Solutions to Practice Midterm
Stat 155: Game Theory

Questions

1. Consider a *modified* game of Nim with 3 piles containing x_1, x_2 and x_3 chips. In a turn a player is allowed to move at most 3 chips from pile one, or at most 4 chips from pile two or at most 5 chips from pile three. In a turn at least one chip has to be removed. The game ends when there is no chip is left on the board and the last player to make a move is the winner.
   (a) Starting with $x_1 = 134$, $x_2 = 51$ and $x_3 = 650$ which player has a winning strategy and why?
   (b) For the above game find the Sprague-Grundy function.
2. Suppose two players I and II call two numbers simultaneously from the set $\{10, 20\}$. If the two numbers match then Player I receives \$30 from Player II, otherwise Player I pays Player II the difference of the two numbers in dollars.

   (a) Find the pay-off matrix for this game.
   (b) Find a pair of optimal strategies for Players I & II. What is the value of the game?
   (c) Suppose the two players call two numbers simultaneously from the set $\{10, 20, 30\}$. If the two numbers match then Player I receives \$30 from Player II, otherwise Player I pays Player II \$10. Write down the new pay-off matrix. Can you give a pair of optimal strategies for the two players?

Solution 1

a) This game is the sum of 3 subtraction games with the subtraction sets being $\{1, 2, 3\}, \{1, 2, 3, 4\}$ and $\{1, 2, 3, 4, 5\}$. Lets write G_1, G_2, G_3 for the 3 games and g_1, g_2, g_3 for their Sprague-Grundry functions.

We will proceed by analyzing each G_i separately and then using the Sum theorem. Taking G_1, it is easy to see by backward induction that the SG function, $g_1(x_1) = x_1 \mod 4$. We omit the details here.

Similar analysis shows that $g_2(x_2) = x_2 \mod 5, g_3(x_3) = x_3 mod 6$.

Then, writing $g(x_1, x_2, x_3)$ for the SG function of $G_1 + G_2 + G_3$, we have by the Sum Theorem, $g(x_1, x_2, x_3) = g_1(x_1) \oplus g_2(x_2) \oplus g_3(x_3) = (x_1 \mod 4) \oplus (x_2 \mod 5) \oplus (x_3 \mod 6)$

a) Plugging in the initial positions given in the question, we get $g(134, 51, 650) = (134 \mod 4) \oplus (51 \mod 5) \oplus (650 \mod 6) = 2 \oplus 1 \oplus 2 = 1$.

As the SG function is non-zero this is a N position and Player I has a winning strategy.

b) As discussed before the SG function is, $g(x_1, x_2, x_3) = (x_1 \mod 4) \oplus (x_2 \mod 5) \oplus (x_3 \mod 6)$

Solution 2

a) The payoff matrix of this Two-Person Zero-Sum Game is as follows,

30	-10
-10	30

b) First we notice that there are no saddle points. So we need to solve this using mixed strategies. We will write $(p, 1-p)$ for a mixed strategy of Player I and $(q, 1-q)$ for a mixed strategy of Player II. Then, equating rows and equating columns, we get

$$30q - 10(1-q) = -10q + 30(1-q)$$
$$30p - 10(1-p) = -10p + 30(1-p)$$

Which gives us the following solution, $p = 1/2, q = 1/2$. Thus the optimal strategies for both Player I and Player II is $(1/2, 1/2)$. The value of the game is 10.

c) The new payoff matrix is

30	-10	-10
-10	30	-10
-10	-10	30

Again, there are no saddle points. So the optimal solutions will be mixed strategies, and notice that the payoff matrix is symmetric so the optimal strategies for Player I and Player II is going to be identical. We will write (p, q, r) for a mixed strategy of both Player I and Player II. Of course $p + q + r = 1$.

Then, equating rows,

$$30p - 10q - 10r = -10p + 30q - 10r = -10p - 10q + 30r$$

From the first and second expressions we get $p = q$ and from the second and third expressions, $q = r$. Thus, $p = q = r = 1/3$ as $p + q + r = 1$. Thus the optimal strategies for both Player I and Player II is $(1/3, 1/3, 1/3)$.