Notes on K-theory

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1 Preliminaries from C^*-algebra theory

1.1 Constructions with C^*-algebras

Direct Sum.

Tensor Product.

1. Form the algebraic tensor product $A \otimes_{alg} B$,

2. put a C^*-norm on $A \otimes_{alg} B$ that obeys $\|a \otimes b\| = \|a\|.\|b\|$ for $a \in A$, $b \in B$. One can put several C^*-norms in general; there is a maximal and minimal one.

3. complete $A \otimes_{alg} B$ with respect to that norm. several choices of norms on the algebraic tensor product and hence several choices of $A \otimes B$ possible.

4. the spatial norm: this is one choice of a C^*-norm on $A \otimes_{alg} B$. By GNS theorem, there exist faithful representations $\pi_A : A \to \mathcal{L}(\mathcal{H}_A)$ and $\pi_B : B \to \mathcal{L}(\mathcal{H}_B)$. Define $\pi : A \otimes_{alg} B \to \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ by $\pi(a \otimes b) = \pi_A(a) \otimes \pi_B(b)$ and define $\|\cdot\|$ by $\|\sum a_i \otimes b_i\| = \|\pi(\sum a_i \otimes b_i)\|$.

5. this norm is independent of the choice of the representations π_A and π_B as long as they are faithful, and is called the spatial norm. This turns out to coincide with the minimal norm on $A \otimes_{alg} B$.

6. For a large class of C^*-algebras A, one can put only one norm (C^* cross norm) on $A \otimes_{alg} B$ for any C^*-algebra B. Such C^*-algebras are called nuclear C^*-algebras. All abelian C^*-algebras and type I C^*-algebras are nuclear.

1.2 Unitization

Let A be a C^*-algebra, and let

$$A^\dagger = \begin{cases} \text{unitization of } A & \text{if } A \text{ is nonunital,} \\ A \oplus \mathbb{C} & \text{if } A \text{ is unital.} \end{cases}$$

Exercise 1.1 Show that if $A \subseteq B$, B is unital but $1_B \notin A$, then $A^\dagger \cong A + \mathbb{C}1_B$.

Exercise 1.2 Let A be a C^*-algebra and let $\pi : A^\dagger \to \mathbb{C}$ be the map $(a,t) \mapsto t$ and $\lambda : \mathbb{C} \to A^\dagger$ be the map

$$\lambda(t) = \begin{cases} (0,t) & \text{if } A \text{ is nonunital,} \\ (t,t) & \text{if } A \text{ is unital.} \end{cases}$$

Show that the following sequence is split exact:

$$0 \longrightarrow A \longrightarrow A^\dagger \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

The map $s := \lambda \circ \pi : A^\dagger \to A^\dagger$ is called the **scalar map**. Thus

$$s(a,t) = \begin{cases} (0,t) & \text{if } A \text{ is nonunital,} \\ (t,t) & \text{if } A \text{ is unital.} \end{cases}$$

Exercise 1.3 Let $\phi : A \to B$ be a morphism. Define a map $\phi^\dagger : A^\dagger \to B^\dagger$ as follows:

$$\phi^\dagger(a,t) = \begin{cases} (\phi(a),t) & \text{if } A,\, B \text{ both unital or both nonunital,} \\ (\phi(a)+t,t) & \text{if } A \text{ nonunital and } B \text{ unital,} \\ (\phi(a-t),t) & \text{if } A \text{ unital and } B \text{ nonunital.} \end{cases}$$

Show that

1. $\phi^\dagger$ is the unique extension of ϕ to a unital morphism $\phi^\dagger$ from $A^\dagger$ to $B^\dagger$.
2. $\phi^\dagger$ is injective if and only if ϕ is injective,
3. $\phi^\dagger$ is surjective if and only if ϕ is injective.

Exercise 1.4 Let $\phi : A \to B$ be a morphism and let s_A and s_B be the scalar maps for $A^\dagger$ and $B^\dagger$ respectively. Show that for any $a \in A^\dagger$, one has $s_B(\phi^\dagger(a)) = \phi^\dagger(s_A(a))$.

Exercise 1.5 Let

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\pi} A/J \longrightarrow 0,$$

be a short exact sequence. Then $\phi^\dagger : M_n(J^\dagger) \to M_n(A^\dagger)$ is injective.

