is the solution with finite moments but, as we shall see that this is not the unique solution, instead there is a one parameter family of solutions.

Proposition 20 *The set of all solutions of the RDE (3.9) is given by*

$$F_a(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{a+x} & \text{if } x \geq 0 \end{cases} \tag{3.10}$$

where $a \geq 0$. In particular for $a = 0$ it is the solution δ_0.

Proof : Let μ be a solution of (3.9). Notice that the points $\left\{(\xi_i; X_i) \,\middle|\, i \geq 1\right\}$ form a Poisson point process, say $\mathfrak{P}$, on $(0,\infty)^2$ with mean intensity $dt\,\mu(dx)$. Thus if $F(x) = \mathbf{P}\left(X \leq x\right)$ then for $x > 0$

$$\begin{aligned} F(x) &= \mathbf{P}\left(\max_{i \geq 1} e^{-\xi_i} X_i \leq x\right) \\ &= \mathbf{P}\left(\text{No points of } \mathfrak{P} \text{ are in } \left\{(t,z) \,\middle|\, e^{-t} z > x\right\}\right) \\ &= \exp\left(-\iint_{e^{-t}z > x} dt\,\mu(dx)\right) \\ &= \exp\left(-\int_x^{\infty} \frac{1-F(u)}{u}\,du\right) \end{aligned} \tag{3.11}$$

We note that F is infinitely differentiable so by differentiating (3.11) we get

$$\frac{dF}{dx} = \frac{F(x)(1-F(x))}{x} \quad \text{for} \quad x > 0. \tag{3.12}$$

It is easy to solve the equation (3.12) to get that the set of all solutions is given by (3.10). □

The following is an immediate corollary of the above proposition.

Corollary 21 *Consider the following RDE*

$$X \overset{d}{=} \min_{i \geq 1} (\xi_i + X_i) \quad \textit{on} \quad S = \mathbb{R}, \tag{3.13}$$

where $(X_i)_{i \geq 1}$ *are i.i.d. copies of* X *and are independent of* $(\xi_i)_{i \geq 1}$ *which are points of a Poisson process of rate* 1 *on* $(0, \infty)$*. The set of all solutions of this RDE is given by*

$$H_c(x) = \frac{1}{1 + e^{-(x+c)}}, \quad x \in \mathbb{R}, \tag{3.14}$$

where $c \in \mathbb{R}$*.*

Proof : There is a one-to-one mapping between the non-zero solutions of the equation (3.9) and the finite solutions of the (3.13) through the map $x \mapsto \exp(-x)$. The rest follows by easy computation. □

In other words if X is a solution of the RDE (3.13) then all the solutions are given by the translates of X.

It is interesting to note that the solution H_0 of (3.13) is the Logistic distribution defined in Chapter 4. The RDE (3.13) looks similar to the Logistic RDE (4.1) but it is genuinely different. For the first place it has one parameter family of solutions given by the translates of the Logistic distribution while the Logistic RDE has unique solution. Secondly we will see that all the solutions of the RDE (3.13) are not endogenous while in Chapter 4 we will prove that the Logistic RDE is endogenous.

Observe that the RDE (3.13) is equivalent (through the map $x \mapsto -x$) to the following RDE

$$X \overset{d}{=} \max_{i \geq 1} (\xi_i + X_i) \quad \text{on} \quad S = \mathbb{R}, \tag{3.15}$$

where $(-\xi_i)_{i\geq 1}$ are points of a Poisson process of rate 1. Such RDEs appear in the study of the *branching random walk* (see Aldous and Bandyopadhyay [2] for a survey).

Finally we will prove that any solution of the RDE (3.15) is not endogenous. This will show that all the non-zero solutions of the original homogeneous RDE (3.9) are not endogenous. We start by proving the following technical lemma.

Lemma 22 *For a r.v. Y and $\delta > 0$ define*

$$\operatorname{conc}\left(\operatorname{dist}(Y), \delta\right) = \max_a P(a \leq Y \leq a + \delta).$$

Suppose (Z_i) are i.i.d. with $P(Z > x) \sim ce^{-\alpha x}$ as $x \to \infty$. Then there exists $\delta > 0$, depending only on the distribution of Z, such that for every countable set (x_i) of reals for which $Y := \max_i (x_i + Z_i) < \infty$ a.s., we have $\operatorname{conc}\left(\operatorname{dist}(Y), \delta\right) \leq 1 - \delta$*.*

Proof : Suppose if possible the statement of the lemma is not true. Then for every $\delta_n \downarrow 0+$ we can find a countable collection of reals $(x_i^n)_{i\geq 1}$ such that $Y_n := \max_{i\geq 1} \left(x_i^n + Z_i\right) < \infty$ a.s. and

$$\mathbf{P}\left(0 \leq Y_n \leq \delta_n\right) \geq 1 - \delta_n. \tag{3.16}$$

Notice that since $Y_n < \infty$ a.s. and we assumed that $P(Z > x) \sim ce^{-\alpha x}$ as $x \to \infty$, thus

$$0 < \sum_{i=1}^{\infty} e^{\alpha x_i^n} < \infty. \tag{3.17}$$

So in particular $x_i^n \to -\infty$ as $i \to \infty$ for every $n \geq 1$. Thus without lose of any generality we can assume that (x_i^n) are in decreasing order.

Let F be the distribution function of Z and we will write $\bar{F}\left(\cdot\right) = 1 - F\left(\cdot\right)$. We calculate,

$$\begin{aligned}
\mathbf{P}\left(0 \leq Y_n \leq \delta_n\right) &= 1 - \mathbf{P}\left(Y_n \notin [0, \delta_n]\right) \\
&= 1 - \mathbf{P}\left(Z_i < -x_i^n \text{ for all } i \geq 1, \text{ or } Z_i > \delta_n - x_i^n \text{ for some } i \geq 1\right) \\
&\leq 1 - \prod_{i=1}^{\infty} F\left(-\lambda - x_i^n\right) - \max_{i\geq 1} \bar{F}\left(\delta_n - x_i^n\right),
\end{aligned}$$

where $\lambda > 0$ is a fixed number. So from (3.16) we get

$$\prod_{i=1}^{\infty} F\left(-\lambda - x_i^n\right) + \max_{i\geq 1} \bar{F}\left(\delta_n - x_i^n\right) \leq \delta_n. \tag{3.18}$$

But $\max_{i\geq 1} \bar{F}\left(\delta_n - x_i^n\right) = \bar{F}\left(\delta_n - x_1^n\right)$, so using (3.18) we get

$$\begin{aligned} & \lim_{n\to\infty} \bar{F}\left(\delta_n - x_1^n\right) = 0 \\ \Longrightarrow \quad & x_1^n \to -\infty \text{ as } \quad n \to \infty. \end{aligned} \tag{3.19}$$

Fix $\varepsilon > 0$, from assumption there exists $M > 0$ such that

$$(1-\varepsilon)\, c\, e^{-\alpha x} \;\leq\; \bar{F}\,(x) \;\leq\; (1+\varepsilon)\, c\, e^{-\alpha x} \;\text{ for all } \quad x > M - \lambda. \tag{3.20}$$

Find $N \geq 1$ such that for all $n \geq N$ we have $x_1^n < -M$, and hence $x_i^n < -M$ for all $i \geq 1$. Now from (3.18)

$$\begin{aligned} \delta_n \;\geq\; \bar{F}\left(\delta_n - x_1^n\right) \;\geq\; (1-\varepsilon)\, c\, e^{-\alpha\left(\delta_n - x_1^n\right)} \\ \Longrightarrow \; e^{\alpha x_1^n} \;\leq\; \frac{1}{c(1-\varepsilon)} \delta_n e^{\alpha \delta_n}. \end{aligned} \tag{3.21}$$

Further

$$\begin{aligned} \prod_{i=1}^{k_n} F\left(-\lambda - x_i^n\right) \;&=\; \prod_{i=1}^{k_n} \left(1 - \bar{F}\left(-\lambda - x_i^n\right)\right) \\ &\geq\; \left(1 - \bar{F}\left(-\lambda - x_1^n\right)\right)^{k_n} \\ &\geq\; \left(1 - (1+\varepsilon) c e^{\alpha\lambda} e^{\alpha x_1^n}\right)^{k_n} \\ &\geq\; \left(1 - \frac{1+\varepsilon}{1-\varepsilon} e^{\alpha\lambda} \delta_n e^{\alpha\delta_n}\right)^{k_n}, \end{aligned} \tag{3.22}$$

the last inequality follows from (3.21). Now take $k_n = \frac{1}{\sqrt{\delta_n}} \uparrow \infty$ to get

$$\liminf_{n\to\infty} \prod_{i=1}^{k_n} F\left(-\lambda - x_i^n\right) \;\geq\; 1.$$

This contradicts (3.18). $\square$

Proposition 23 *Suppose X is a solution of the RDE (3.15), then the invariant RTP associated with this solution is not endogenous.*

Proof : Consider the RTF associated with the RDE (3.15). Let $(Q_{\mathbf{i}}, \mathbf{i} \in \mathcal{T})$ be the associated BRW; that is, $\mathcal{T}$ is the family tree of the progenitor, and $Q_{\mathbf{i}}$ is the position of the $\mathbf{i}^{\text{th}}$ individual on $\mathbb{R}$, with $Q_{\emptyset} = 0$. Fix $d \geq 1$, let $\left\{ Z_{\mathbf{i}}^{(d)} \,\middle|\, \text{gen}(\mathbf{i}) = d \right\}$ be i.i.d. with the invariant distribution as of X. For $\mathbf{i} \in \mathcal{T}$ define

- $Y_{\mathbf{i}}^{(d)} = Z_{\mathbf{i}}^{(d)}$, when $\text{gen}(\mathbf{i}) = d$;
- $Y_{\mathbf{i}}^{(d)} = \max\left\{ Q_{\mathbf{j}} - Q_{\mathbf{i}} + Z_{\mathbf{j}}^{(d)} \,\middle|\, \text{gen}(\mathbf{j}) = d \text{ and } \mathbf{j} \text{ is a descendant of } \mathbf{i} \right\}$, when $\text{gen}(\mathbf{i}) \in \{d-1, d-2, \ldots, 1, 0\}$.

It is easy to check that $\left(Y_{\mathbf{i}}^{(d)}\right)$ defines an invariant RTP of depth d.

Let $\mathcal{G}_d$ be the σ-field generated by the first d generations of the BRW, naturally $\mathcal{G}_d \uparrow \mathcal{G}$, the σ-field generated by all the $\xi_{\mathbf{i}}$'s. Observe that

$$Y_{\emptyset}^{(d)} = \max\left\{ Q_{\mathbf{j}} + Z_{\mathbf{j}}^{(d)} \,\middle|\, \text{gen}(\mathbf{j}) = d \right\}.$$

So under the conditional distribution given $\mathcal{G}_d$, the random variable $Y_{\emptyset}^{(d)}$ has the same form as in the Lemma 22 with the role of the (x_i) being played by the $\mathcal{G}_d$-measurable random variables $(Q_{\mathbf{j}}, \text{gen}(\mathbf{j}) = d)$, and the role of the (Z_i) being played by the i.i.d. random variables $\left(Z_{\mathbf{j}}^{(d)}, \text{gen}(\mathbf{j}) = d\right)$. Now from Corollary 21 we know that X has exponential right tail, so Lemma 22 implies that there exists $\delta > 0$ depending only on the distribution of X such that,

$$\text{conc}\left(\text{dist}\left(Y_{\emptyset}^{(d)} \middle| \mathcal{G}_d\right), \delta\right) \leq 1 - \delta. \tag{3.23}$$

This inequality is true for any invariant RTP of depth at least d, so in particular true for the invariant RTP of infinite depth. Thus we have

$$\begin{aligned} & \text{conc}\left(\text{dist}\left(Y_{\emptyset} \middle| \mathcal{G}_d\right), \delta\right) \leq 1 - \delta \\ \Rightarrow \quad & \max_{-\infty < a < \infty} \mathbf{P}\left(a \leq Y_{\emptyset} \leq a + \delta \,\middle|\, \mathcal{G}_d\right) \leq 1 - \delta \\ \Rightarrow \quad & \max_{-\infty < a < \infty} \mathbf{E}\left[\mathbf{I}\left(a \leq Y_{\emptyset} \leq a + \delta\right) \,\middle|\, \mathcal{G}_d\right] \leq 1 - \delta \end{aligned}$$

Now assume that the invariant RTP is endogenous, that is, $Y_{\emptyset}$ is $\mathcal{G}$-measurable, then using the martingale convergence theorem we get for each $a \in \mathbb{R}$

$$\mathbf{I}\left(a \leq Y_{\emptyset} \leq a + \delta\right) \leq 1 - \delta \quad \text{a.s.}$$

and hence

$$\mathbf{P}\left(Y \notin [q, q + \delta] \text{ for all } q \text{ rational }\right) = 1,$$

this is clearly a contradiction. So the invariant RTP associated with the solution X is not endogenous. $\square$

Chapter 4

Logistic RDE

Consider the following RDE

$$X \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j \right), \tag{4.1}$$

where $(\xi_j)_{j\geq 1}$ are points of a Poisson point process of rate 1 on $(0, \infty)$, and are independent of $(X_j)_{j\geq 1}$, which are independent and identically distributed with same law as of X. Aldous [5] showed that this RDE has unique solution as the Logistic distribution defined below.

Definition 24 We say a real valued random variable Z has *Logistic* distribution if its distribution function is given by

$$H(x) := \mathbf{P}\left(Z \leq x\right) = \frac{1}{1 + e^{-x}}, \;\; x \in \mathbb{R} \tag{4.2}$$

In this chapter we will study the endogenous property of the Logistic RDE (4.1). The following theorem which is the main result of this chapter shows that the associated RTP has the bivariate uniqueness property of 1st kind and hence endogeny follows from Theorem 12 of Chapter 2.

Theorem 25 *Let $0 < \xi_1 < \xi_2 < \cdots$ be points of a Poisson point process of rate 1 on $(0, \infty)$. Let $(X, Y), ((X_j, Y_j))_{j\geq 1}$ be independent random variables with some common distribution ν on $\mathbb{R}^2$, which are independent of $(\xi_j)_{j\geq 1}$. Then*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \min\limits_{j\geq 1} \left(\xi_j - X_j\right) \\ \min\limits_{j\geq 1} \left(\xi_j - Y_j\right) \end{pmatrix}, \tag{4.3}$$

if and only if $\nu = \mu^{\nearrow}$, where $\mu^{\nearrow}$ is defined as the joint distribution of (Z, Z) on $\mathbb{R}^2$, with $Z \sim$ Logistic distribution.

Theorem 25 is a concrete result falling within the framework of *recursive distributional equations* discussed in Chapter 2. This particular problem arose in the study of a classical problem of combinatorial optimization, namely, the *mean-field random assignment problem.* The following section develops background and provides the details of our motivation for Theorem 25. In Section 4.2 we give a proof of Theorem 25 using analytic techniques. Some technical results which are not terribly important, but are needed for the proof, are given separately in Section 4.3 and in Section 4.4 we recall some basic facts about the Logistic distribution which are useful for the proofs.

4.1 Background and Motivation

For a given $n \times n$ matrix of costs (C_{ij}), consider the problem of assigning n jobs to n machines in the most "cost effective" way. Thus the task is to find a permutation π of $\{1, 2, \ldots, n\}$, which solves the following minimization problem

$$A_n := \min_{\pi} \sum_{i=1}^{n} C_{i,\pi(i)}. \tag{4.4}$$

This problem has been extensively studied in literature for a fixed cost matrix, and there are various algorithms to find the optimal permutation π. A probabilistic model for the assignment problem can be obtained by assuming that the costs are independent random variables each with distribution Uniform$[0, 1]$. Although this model appears to be rather simple, careful investigations of it in the last few decades have proven that, it has enormous richness in its structure. For a careful survey and other related works see [29, 3].

Our interest in this problem is from another perspective. In 2001 Aldous [5] showed that

$$\lim_{n \to \infty} \mathbf{E}[A_n] = \zeta(2) = \frac{\pi^2}{6}, \tag{4.5}$$

confirming the earlier work of Mézard and Parisi [23], where they computed the same limit using some non-rigorous arguments based on the *replica method* [24]. In an earlier work Aldous [1] showed that the limit of $\mathbf{E}[A_n]$ as $n \to \infty$ exists for any cost distribution, and does not depend on the specifics of it, except only on the value of its density at 0, provided it exists and is strictly positive. So for calculation of the limiting constant one can assume that C_{ij}'s are independent and each has Exponential distribution with mean n, and re-write the objective function A_n in the normalized form,

$$A_n = \min_{\pi} \frac{1}{n} \sum_{i=1}^{n} C_{i,\pi(i)}. \tag{4.6}$$

Aldous [5] identified the limit constant $\zeta(2)$ in terms of an optimal matching problem on a limit infinite tree with random edge weights. This structure is called *Poisson Weighted Infinite Tree*, or, *PWIT*, it is described as follows (see the survey of Aldous and Steele [3] for a more friendly account).

> Let $\mathbb{T} := (\mathcal{V}, \mathcal{E})$ be the canonical infinite rooted tree as defined in Chapter 2, Section 2.3. For every vertex $\mathbf{i} \in \mathcal{V}$, let $(\xi_{\mathbf{i}j})_{j\geq 1}$ be points of independent Poisson process of rate 1 on $(0, \infty)$. Define the weight of the edge $e = (\mathbf{i}, \mathbf{i}j)$ as $\xi_{\mathbf{i}j}$.