An element $a \in M_n(A^\dagger)$ is in $\phi^\dagger(M_n(J^\dagger))$ if and only if $\pi^\dagger(a) = s(\pi^\dagger(a))$.

Inductive limits of C^*-algebras. Let (A_i, ϕ_{jk}) be an inductive system of C^*-algebras, i.e. $\phi_{jk} : A_k \to A_j$ are morphisms for $k \le j$, and $\phi_{ij}\phi_{jk} = \phi_{ik}$, $\phi_{ii} = id$.

Define

$$\begin{aligned} B_\infty &= \{(a_i) : a_i \in A_i \text{ for all } i, \text{ there exists } k \text{ such that } a_j = \phi_{kj}(a_k) \text{ for } j \ge k\}, \\ B &= \{(a_i) : \sup \|a_i\| < \infty\}, \\ J_\infty &= \{(a_i) \in B_\infty : a_i = 0 \text{ for all but finitely many } i\} \\ J &= \text{ closure of } J_\infty \text{ in } B, \\ \pi &: \text{ canonical projection } B \to B/J, \\ A_\infty &= \pi(B_\infty), \\ A &= \text{ closure of } A_\infty \text{ in } B/J, \end{aligned}$$

Note that forming B/J is same as putting the seminorm $\|(a_i)\|_1 := \limsup \|a_i\|$ on B and quotienting by elements of length zero.

Define $\phi_j : A_j \to A$ by

$$\phi_j(a) = (\underbrace{0, \ldots, 0}_{j-1}, a, \phi_{j+1,j}(a), \phi_{j+2,j}(a), \ldots).$$

Then

1. the following diagram commutes:

$$\begin{array}{ccc} A_j & \xrightarrow{\phi_j} & A \\ {\scriptstyle \phi_{kj}}\downarrow & \nearrow_{\phi_k} & \\ A_k & & \end{array}$$

2. $A_\infty = \cup_j \phi_j(A_j)$,

3. if D is a C^*-algebra such that for each i, there is a morphism $\psi_i : A_i \to D$ with

$$\begin{array}{ccc} A_j & \xrightarrow{\psi_j} & D \\ {\scriptstyle \phi_{kj}}\downarrow & \nearrow_{\psi_k} & \\ A_k & & \end{array}$$

   then there is a unique morphism $\psi : A \to D$ such that

$$\begin{array}{ccc} A_j & \xrightarrow{\phi_j} & A \\ & {\scriptstyle \psi_j}\searrow & \downarrow{\scriptstyle \psi} \\ & & D \end{array}$$

   If the ψ_i's are all one-one, then ψ is one-one.

4. if $a \in A$, then for any $\epsilon > 0$, there is a $k \in \mathbb{N}$ and $a_k \in A_k$ such that

$$\|a - \phi_k(a_k)\| < \epsilon.$$

2 K-theory

2.1 Vector bundles

Let X be a compact hausdorff space and E be a complex vector bundle over X of rank n. Let $\Gamma(E)$ be the space of sections of E.

1. $\Gamma(E)$ is a vector space with pointwise addition.

2. It is a $C(X)$-module with pointwise multiplication.

3. If $E = X \times \mathbb{C}^n$, then $\Gamma(E) = C(X, \mathbb{C}^n) \cong C(X) \otimes \mathbb{C}^n$ is the direct sum of n copies of $C(X)$.

4. $\Gamma(E \oplus F) = \Gamma(E) \oplus \Gamma(F)$.

5. Theorem (Swan): If E is a locally trivial complex vector bundle over a compact Hausdorff space X, then there is another locally trivial complex vector bundle F over X such that $E \oplus F$ is trivial.

6. Thus $\gamma(E) \oplus \Gamma(F) \cong C(X) \oplus \ldots \oplus C(X)$. Observe that $\mathcal{L}(C(X) \oplus \ldots \oplus C(X)) = M_n(C(X))$. So $\gamma(E)$ can be identified with the projection p_E in $M_n(C(X))$ onto $\gamma(E)$.

$K_0(A)$: Grothendieck group of the semigroup of projections in $\cup_n M_n(A)$ modulo homotopy.

2.2 K_0 group

2.2.1 Equivalence relations on projections

Murray-von Neumann equivalence. Let $p, q \in Proj(A)$. Define $p \sim_{MvN} q$ if there is a partial isometry $v \in A$ such that $p = vv^*$ and $q = v^*v$.

Exercise 2.1 Show that $\sim$ is an equivalence relation on A.

Exercise 2.2 Show that $p \sim_{MvN} q$ if and only if there are elements $x, y \in A$ such that $p = xy$ and $q = yx$.

Unitary equivalence. Let $p, q \in Proj(A)$. Define $p \sim_u q$ if there is a unitary $u \in A^\dagger$ such that $q = upu^*$.

Exercise 2.3 Show that $\sim_u$ is an equivalence relation on A.

Exercise 2.4 Show that $p \sim_u q$ if and only if there is an element $z \in GL_1(A^\dagger)$ such that $q = zpz^{-1}$.

Exercise 2.5 Let $p, q \in Proj(A)$. Show that $\|p - q\| \leq 1$.

Lemma 2.1 *If* $\|p - q\| < 1$ *then* $p \sim_u q$.

Proof: Write $x = qp + (1-q)(1-p)$. Then $x - 1 = 2qp - q - p = (2q-1)(p-q)$, so that $\|x - 1\| < 1$. Therefore x is invertible. It is easy to see now that $xpx^{-1} = q$. By the previous exercise, the result follows. □

Exercise 2.6 Let $p(t)$ be a continuous path of projections in a unital C^*-algebra A. Then there is a continuous path of unitaries $u(t)$ with $u(0) = I$ such that $p(t) = u(t)p(0)u(t)^*$ for all t.

(Use the proofs of the lemma above and exercise 2.4)

Homotopy. Let $p, q \in Proj(A)$. p and q are said to be homotopic if there is a norm continuous path $t \mapsto P(t)$ in A such that $P(t)^* = P(t) = P(t)^2$ for all t and $P(0) = p$, $P(1) = q$. One writes $p \sim_h q$ in such a case.

Exercise 2.7 Show that $\sim_h$ is an equivalence relation on A.

Exercise 2.8 Let $p, q \in Proj(A)$. Suppose there is a homotopy of idempotents from p to q. Show that $p \sim_h q$.

Lemma 2.2 *Let $p, q \in Proj(A)$ and $\|p - q\| < 1$. Then show that $p \sim_h q$.*

Proof: Write $P(t) = tp + (1-t)q$ for $0 \leq t \leq 1$. Let $\delta = \frac{1}{2}\|p - q\|$. Then $\|P(t) - p\| = (1-t)\|p - q\| \leq \delta$ for $\frac{1}{2} \leq t \leq 1$, and $\|P(t) - q\| = t\|p - q\| \leq \delta$ for $0 \leq t \leq \frac{1}{2}$. Thus for all $t \in [0, 1]$, one has $\sigma(P(t)) \subseteq [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$. Let $f : [-\delta, \delta] \cup [1 - \delta, 1 + \delta] \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 0 & \text{if } |x| \leq \delta, \\ 1 & \text{otherwise.} \end{cases}$$

Then $f(P(t))$ gives a required homotopy. □

Proposition 2.3 *Let $p, q \in Proj(A)$. Then $p \sim_h q \Rightarrow p \sim_u q \Rightarrow p \sim_{MvN} q$.*

Proof: Let $P : [0, 1] \to A$ be a homotopy from p to q. Let $0 < t_1 < \ldots < t_k < 1$ be such that $\|P(t_i) - P(t_{i+1})\| < 1$ for each i. Now use exercise 2.1 for each pair to conclude that $p \sim_u q$.