Aldous [5] showed that on a PWIT one can construct random variables $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ taking values in $\mathbb{R}$, such that

- $X_{\mathbf{i}} = \min_{j\geq 1} (\xi_{\mathbf{i}j} - X_{\mathbf{i}j})\,, \;\; \forall \;\mathbf{i} \in \mathcal{V}.$
- $X_{\mathbf{i}}$ is independent of $\left\{ (\xi_{\mathbf{i}'j})_{j\geq 1} \,\middle|\, \mathrm{gen}(\mathbf{i}') < \mathrm{gen}(\mathbf{i}) \right\}$, for all $\mathbf{i} \in \mathcal{V}\backslash\{\emptyset\}$.
- $X_{\mathbf{i}} \sim$ Logistic distribution.

In the abstract setting of Chapter 2 we observe that this construction of $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is nothing but the standard construction of an invariant RTP associated with the Logistic RDE (4.1). In [5] Aldous has a heuristic interpretation of $X_{\mathbf{i}}$'s through the edge weights. Thus it is natural to ask if they are actually measurable with respect to the sigma-field generated by the edge weights, this is preciously asking about the endogenous property of the Logistic RTP. This will also answer the question of Aldous (see remarks (4.2.d) and (4.2.e) in [5]). From the first equivalence theorem (Theorem 12) discussed in Chapter 2 we know that the RTP is endogenous if and only if the "bivariate uniqueness" property of Theorem 25 holds. Of course one will need to check the technical condition of the part (b) of the Theorem 12. In our current setting the Theorem 12 specializes as follows

Theorem 26 *Suppose $\mathfrak{S}$ is the set of all probabilities on $\mathbb{R}^2$ and we define $\Lambda : \mathfrak{S} \to \mathfrak{S}$ as*

$$\Lambda(\nu) \stackrel{d}{=} \left(\begin{array}{c} \min_{j \geq 1} (\xi_j - X_j) \\ \min_{j \geq 1} (\xi_j - Y_j) \end{array} \right), \tag{4.7}$$

where $(\xi_j)_{j\geq 1}$ are points of a Poisson process with mean intensity 1 on $(0, \infty)$, and are independent of $(X_j, Y_j)_{j\geq 1}$, which are i.i.d with distribution ν on $\mathbb{R}^2$. Suppose further that Λ is continuous with respect to the weak convergence topology when restricted to the subspace $\mathfrak{S}^\star$ of $\mathfrak{S}$ defined as

$$\mathfrak{S}^\star := \left\{ \nu \in \mathfrak{S} \,\middle|\, \textit{both the marginals of } \nu \textit{ are Logistic distribution} \right\}. \tag{4.8}$$

If the fixed-point equation $\Lambda(\nu) = \nu$ has unique solution as $\mu^{\nearrow}$ (as defined in Theorem 25) then $X_\emptyset$ as defined above is measurable with respect to the σ-field $\mathcal{G}$.

Notice that Theorem 25 basically states that $\Lambda(\nu) = \nu$ has unique solution as $\mu^{\nearrow}$. Further it is easy to see that the operator Λ is continuous with respect to the weak convergence topology when restricted to the subspace $\mathfrak{S}^\star$ (see Proposition 33 of Section 4.3 for a proof). Thus we have the following immediate corollary of Theorem 26 and Theorem 25

Corollary 27 *The RTP associated with the Logistic RDE (4.1) is endogenous.*

4.2 Proof of The Bivariate Uniqueness

First observe that if the equation (4.3) has a solution then, the marginal distributions of X and Y solve the Logistic RDE, and hence they are both Logistic. Further by inspection $\mu^{\nearrow}$ is a solution of (4.3). So it is enough to prove that $\mu^{\nearrow}$ is the only solution of (4.3).

Let ν be a solution of (4.3). Notice that the points $\{(\xi_j; (X_j, Y_j)) \mid j \geq 1\}$ form a Poisson point process, say $\mathcal{P}$, on $(0, \infty) \times \mathbb{R}^2$, with mean intensity $\rho(t; (x, y))\, dt\, d(x, y) :=$

$dt\,\nu(d(x,y))$. Thus if $G(x,y) := \mathbf{P}\left(X > x, Y > y\right)$, for $x, y \in \mathbb{R}$, then

$$
\begin{aligned}
G(x,y) &= \mathbf{P}\left(\min_{j\geq 1}(\xi_j - X_j) > x, \text{ and, } \min_{j\geq 1}(\xi_j - Y_j) > y\right) \\
&= \mathbf{P}\left(\text{No points of } \mathfrak{P} \text{ are in } \left\{(t;(u,v)) \,\middle|\, t-u \leq x, \text{ or, } t-v \leq y\right\}\right) \\
&= \exp\left(-\iiint_{t-u\leq x, \text{ or, } t-v\leq y} \rho(t;(u,v))\,dt\,d(u,v)\right) \\
&= \exp\left(-\int_0^\infty \left[\overline{H}(t-x) + \overline{H}(t-y) - G(t-x,t-y)\right]\,dt\right) \\
&= \overline{H}(x)\,\overline{H}(y)\,\exp\left(\int_0^\infty G(t-x,t-y)\,dt\right). \qquad (4.9)
\end{aligned}
$$

The last equality follows from properties of the Logistic distribution (see Fact 36 of Section 4.4). For notational convenience we will write $\overline{F}(\cdot) := 1 - F(\cdot)$, for any distribution function F.

The following simple observation reduces the bivariate problem to a univariate problem.

Lemma 28 *For any two random variables U and V, $U = V$ a.s. if and only if $U \stackrel{d}{=} V \stackrel{d}{=} U \wedge V$.*

Proof : First of all if $U = V$ a.s. then $U \wedge V = U$ a.s.

Conversely suppose that $U \stackrel{d}{=} V \stackrel{d}{=} U \wedge V$. Fix a rational q, then under our assumption,

$$
\begin{aligned}
\mathbf{P}\left(U \leq q < V\right) &= \mathbf{P}\left(V > q\right) - \mathbf{P}\left(U > q,\, V > q\right) \\
&= \mathbf{P}\left(V > q\right) - \mathbf{P}\left(U \wedge V > q\right) \\
&= 0
\end{aligned}
$$

A similar calculation will show that $\mathbf{P}\left(V \leq q < U\right) = 0$. These are true for any rational q, thus $\mathbf{P}\left(U \neq V\right) = 0$. □

Thus if we can show that $X \wedge Y$ also has Logistic distribution, then from the lemma above we will be able to conclude that $X = Y$ a.s., and hence the proof will be complete. Put $g(\cdot) := \mathbf{P}\left(X \wedge Y > \cdot\right)$, we will show $g = \overline{H}$. Now, for every fixed $x \in \mathbb{R}$, $g(x) = G(x,x)$ by definition. So using (4.9) we get

$$
g(x) = \overline{H}^2(x)\,\exp\left(\int_{-x}^\infty g(s)\,ds\right),\ x \in \mathbb{R}. \qquad (4.10)
$$

Notice that from (4.29) (see Fact 38 of Section 4.4) $g = \overline{H}$ is a solution of this non-linear integral equation (4.10), which corresponds to the solution $\nu = \mu^{\nearrow}$ of the original equation (4.3). To complete the proof of Theorem 25 we need to show that this is the only solution. For that we will prove that the operator associated with (4.10) (defined on an appropriate space) is monotone and has unique fixed-point as $\overline{H}$. The techniques we will use here are similar to Eulerian recursion [28], and are based heavily on analytic arguments.

Let $\mathfrak{F}$ be the set of all functions $f : \mathbb{R} \to [0,1]$ such that

- $\overline{H}^2(x) \leq f(x) \leq \overline{H}(x), \ \forall \ x \in \mathbb{R}$,
- f is a tail of a distribution, that is, $\exists$ random variable say W such that $f(x) = \mathbf{P}\left(W > x\right), \, x \in \mathbb{R}$.

Observe that by definition $\overline{H} \in \mathfrak{F}$. Further, from (4.10) it follows that $g(x) \geq \overline{H}^2(x), \ \forall \ x \in \mathbb{R}$, as well as, $g(x) = \mathbf{P}\left(X \wedge Y > x\right) \leq \mathbf{P}\left(X > x\right) = \overline{H}(x), \ \forall \ x \in \mathbb{R}$. So it is appropriate to search for solutions of (4.10) in $\mathfrak{F}$.