Next assume that u is a unitary such that $p = uqu^*$. Write $v = uq$. Then $vv^* = uqu^* = p$ and $v^*v = qu^*uq = q$. Thus $p \sim_{MvN} q$. □

Lemma 2.4 *Let $p, q \in Proj(A)$. If $p \sim_{MvN} q$ and $1 - p \sim_{MvN} 1 - q$, then $p \sim_u q$.*

Proof: Let v and w be partial isometries in A with $v^*v = p$, $vv^* = q$, $w^*w = 1 - p$, $ww^* = 1 - q$. Then $1 - v^*v = w^*w$. Multiplying both sides from the left by w and from the right by w^*, one gets $wv^*vw^* = 0$, so that $vw^* = 0$. A similar argument shows that $v^*w = 0$. It follows then that $u = v + w$ is unitary and $upu^* = q$. □

Corollary 2.5 *Let $p, q \in Proj(A)$. $p \sim_u q$ if and only if $p \sim_{MvN} q$ and $1 - p \sim_{MvN} 1 - q$.*

Example 2.6 *Example where $p \sim_{MvN} q$ but $p \not\sim_u q$: Take $P \in L_2(\mathbb{N})$ to be the projection onto $L_2(\mathbb{N}\backslash\{0\})$ and Q to be the identity operator.*

Example 2.7 *Example where $p \sim_u q$ but $p \not\sim_h q$: exists in $M_2(C(S^3))$!*

Proposition 2.8 *Let $p, q \in Proj(A)$. If $p \sim_{MvN} q$, then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.*

Proof: Let v be a partial isometry with $v^*v = p$ and $vv^* = q$. Then $u := \begin{pmatrix} v & 1 - vv^* \\ v^*v - 1 & v^* \end{pmatrix}$ is a unitary and $u \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$. □

Proposition 2.9 *Let $p, q \in Proj(A)$. If $p \sim_{MvN} q$, then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.*

Proof: Let v and u be as in the previous proof. The path

$$t \mapsto \begin{pmatrix} \cos(\frac{\pi}{2}t)v & 1 - (1 - \sin(\frac{\pi}{2}t))vv^* \\ (1 - \sin(\frac{\pi}{2}t))v^*v - 1 & \cos(\frac{\pi}{2}t)v^* \end{pmatrix}$$

connects u to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The path

$$t \mapsto \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}$$

connects $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let u_t be a continuous path of unitaries that connect u to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $t \mapsto u_t \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u_t^*$ is a continuous path of projections that connect $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ with $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$. □

Proposition 2.10 *Let $p, q \in Proj(A)$. If $p \sim_u q$, then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.*

Proof: This is a corollary of the previous proposition. □

2.2.2 K_0 group for unital C^*-algebras

Exercise 2.9 Let $p, p' \in Proj(M_n(A))$, $q, q' \in Proj(M_k(A))$. Assume $p \sim_{MvN} p'$ and $q \sim_{MvN} q'$. Show that

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \sim_{MvN} \begin{pmatrix} p' & 0 \\ 0 & q' \end{pmatrix} \quad \text{in } M_{n+k}(A).$$

Exercise 2.10 Let $p, q \in Proj(M_n(A))$. Show that

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}.$$

Let $P_\infty(A)$ denote $Proj(\cup_n M_n(A))$ modulo the equivalence $p \sim_0 \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$. On $P_\infty(A)$, define an equivalence relation $\sim$ by declaring $[p]_0 \sim [q]_0$ if there is an $n \in \mathbb{N}$, $p' \in [p]_0$, $q' \in [q]_0$ with $p', q' \in M_n(A)$ and $p' \sim_{MvN} q'$ in $M_n(A)$. Let $V(A) := P_\infty(A)/\sim$.

Exercise 2.11 Define a relation $\sim_1$ on $\sqcup_n Proj(M_n(A))$ as follows:
for $p \in M_n(A)$ and $q \in M_k(A)$, $p \sim_1 q$ if there exists a partial isometry $v \in M_{n,k}(A)$ such that $p = vv^*$, $q = v^*v$.

Show that this is an equivalence relation and $\sqcup_n Proj(M_n(A))/\sim_1 = V(A)$.

Define an operation on $V(A)$ by

$$[p] + [q] := \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right].$$

This is well-defined and turns it into an abelian semigroup. We will denote this semigroup by $V(A)$.

Exercise 2.12 Recall that if $p, q \in Proj(A)$ obey $\|p - q\| < 1$, then $p \sim_u q$. Use this to show that if A is separable, then $V(A)$ is countable.

Exercise 2.13 Let $p, q \in Proj(A)$ with $pq = 0 = qp$. Show that $\begin{pmatrix} p+q & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$.

Exercise 2.14 Let $(S, +)$ be a cancellative abelian semigroup. Define a relation $\sim$ on $S \times S$ by declaring $(a, b) \sim_{MvN} (a', b')$ if $a + b' = a' + b$. Show that this is an equivalence relation.

Define an operation $+$ on $S \times S$ by $(a, b) + (a', b') = (a + a', b + b')$. Show that if $(a, b) \sim_{MvN} (a', b')$ and $(c, d) \sim_{MvN} (c', d')$, then $(a, b) + (c, d) \sim_{MvN} (a', b') + (c', d')$. Thus the operation $+$ lifts to a well-defined operation on $S \times S/\sim$.

Show that $(S \times S/\sim, +)$ is an abelian group with identity $[(a, a)]$ and $-[(a, b)] = [(b, a)]$.

If $(S, +)$ is an abelian semigroup possibly without cancellation, the relation defined in the above exercise need not be an equivalence relation. So in general, one needs to define the relation on $S \times S$ slightly differently.

Exercise 2.15 Let $(S, +)$ be an abelian semigroup. Define a relation $\sim$ on $S \times S$ by declaring $(a, b) \sim_{MvN} (a', b')$ if there exists a $c \in S$ such that $a + b' + c = a' + b + c$. Show that this is an equivalence relation.

Show that the natural addition on $S \times S$ lifts to an operation on $S \times S/\sim$, and $(S \times S/\sim, +)$ is an abelian group. (this is called the **Grothendieck group** of $(S, +)$ and will be denoted by $G(S)$.)

Exercise 2.16 Let $(S, +)$ be a semigroup and let $\sim$ be as above. Show that $[(x + y, y)]$ is independent of y.

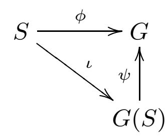
Choose and fix an $y \in S$. Show that $\iota : x \mapsto [(x + y, y)]$ gives a semigroup homomorphism from S into $G(S)$.

ι is injective if and only if S is cancellative.

Exercise 2.17 Let S and S' be two semigroups and let $\phi : S \to S'$ be a homomorphism. Then there is a unique group homomorphism $\psi : G(S) \to G(S')$ such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\phi} & S' \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ G(S) & \xrightarrow{\psi} & G(S') \end{array}$$

Exercise 2.18 Let S be a semigroup, G be a group and let $\phi : S \to G$ be a homomorphism. Then there is a unique group homomorphism $\psi : G(S) \to G$ such that the following diagram commutes:



Exercise 2.19 Let $S = \mathbb{N} \cup \{\infty\}$, with an operation $+$ that gives the usual addition when restricted to $\mathbb{N}$ and for $n \in S$, one has $n + \infty = \infty = \infty + n$.

Show that the Grothendieck group of $(S, +)$ is the trivial group.

Definition 2.11 Let A be a unital C^*-algebra. The K_0 **group of** A is defined to be the Grothendieck group of $V(A)$.

Exercise 2.20 Let A be a unital C^*-algebra. Let S be the set $Proj(M_\infty(A))$ modulo the equivalence relation $\sim$. Let $\widetilde{K}_0(A)$ be the abelian group with generators $[p] \in S$ and satisfying the relation $[p] + [q] = [p \oplus q]$. Show that $\widetilde{K}_0(A) = K_0(A)$.

Exercise 2.21 Show that two projections p and q in $M_n(\mathbb{C})$ are equivalent if and only if $\mathrm{Trace}\, p = \mathrm{Trace}\, q$. Use this to prove that $V(\mathbb{C}) = (\mathbb{N}, +)$ and hence conclude that $K_0(\mathbb{C}) = \mathbb{Z}$.

Exercise 2.22 Use exercise 2.21 to show that $K_0(M_n(\mathbb{C})) = \mathbb{Z}$.

Exercise 2.23 Let A and B be two unital C^*-algebras and let $\phi : A \to B$ be a *-homomorphism. Denote by the same symbol the induced homomorphism $M_n(A)$ to $M_n(B)$. Let $p, q \in M_n(A)$. Show that if p and q are homotopic, then $\phi(p)$ and $\phi(q)$ are also homotopic.

Define $K_0(\phi) : V(A) \to V(B)$ by $K_0(\phi)[p] = [\phi(p)]$. Show that this induces a homomorphism from $K_0(A)$ to $K_0(B)$.