Let $T : \mathfrak{F} \to \mathfrak{F}$ be defined as

$$T(f)(x) := \overline{H}^2(x) \, \exp\left(\int_{-x}^{\infty} f(s)\, ds\right), \ x \in \mathbb{R}. \tag{4.11}$$

Proposition 34 of Section 4.3 shows that T does indeed map $\mathfrak{F}$ into itself. Observe that the equation (4.10) is nothing but the fixed-point equation associated with the operator T, that is,

$$g = T(g) \ \text{ on } \ \mathfrak{F}. \tag{4.12}$$

We here note that using (4.29) (see Fact 38 of Section 4.4) T can also be written as

$$T(f)(x) := \overline{H}(x) \, \exp\left(-\int_{-x}^{\infty} \left(\overline{H}(s) - f(s)\right) \, ds\right), \ x \in \mathbb{R}, \tag{4.13}$$

which will be used in the subsequent discussion.

Define a partial order $\preccurlyeq$ on $\mathfrak{F}$ as, $f_1 \preccurlyeq f_2$ in $\mathfrak{F}$ if $f_1(x) \leq f_2(x), \ \forall \ x \in \mathbb{R}$, then the following result holds.

Lemma 29 *T is a monotone operator on the partially ordered set $(\mathfrak{F}, \preccurlyeq)$.*

Proof : Let $f_1 \preccurlyeq f_2$ be two elements of $\mathfrak{F}$, so from definition $f_1(x) \leq f_2(x)$, $\forall\ x \in \mathbb{R}$. Hence

$$\begin{array}{rcl} \int\limits_{-x}^{\infty} f_1(s)\,ds & \leq & \int\limits_{-x}^{\infty} f_2(s)\,ds, \quad \forall\ x \in \mathbb{R} \\ \Rightarrow \quad T(f_1)(x) & \leq & T(f_2)(x), \quad \forall\ x \in \mathbb{R} \\ \Rightarrow \quad T(f_1) & \preccurlyeq & T(f_2). \end{array}$$

$\square$

Put $f_0 = \overline{H}^2$, and for $n \in \mathbb{N}$, define $f_n \in \mathfrak{F}$ recursively as, $f_n = T(f_{n-1})$. Now from Lemma 29 we get that if g is a fixed-point of T in $\mathfrak{F}$ then,

$$f_n \preccurlyeq g, \ \ \forall\ \ n \geq 0. \tag{4.14}$$

If we can show $f_n \to \overline{H}$ pointwise, then using (4.14) we will get $\overline{H} \preccurlyeq g$, so from definition of $\mathfrak{F}$ it will follow that $g = \overline{H}$, and our proof will be complete. For that, the following lemma gives an explicit recursion for the functions $\{f_n\}_{n\geq 0}$.

Lemma 30 *Let $\beta_0(s) = 1 - s$, $0 \leq s \leq 1$. Define recursively*

$$\beta_n(s) := \int_s^1 \frac{1}{w}\left(1 - e^{-\beta_{n-1}(1-w)}\right)\,dw, \ \ 0 < s \leq 1. \tag{4.15}$$

Then for $n \geq 1$,

$$f_n(x) = \overline{H}(x)\,\exp\left(-\beta_{n-1}(\overline{H}(x))\right), \ \ x \in \mathbb{R}. \tag{4.16}$$

Proof : We will prove this by induction on n. Fix $x \in \mathbb{R}$, for $n = 1$ we get

$$\begin{aligned} f_1(x) &= T(f_0)(x) \\ &= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}\left(\overline{H}(s) - \overline{H}^2(s)\right)\,ds\right) \qquad \text{[using (4.13)]} \\ &= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}\overline{H}(s)\,\left(1 - \overline{H}(s)\right)\,ds\right) \\ &= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}\overline{H}(s)\,H(s)\,ds\right) \\ &= \overline{H}(x)\,\exp\left(-\int_{-x}^{\infty}H'(s)\,ds\right) \qquad \text{[using Fact 36 of Section 4.4]} \\ &= \overline{H}(x)\,\exp\left(-H(x)\right) \\ &= \overline{H}(x)\,\exp\left(-\beta_0(\overline{H}(x))\right) \end{aligned}$$

Now, assume that the assertion of the Lemma is true for $n \in \{1, 2, \ldots, k\}$, for some $k \geq 1$, then from definition we have

$$\begin{aligned} f_{k+1}(x) &= T(f_k)(x) \\ &= \overline{H}(x)\, \exp\left(-\int_{-x}^{\infty} \left(\overline{H}(s) - f_k(s)\right)\, ds\right) \qquad [\text{ using (4.13) }] \\ &= \overline{H}(x)\, \exp\left(-\int_{-x}^{\infty} \overline{H}(s)\left(1 - e^{-\beta_{k-1}(\overline{H}(s))}\right)\, ds\right) \\ &= \overline{H}(x)\, \exp\left(-\int_{\overline{H}(x)}^{1} \frac{1}{w}\left(1 - e^{-\beta_{k-1}(1-w)}\right)\, dw\right) \end{aligned} \tag{4.17}$$

The last equality follows by substituting $w = H(s)$ and thus from Fact 36 and Fact 37 of Section 4.4 we get that $\frac{dw}{w} = \overline{H}(s)\, ds$ and $H(-x) = \overline{H}(x)$. Finally by definition of β_n's and using (4.17) we get $f_{k+1} = T(f_k)$. □

To complete the proof it is now enough to show that $\beta_n \to 0$ pointwise, which will imply by Lemma 30 that $f_n \to \overline{H}$ pointwise, as $n \to \infty$. Using Proposition 35 (see Section 4.3) we get the following characterization of the pointwise limit of these β_n's.

Lemma 31 *There exists a function $L : [0,1] \to [0,1]$ with $L(1) = 0$, such that*

$$L(s) = \int_{s}^{1} \frac{1}{w}\left(1 - e^{-L(1-w)}\right)\, dw, \;\forall\, s \in [0,1), \tag{4.18}$$

and $L(s) = \lim_{n\to\infty} \beta_n(s)$, $\forall\; 0 \leq s \leq 1$.

Proof : From part (b) of Proposition 35 we know that for any $s \in [0,1]$ the sequence $\{\beta_n(s)\}$ is decreasing, and hence $\exists$ a function $L : [0,1] \to [0,1]$ such that $L(s) = \lim_{n\to\infty} \beta_n(s)$. Now observe that $\beta_n(1-w) \leq \beta_0(1-w) = w$, $\forall\; 0 \leq w \leq 1$, and hence

$$0 \leq \frac{1}{w}\left(1 - e^{-\beta_n(1-w)}\right) \leq \frac{\beta_n(1-w)}{w} \leq 1, \;\forall\;\, 0 \leq w \leq 1.$$

Thus by taking limit as $n \to \infty$ in (4.15) and using the *dominated convergence theorem* along with part (a) of Proposition 35 we get that

$$L(s) = \int_{s}^{1} \frac{1}{w}\left(1 - e^{-L(1-w)}\right)\, dw, \;\forall\;\, 0 \leq s < 1.$$

□

The above lemma basically translates the non-linear integral equation (4.10) to the non-linear integral equation (4.18), where the solution $g = \overline{H}$ of (4.10) is given by the solution

$L \equiv 0$ of (4.18). So at first sight this may not lead us to the conclusion. But fortunately, something nice happens for equation (4.18), and we have the following result which is enough to complete the proof of Theorem 25.

Lemma 32 *If $L : [0,1] \to [0,1]$ is a function which satisfies the non-linear integral equation (4.18), namely,*

$$L(s) = \int_s^1 \frac{1}{w} \left(1 - e^{-L(1-w)}\right) dw, \ \forall \ 0 \leq s < 1,$$

and if $L(1) = 0$, then $L \equiv 0$.

Proof : First note that $L \equiv 0$ is a solution. Now let L be any solution of (4.18), then L is infinitely differentiable on the open interval $(0,1)$, by repeated application of the Fundamental Theorem of Calculus.

Consider,

$$\eta(w) := (1-w)e^{L(1-w)} + we^{-L(w)} - 1, \ w \in [0,1]. \tag{4.19}$$

Observe that $\eta(0) = \eta(1) = 0$ as $L(1) = 0$. Now, from (4.18) we get that

$$L'(w) = -\frac{1}{w}\left(1 - e^{-L(1-w)}\right), \ w \in (0,1). \tag{4.20}$$

Thus differentiating the function η we get

$$\eta'(w) = e^{-L(w)} \left[2 - \left(e^{L(1-w)} + e^{-L(1-w)}\right)\right] \leq 0, \ \forall \ w \in (0,1). \tag{4.21}$$

So the function η is decreasing in $(0,1)$ and is continuous in $[0,1]$ with boundary values as 0, hence $\eta \equiv 0 \Leftrightarrow L \equiv 0$. $\square$

4.3 Some Technical Details

In this section we prove some results which were used in Sections 4.1 and 4.2 for proving Theorems 25 and Corollary 27. These results are mainly technical details and hence have been omitted in the previous sections.

Proposition 33 *The operator Λ defined in Theorem 26 is weakly continuous when restricted to the subspace $\mathfrak{S}^\star$ as defined in (4.8).*

Proof : Let $\{\nu_n\}_{n=1}^{\infty} \subseteq \mathfrak{S}^{\star}$ and suppose that $\nu_n \xrightarrow{d} \nu \in \mathfrak{S}^{\star}$. We will show that $\Lambda(\nu_n) \xrightarrow{d} \Lambda(\nu)$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space such that, $\exists \, \{(X_n, Y_n)\}_{n=1}^{\infty}$ and (X, Y) random vectors taking values in $\mathbb{R}^2$, with $(X_n, Y_n) \sim \nu_n, \; n \geq 1$, and $(X, Y) \sim \nu$. Notice that by definition $X_n \stackrel{d}{=} Y_n \stackrel{d}{=} X \stackrel{d}{=} Y$, and each has Logistic distribution.

Fix $x, y \in \mathbb{R}$, then using similar calculations as in (4.9) we get

$$\begin{aligned} G_n(x,y) &:= \Lambda(\nu_n)\left((x,\infty)\times(y,\infty)\right) \\ &= \overline{H}(x)\overline{H}(y)\,\exp\left(-\int_0^{\infty}\mathbf{P}\left(X_n > t-x, Y_n > t-y\right)\,dt\right) \\ &= \overline{H}(x)\overline{H}(y)\,\exp\left(-\int_0^{\infty}\mathbf{P}\left((X_n+x)\wedge(Y_n+y) > t\right)\,dt\right) \\ &= \overline{H}(x)\overline{H}(y)\,\exp\left(-\mathbf{E}\left[(X_n+x)^+\wedge(Y_n+y)^+\right]\right), \end{aligned} \tag{4.22}$$

and a similar calculation will also give that

$$\begin{aligned} G(x,y) &:= \Lambda(\nu)\left((x,\infty)\times(y,\infty)\right) \\ &= \overline{H}(x)\overline{H}(y)\,\exp\left(-\mathbf{E}\left[(X+x)^+\wedge(Y+y)^+\right]\right). \end{aligned} \tag{4.23}$$

Now to complete the proof all we need is to show

$$\mathbf{E}\left[(X_n+x)^+\wedge(Y_n+y)^+\right] \longrightarrow \mathbf{E}\left[(X+x)^+\wedge(Y+y)^+\right].$$

Since we assumed that $(X_n, Y_n) \xrightarrow{d} (X, Y)$ thus

$$(X_n+x)^+\wedge(Y_n+y)^+ \xrightarrow{d} (X+x)^+\wedge(Y+y)^+, \;\; \forall \;\, x,y\in\mathbb{R}. \tag{4.24}$$

Fix $x, y \in \mathbb{R}$, define $Z_n^{x,y} := (X_n+x)^+ \wedge (Y_n+y)^+$, and $Z^{x,y} := (X+x)^+ \wedge (Y+y)^+$. Observe that

$$0 \,\leq\, Z_n^{x,y} \,\leq\, (X_n+x)^+ \leq |X_n+x|\,, \;\; \forall \;\, n\geq 1. \tag{4.25}$$

But, $|X_n + x| \stackrel{d}{=} |X + x|\,, \; \forall \, n \geq 1$. So clearly $\{Z_n^{x,y}\}_{n=1}^{\infty}$ is uniformly integrable. Hence we conclude (using Theorem 25.12 of Billingsley [11]) that

$$\mathbf{E}\left[Z_n^{x,y}\right] \longrightarrow \mathbf{E}\left[Z^{x,y}\right].$$

This completes the proof. $\square$

Proposition 34 *The operator T maps $\mathfrak{F}$ into $\mathfrak{F}$.*

Proof : First note that if $f \in \mathfrak{F}$, then by definition $T(f)(x) \geq \overline{H}^2(x)$, $\forall\, x \in \mathbb{R}$. Next by definition of $\mathfrak{F}$ we get that $f \in \mathfrak{F} \Rightarrow f \preccurlyeq \overline{H}$, thus

$$\begin{aligned} \int_{-x}^{\infty} f(s)\,ds \;&\leq\; \int_{-x}^{\infty} \overline{H}(s)\,ds, \;\forall\, x \in \mathbb{R} \\ \Rightarrow \quad T(f)(x) \;&\leq\; \overline{H}^2(x)\,\exp\left(\int_{-x}^{\infty} \overline{H}(s)\,ds\right) \;=\; \overline{H}(x), \;\forall\, x \in \mathbb{R} \end{aligned}$$

The last equality follows from (4.29) (see Fact 38 of Section 4.4). So,

$$\overline{H}^2(x) \leq T(f)(x) \leq \overline{H}(x), \;\; \forall \;\; x \in \mathbb{R}. \tag{4.26}$$

Now we need to show that for $f \in \mathfrak{F}$, $T(f)$ is a tail of a distribution. From the definition $T(f)$ is continuous (in fact infinitely differentiable). Further using (4.26) and the fact that $\overline{H}$ is a tail of a distribution we get that

$$\lim_{x\to\infty} T(f)(x) = 0, \quad \text{and} \quad \lim_{x\to-\infty} T(f)(x) = 1. \tag{4.27}$$

Finally let $x \leq y$ be two real numbers, then

$$\int_{-x}^{\infty} \left(\overline{H}(s) - f(s)\right)\,ds \leq \int_{-y}^{\infty} \left(\overline{H}(s) - f(s)\right)\,ds,$$

because $f \preccurlyeq \overline{H}$. Also $\overline{H}(x) \geq \overline{H}(y)$, thus using (4.13) we get

$$T(f)(x) \geq T(f)(y) \tag{4.28}$$

So using (4.26), (4.27), (4.28) we conclude that $T(f) \in \mathfrak{F}$ if $f \in \mathfrak{F}$. $\square$

Proposition 35 *The following are true for the sequence of functions $\{\beta_n\}_{n\geq 0}$ as defined in (4.15).*

(a) For every $n \geq 1$, $\lim_{s\to 0+} \beta_n(s)$ exists, and is given by

$$\int_0^1 \tfrac{1}{w}\left(1 - e^{-\beta_{n-1}(1-w)}\right)\,dw,$$

we will write this as $\beta_n(0)$.

(b) For every fixed $s \in [0,1]$, the sequence $\{\beta_n(s)\}$ is decreasing.

Proof : (a) Note that for $n = 1$,

$$\beta_1(s) = \int_s^1 \frac{1}{w}\left(1 - e^{w}\right)\, dw,\ \forall\ s \in (0, 1],$$

Thus $\lim_{s \to 0+} \beta_1(s)$ exists and is given by

$$\int_0^1 \frac{1}{w}\left(1 - e^{-\beta_0(1-w)}\right)\, dw.$$

Now we assume that the assertion is true for $n \in \{1, 2, \ldots, k\}$ for some $k \geq 1$, we will show that it is true for $n = k + 1$. For that note

$$\beta_{k+1}(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-\beta_k(1-w)}\right)\, dw,\ \forall\ s \in (0, 1].$$

But,

$$\begin{aligned}
& \lim_{w \to 0+} \frac{1}{w}\left(1 - e^{-\beta_k(1-w)}\right) \\
= & \lim_{w \to 0+} \frac{1 - e^{-\beta_k(1-w)}}{\beta_k(1-w)} \times \frac{\beta_k(1-w)}{w} \\
= & \lim_{w \to 0+} \frac{1}{w} \int_{1-w}^1 \frac{1}{v}\left(1 - e^{-\beta_{k-1}(1-v)}\right)\, dv \\
= & \ 1 - e^{-\beta_{k-1}(0)}
\end{aligned}$$

The last equality follows from *mean-value theorem* and the induction hypothesis. The rest follows from the definition.

(b) Notice that $\beta_0(s) = 1 - s$ for $s \in [0, 1]$, thus

$$\beta_1(s) = \int_s^1 \frac{1 - e^{-w}}{w}\, dw < 1 - s = \beta_0(s),\ \ \forall\ \ s \in [0, 1].$$

Now assume that for some $n \geq 1$ we have $\beta_n(s) < \beta_{n-1}(s) < \cdots < \beta_0(s),\ \ \forall\ s \in [0, 1]$, if we show that $\beta_{n+1}(s) < \beta_n(s),\ \ \forall\ \ s \in [0, 1]$ then by induction the proof will be complete. For that, fix $s \in [0, 1]$ then

$$\begin{aligned}
\beta_{n+1}(s) &= \int_s^1 \frac{1}{w}\left(1 - e^{-\beta_n(1-w)}\right)\, dw \\
&< \int_s^1 \frac{1}{w}\left(1 - e^{-\beta_{n-1}(1-w)}\right)\, dw \\
&= \beta_n(s)
\end{aligned}$$

Hence the proof of the proposition. $\square$

4.4 Some Basic Properties of the Logistic Distribution

Here we provide some known basic facts about the Logistic distribution which are used in various places in the proofs.

Recall that we say a real valued random variable Z has Logistic distribution if its distribution function is given by (4.2), namely,

$$H(x) = \mathbf{P}\left(Z \leq x\right) = \frac{1}{1+e^{-x}}, \;\; x \in \mathbb{R}.$$

The following facts hold for the function H.

Fact 36 *H is infinitely differentiable, and $H'(\cdot) = H(\cdot)\overline{H}(\cdot)$, where $\overline{H}(\cdot) = 1 - H(\cdot)$.*

Proof : From the definition it follows that H is infinitely differentiable on $\mathbb{R}$. Further,

$$\begin{aligned} H'(x) &= \frac{1}{1+e^{-x}} \times \frac{e^{-x}}{1+e^{-x}} \\ &= H(x)\,\overline{H}(x) \;\; \forall \;\, x \in \mathbb{R} \end{aligned}$$

□

Fact 37 *H is symmetric around 0, that is, $H(-x) = \overline{H}(x) \; \forall \; x \in \mathbb{R}$.*

Proof : From the definition we get that for any $x \in \mathbb{R}$,

$$H(-x) = \frac{1}{1+e^{x}} = \frac{e^{-x}}{1+e^{-x}} = \overline{H}(x).$$

□

Fact 38 *$\overline{H}$ is the unique solution of the non-linear integral equation*

$$\overline{H}(x) = \exp\left(-\int_{-x}^{\infty} \overline{H}(s)\,ds\right), \;\; \forall \;\, x \in \mathbb{R}. \tag{4.29}$$

Proof : Notice that the equation (4.29) is nothing but Logistic RDE, since $\mathfrak{A}(H)(x) = \exp\left(-\int_{-x}^{\infty}\overline{H}(s)\,ds\right)$, $\forall \; x \in \mathbb{R}$ (see proof of Lemma 5 in Aldous [5]). Thus from the fact that $\overline{H}$ is the unique solution of Logistic RDE (Lemma 5 of Aldous [5]) we conclude that $\overline{H}$ is unique solution of equation (4.29). □

4.5 Final Remarks

4.5.1 Comments on the proof of Theorem 25

(a) Intuitively, a natural approach to show that the fixed-point equation $\Lambda(\nu) = \nu$ on $\mathfrak{S}$ has unique solution, would be to specify a metric ρ on $\mathfrak{S}$ such that the operator Λ becomes a contraction with respect to it. Unfortunately, this approach seems rather hard or may even be impossible. For this reason we have taken a complicated route of proving the bivariate uniqueness using analytic techniques similar to Eulerian recursion.

(b) Although at first glance it seems that the operator T as defined in (4.11) is just an analytic tool to solve the equation (4.10), it has a nice interpretation through the Logistic RDE (4.1). Suppose $\mathfrak{A}$ is the operator associated with Logistic RDE, that is,

$$\mathfrak{A}(\mu) \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j\right), \tag{4.30}$$

where $\left(\xi_j\right)_{j\geq 1}$ are points of a Poisson point process of mean intensity 1 on $(0, \infty)$, and are independent of $\left(X_j\right)_{j\geq 1}$, which are i.i.d with distribution μ on $\mathbb{R}$. It is easy to check that the domain of definition of $\mathfrak{A}$ is the space

$$\mathcal{A} := \left\{ F \,\middle|\, F \text{ is a distribution function on } \mathbb{R} \text{ and } \int_0^\infty \overline{F}(s)\, ds < \infty \right\}. \tag{4.31}$$

Note that in probabilistic terminology the condition $\int_0^\infty \overline{F}(s)\, ds < \infty$ means $\mathbf{E}_F\left[X^+\right] < \infty$.

Notice that from definition $\mathfrak{F} \subseteq \mathcal{A}$, and T can be naturally extended to the whole of $\mathcal{A}$. In that case the following identity holds

$$\frac{\overline{T(\mu)}(\cdot)}{\overline{H}(\cdot)} \times \frac{\overline{\mathfrak{A}(\mu)}(\cdot)}{\overline{H}(\cdot)} = 1, \;\; \forall \;\, \mu \in \mathcal{A}. \tag{4.32}$$

This at least explains the monotonicity of T through anti-monotonicity property (easy to check) of the Logistic operator $\mathfrak{A}$. The identity (4.32) seems rather interesting and might be useful for deeper understanding of the bivariate uniqueness property of the Logistic fixed-point equation.

4.5.2 Domain of attraction

Related to any fixed-point equation there is always the natural question of its domain of attraction. From the recursion proof we can clearly see that the equation (4.10) has the whole of $\mathfrak{F}$ within its domain of attraction. Thus it is natural to believe that one might be able to derive the uniqueness by a contraction argument.

It still remains an open problem to determine the exact domain of attraction of Logistic RDE. Unfortunately, the identity (4.32) does not seem to be useful in that regard.

4.5.3 Everywhere discontinuity of the operators Λ and $\mathfrak{A}$

From Proposition 33 we get that the operator Λ is continuous with respect to the weak convergence topology when restricted to the subspace $\mathfrak{S}^\star$ of its domain of definition, and we saw that this is enough regularity to conclude nice result like Corollary 27. It is still interesting to see if Λ is continuous on whole of its domain of definition. Unfortunately, it is just the opposite. Λ is discontinuous everywhere on its domain of definition. In fact, even the operator $\mathfrak{A}$ associated with the Logistic RDE as defined in (4.30), is discontinuous everywhere on its domain of definition $\mathcal{A}$. To see this we note that if $\mu \in \mathcal{A}$, and $\mu_n \stackrel{d}{\rightarrow} \mu$, where $\{\mu_n\}_{n=1}^\infty \subseteq \mathcal{A}$, then $\mathfrak{A}(\mu_n) \stackrel{d}{\rightarrow} \mathfrak{A}(\mu)$ only if $\mathbf{E}_{\mu_n}[X_n^+] \longrightarrow \mathbf{E}_\mu[X^+]$. Thus clearly $\mathfrak{A}$ is discontinuous everywhere on $\mathcal{A}$ and hence so is Λ.

Chapter 5

Frozen Percolation RDE

In this chapter we will study another example of a "max-type" RDE which appears as a crucial tool to establish the existence of an automorphism invariant process on the infinite binary tree, called the *frozen-percolation* process, first studied by Aldous [4]. Unfortunately here we do not have a rigorous answer to the question of endogey, but we provide numerical results which suggest that the solution of interest of the RDE is not endogenous. Also this will illustrate the need of developing further analytic methods.

5.1 Background and Motivation

Let $\mathbf{T}_3$ be the infinite binary tree, where each vertex has degree 3. We will write $\mathcal{E}$ for the edge set of $\mathbf{T}_3$. Suppose each edge $e \in \mathcal{E}$ has a Uniform$[0,1]$ random edge weight U_e, and they are independent as e varies. Aldous [4] studied the following process and called it a *frozen-percolation* process

> For $0 \le t \le 1$ define a random collection of edges $\mathcal{A}_t \subseteq \mathcal{E}$ such that, $\mathcal{A}_0 = \emptyset$, and for each edge $e \in \mathcal{E}$, at time instance $t = U_e$ set $\mathcal{A}_t = \mathcal{A}_{t-} \cup \{e\}$ if and only if each end-vertex of e is in a finite cluster of $\mathcal{A}_t$; otherwise set $\mathcal{A}_t = \mathcal{A}_{t-}$.

Here formally a cluster is a connected subgraph of $\mathbf{T}_3$. A more familiar process of similar kind is the standard percolation process on $\mathbf{T}_3$ defined as $\mathcal{B}_t := \{ e \in \mathcal{E} \,\big|\, U_e \le t \}$ for $0 \le t \le 1$ [18]. From definition we see that $\mathcal{A}_t \subseteq \mathcal{B}_t$ for all $0 \le t \le 1$. We note that

$\mathcal{B}_t$ has no infinite cluster for $t \leq 1/2$, since all the connected components are then sub-critical or critical (when $t = 1/2$) Galton-Watson branching process trees. Thus from above description we get that $\mathcal{A}_t = \mathcal{B}_t$ for all $t \leq 1/2$. Qualitatively, in the process $(\mathcal{A}_t)$ the clusters may grow to infinite size but, at the instant of becoming infinite, they are "frozen" in the sense that no extra edge may be connected to an infinite cluster.

Although this process is apparently novel and intuitively natural, rigorously speaking it is not at all clear from its description that the process exists or if it does whether it is unique or not. As mentioned Aldous [4] showed the existence of the process on $\mathbf{T}_d$, the infinite d-array tree, and a personal communication from Benjamini and Schramm to Aldous (2000) proves the non-existence of the process for $\mathbb{Z}^2$ square lattice. Naturally our interest for this process is because it involves an interesting "max-type" RDE. The key ingredient of the proof of existence of the process [4] is the following RDE

$$X \stackrel{d}{=} \Phi\left(X_1 \wedge X_2, U\right) \quad \text{on} \quad S = \left(\tfrac{1}{2}, 1\right] \cup \{\infty\}, \tag{5.1}$$

where (X_1, X_2) are i.i.d. copies of X which are independent of $U \sim$ Uniform$[0,1]$, and $\Phi : S \to S$ defined as

$$\Phi\left(x, u\right) = \begin{cases} x & \text{if } x > u \\ \infty & \text{if } x \leq u \end{cases}. \tag{5.2}$$

It is easy to show [4] that (5.1) has unique solution with full support on S given by

$$\mu(dx) = \frac{dx}{2x^2}, \quad \text{when} \quad \frac{1}{2} < x \leq 1; \text{ and } \quad \mu(\{\infty\}) = \frac{1}{2}. \tag{5.3}$$

Further one can show (see Lemma 3 of [4]) that the frozen-percolation RDE (5.1) has a one parameter family of solutions which are non-atomic on $(1/2, 1]$ given by

$$\mu_a(dx) = \frac{dx}{2x^2}, \quad \text{when} \quad \frac{1}{2} < x \leq a; \text{ and } \quad \mu(\{\infty\}) = \frac{1}{2a}. \tag{5.4}$$

where $1/2 < a \leq 1$.

One can then construct the invariant RTP $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ with marginal μ. We here note that in this case $N \equiv 2$. In [4] Aldous constructed the frozen-percolation process using the $X_{\mathbf{i}}$'s and also the edge weights. Thus it is natural to ask if the RTP is measurable with respect to the edge weights, in other words if the RTP is endogenous. This will then also resolve the question raised by Aldous in [4] (see remark (5.7)) regarding validity of the informal description of the frozen percolation process. This is one of our main motivation to study the RDE (5.1).

5.2 Study of Bivariate Uniqueness of 1st Kind

In this section we try to see whether the RTP with marginal μ associated with the RDE (5.1) has bivariate uniqueness property of 1st kind. For that we consider the following bivariate version of the RDE (see Section 2.5)

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Phi\left(X_1 \wedge X_2, U\right) \\ \Phi\left(Y_1 \wedge Y_2, U\right) \end{pmatrix}, \tag{5.5}$$

where (X_1, Y_1) and (X_2, Y_2) are independent copies of (X, Y) and are independent of $U \sim$ Uniform$[0, 1]$. Also we assume that the marginal distributions of X and Y are μ as given in (5.3). The following Lemma gives a characterization of bivariate uniqueness in this particular case.

Lemma 39 *For a solution (X, Y) of the bivariate equation (5.5) $X = Y$ a.s. if and only if $[X = \infty] = [Y = \infty]$ a.s.*

Proof : Trivially, if $X = Y$ a.s. then $[X = \infty] \stackrel{\text{a.s.}}{=} [Y = \infty]$. Conversely, suppose that ν is a solution of (5.5) with marginal μ such that $X = \infty \Leftrightarrow Y = \infty$ a.s. Let $(X_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ be the bivariate RTP with marginal ν. Recall that

$$X_{\emptyset} = \begin{cases} X_1 \wedge X_2 & \text{if } X_1 \wedge X_2 > U \\ \infty & \text{if } X_1 \wedge X_2 \leq U \end{cases},$$

and similarly for $Y_{\emptyset}$. Thus

$$X_1 \wedge X_2 \leq U \Longleftrightarrow Y_1 \wedge Y_2 \leq U \ \text{ a.s.}, \tag{5.6}$$

Let $Z = X_{\emptyset} \wedge Y_{\emptyset}$ and similarly define Z_1 and Z_2 then using (5.6) we get

$$Z = \begin{cases} Z_1 \wedge Z_2 & \text{if } Z_1 \wedge Z_2 > U \\ \infty & \text{if } Z_1 \wedge Z_2 \leq U \end{cases}. \tag{5.7}$$

Hence Z also satisfies the RDE (5.1). Clearly $\mathbf{P}\left(Z = \infty\right) = \mathbf{P}\left(X_{\emptyset} = Y_{\emptyset} = \infty\right) = \mathbf{P}\left(X_{\emptyset} = \infty\right) = 1/2$. Further as both $X_{\emptyset}$ and $Y_{\emptyset}$ are non-atomic and hence so is Z. Thus from (5.4) we conclude that Z has the distribution μ. Using the Lemma 28 it follows that $X_{\emptyset} = Y_{\emptyset}$ a.s. $\square$

Now let $F(x,y) := \mathbf{P}(X \leq x, Y \leq y)$, for $x, y \in [0,1]$, so from (5.5) we get

$$\begin{aligned} F(x,y) &= \mathbf{P}\left(\Phi(X_1 \wedge X_2, U) \leq x,\, \Phi(Y_1 \wedge Y_2, U) \leq y\right) \\ &= \mathbf{P}\left(U < X_1 \wedge X_2 \leq x,\, U < Y_1 \wedge Y_2 \leq y\right) \\ &= \int_0^{x \wedge y} \left(G^2(x,y) - G^2(x,u) - G^2(u,y) + G^2(u,u)\right) du \qquad (5.8) \end{aligned}$$

where $G(x,y) := \mathbf{P}(X > x, Y > y)$, which can be written as

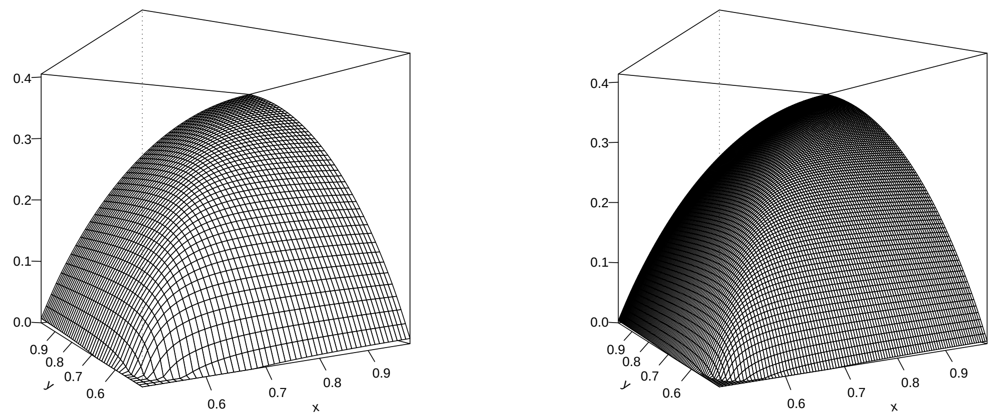
$$\begin{aligned} G(x,y) &= F(x,y) - \mathbf{P}(X \leq x) - \mathbf{P}(Y \leq y) + 1 \\ &= F(x,y) + \tfrac{1}{2x} + \tfrac{1}{2y} - 1 \qquad (5.9) \end{aligned}$$

The last equation is valid only when $x, y \in (1/2, 1]$. Unfortunately, the integral equation (5.8) is too complicated and we have been unable to show that $\mu^{\nearrow}$ is the only solution of it, neither we have been able to find another solution.

From part (c) of Theorem 12 we know that the solution μ of the RDE (5.1) will be endogenous if and only if $T^{(2)^n}(\mu \otimes \mu) \xrightarrow{d} \mu^{\nearrow}$. Further the equation (5.11) provides a formula for computing $T^{(2)^n}(\mu_0)$ for any distribution μ_0 on S. Thus one can do numerical computation by repeating this formula starting from $\mu \otimes \mu$ to check whether the limit exists and if so whether or not it is the same as $\mu^{\nearrow}$. The next section gives the results of our numerical computations which suggests that the the integral equation (5.8) possibly has solutions other than $\mu^{\nearrow}$.

5.3 Results of Numerical Computations for Bivariate Uniqueness of 1st kind

In this section we provide results of our numerical computation to check whether the limit in part (c) of Theorem 12 holds. In all the cases we discretize the set $[1/2, 1] \times [1/2, 1]$ as a lattice grid with some small grid length. The results are provided for two grid lengths, namely $h = 0.01$ and $h = 0.005$.



(a) $h = 0.01$

(b) $h = 0.005$

Figure 5.1: Graphs of the distribution function of $T^{(2)^n}(\mu \otimes \mu)$ for $n = 10000$

We did computation of the distribution $T^{(2)^n}(\mu \otimes \mu)$ for $n = 10000$ iterations, and we observed that the distributions almost do not change after $n = 1000$, suggesting that there is possibly a limiting distribution. The Figure 5.1 gives the graph of the distribution function of $T^{(2)^n}(\mu \otimes \mu)$ for $n = 10000$ in the two cases $h = 0.01$ and $h = 0.005$

To compare the possible limit with $\mu^{\nearrow}$ we provide the graphs of the distribution functions of $\mu^{\nearrow}$ and $T^{(2)^n}(\mu \otimes \mu)$ for $n = 10000$ and $h = 0.005$ in Figures 5.2 and 5.3 respectively. This strongly suggest that the possible limit might not be $\mu^{\nearrow}$.

Further from Lemma 39 we get that if the RTP marginal μ is endogenous then the limit will be 0.5. The following table gives the numerically computed values of the $\mathbf{P}(X_n = \infty, Y_n = \infty)$ where $(X_n, Y_n) \sim T^{(2)^n}(\mu \otimes \mu)$.

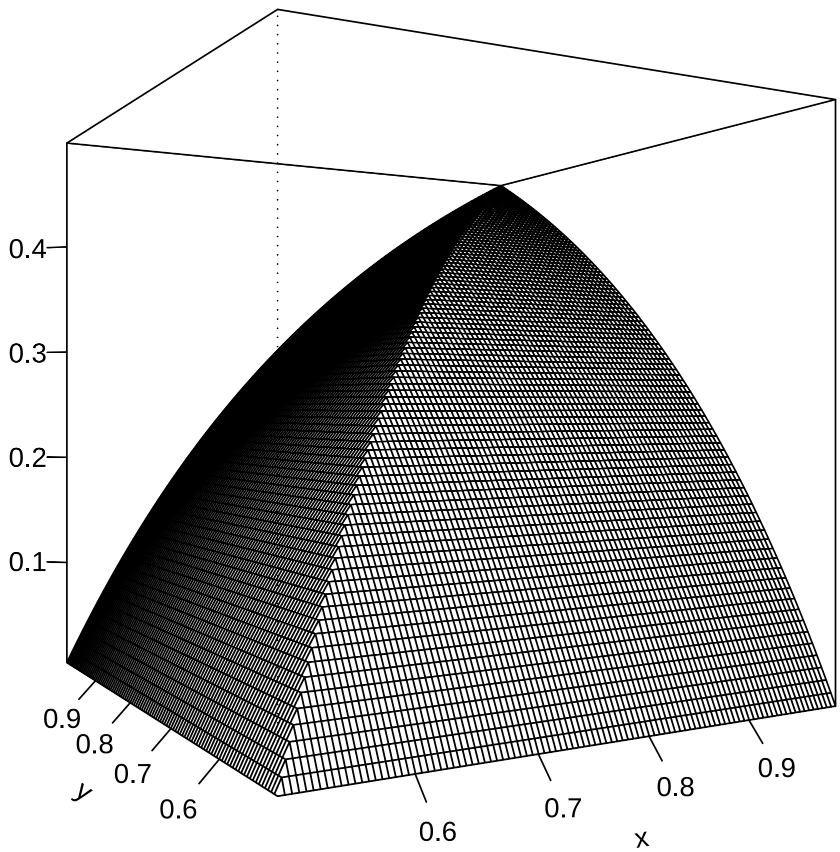


Figure 5.2: Graph of the distribution function of $\mu^{\nearrow}$

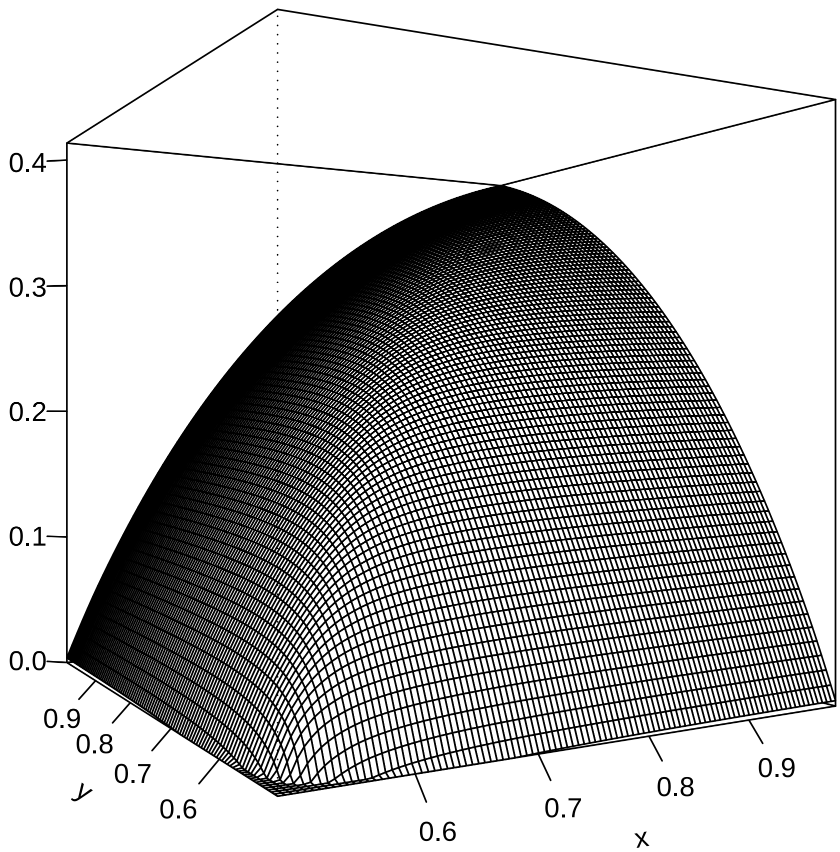


Figure 5.3: Graph of the distribution function of ${T^{(2)}}^n\,(\mu \otimes \mu)$ for $n = 10000$

	$n = 5000$	$n = 10000$
$h = 0.01$	0.404878	0.404878
$h = 0.005$	0.413543	0.413543

This also suggests that perhaps the limit is not $\mu^{\nearrow}$.

So it seems that there are strong numerical evidence to believe that the solution μ of the frozen-percolation RDE (5.1) is not endogenous.

5.4 Study of Bivariate Uniqueness of 2nd Kind

Since the numerical results suggest that the bivariate RDE (5.5) may have solutions other than $\mu^{\nearrow}$ so we now try to show that the bivariate uniqueness of 2nd kind fails (see Section 2.6.2). For that we consider the following bivariate version of (5.1)

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Phi\left(X_1 \wedge X_2, U\right) \\ \Phi\left(Y_1 \wedge Y_2, V\right) \end{pmatrix}, \tag{5.10}$$

where (X_1, Y_1) and (X_2, Y_2) are independent copies of (X, Y) and are independent of U and V which are i.i.d Uniform$[0,1]$. Also we assume that X and Y has marginal distribution μ as defined in (5.3). As before $F(x,y) = \mathbf{P}\left(X \leq x, Y \leq y\right)$ and $G(x,y) = \mathbf{P}\left(X > x, Y > y\right)$, for $0 \leq x, y \leq 1$. So now using (5.10) we get

$$\begin{aligned} F(x,y) &= \mathbf{P}\left(\Phi\left(X_1 \wedge X_2, U\right) \leq x,\, \Phi\left(Y_1 \wedge Y_2, V\right) \leq y\right) \\ &= \mathbf{P}\left(U < X_1 \wedge X_2 \leq x,\, V < Y_1 \wedge Y_2 \leq y\right) \\ &= \int_0^x \int_0^y \left(G^2(x,y) - G^2(x,v) - G^2(u,y) + G^2(u,v)\right) du\, dv \\ &= xy\, G^2(x,y) - x \int_0^y G^2(x,v)\, dv \\ &\qquad - y \int_0^x G^2(u,y)\, du + \int_0^x \int_0^y G^2(u,v)\, du\, dv \end{aligned} \tag{5.11}$$

Assume F is twice differentiable (and hence so is G), using (5.9) we get

$$\frac{\partial^2 F}{\partial x\, \partial y} = \frac{\partial^2 G}{\partial x\, \partial y}$$

for $1/2 < x, y < 1$. Hence from (5.11) we get the following PDE

$$\frac{\partial^2}{\partial x \, \partial y} G(x,y) = xy \, \frac{\partial^2}{\partial x \, \partial y} \left(G^2(x,y) \right), \tag{5.12}$$

where $1/2 < x, y < 1$ and the boundary conditions are given by

$$G(x, 1/2) = \frac{1}{2x} \quad \text{and} \quad G(1/2, y) = \frac{1}{2y}. \tag{5.13}$$

Clearly $G_0(x,y) = \frac{1}{4xy}$ for $1/2 \leq x, y \leq 1$ is a solution of the boundary value problem (5.12, 5.13) corresponding to the solution $\mu \otimes \mu$ of the equation (5.10). The following proposition is an immediate consequence of the part (a) of Theorem 18.

Proposition 40 *The invariant RTP with marginal μ associated with the RDE (5.1) is not endogenous if the boundary value problem (5.12, 5.13) has a* feasible *solution other than G_0.*

Note that we say a solution G of the boundary value problem is *feasible* if there exists random variables (X, Y) taking values in S such that $G(x,y) = \mathbf{P}\left(X > x, Y > y\right)$, for all $1/2 < x, y \leq 1$.

We believe that the above boundary value problem has feasible solution(s) other than G_0 but we have been unable to prove it.

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