Show that $K_0(id) = id$.

Exercise 2.24 Let A, B, C be unital C^*-algebras and let $\phi : A \to B$ and $\psi : B \to C$ be $*$-homomorphisms. Show that $K_0(\psi \circ \phi) = K_0(\psi) \circ K_0(\phi)$.

Let A, B be C^*-algebras. Two homomorphisms $\phi, \psi : A \to B$ are said to be **homotopic** if there exist a family of $*$-homomorphisms $\phi_t : A \to B$, $t \in [0, 1]$ such that $\phi_0 = \phi$, $\phi_1 = \psi$ and for each $a \in A$, the map $t \mapsto \phi_t(a)$ is norm continuous.

Exercise 2.25 Show that if two homomorphisms $\phi, \psi : A \to B$ are homotopic, then $K_0(\phi) = K_0(\psi)$.

Two C^*-algebras A and B are said to be **homotopy equivalent** if there exist homomorphisms $\phi : A \to B$ and $\psi : B \to A$ such that $\phi \circ \psi$ is homotopic to id_B and $\psi \circ \phi$ is homotopic to id_A.

Exercise 2.26 In such a case, one has $K_0(A) = K_0(B)$ and $K_0(\phi)^{-1} = K_0(\psi)$.

Exercise 2.27 Let X be a contractible compact Hausdorff space. Show that $K_0(C(X)) = \mathbb{Z}$.

Exercise 2.28 Find $V(\mathcal{L}(\mathcal{H}))$ where $\mathcal{H}$ is infinite dimensional. Use this to show that $K_0(\mathcal{L}(\mathcal{H})) = 0$.

Exercise 2.29 Let A be a unital C^*-algebra, and let $n \in \mathbb{N}$. Show that the map $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ induces an isomorphism between $K_0(M_n(A))$ and $K_0(A)$.

2.2.3 K_0 group for nonunital C^*-algebras

Suppose we have the short exact sequence

$$0 \longrightarrow A \longrightarrow A^\dagger \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$$

Then we have a group homomorphism $K_0(\pi)$ from $K_0(A^\dagger)$ to $K_0(\mathbb{C}) = \mathbb{Z}$. Define the K_0 **group of** A to be the kernel of this homomorphism.

Exercise 2.30 Let A and B be two C^*-algebras and let $\phi : A \to B$ be a $*$-homomorphism. Then ϕ extends uniquely to a unital $*$-homomorphism $\phi^\dagger : A^\dagger \to B^\dagger$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & A^\dagger & \xrightarrow{\pi_A} & \mathbb{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi^\dagger} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & B & \longrightarrow & B^\dagger & \xrightarrow{\pi_B} & \mathbb{C} & \longrightarrow & 0 \end{array}$$

Show that

1. $K_0(\phi^\dagger)$ maps $\ker K_0(\pi_A)$ into $\ker K_0(\pi_B)$.
2. if A is unital, then $\ker K_0(\pi_B) \cong K_0(A)$.
3. if A and B are unital, then the restriction of $K_0(\phi^\dagger)$ to $\ker K_0(\pi_A)$ is same as the map $K_0(\phi)$.

Let ϕ_t be a family of homomorphisms from A to B and let $\phi_t^\dagger$ be its unique extension to a homomorphism from $A^\dagger$ to $B^\dagger$. Show that if ϕ_t is a homotopy, then $\phi_t^\dagger$ is also a homotopy.

If A and/or B is nonunital, define $K_0(\phi)$ to be the restriction of $K_0(\phi^\dagger)$ to $\ker K_0(\pi_A)$.

Exercise 2.31 Let A, B, C be C^*-algebras and let $\phi : A \to B$ and $\psi : B \to C$ be $*$-homomorphisms. Show that $K_0(id_A) = id$ and $K_0(\psi \circ \phi) = K_0(\psi) \circ K_0(\phi)$.

Exercise 2.32 Suppose two C^*-algebras A and B are homotopic, i.e. there are homomorphisms $\phi : A \to B$ and $\psi : B \to C$ such that $\phi \circ \psi$ is homotopic to id_B and $\psi \circ \phi$ is homotopic to id_A. Then $K_0(A) = K_0(B)$ and $K_0(\phi)^{-1} = K_0(\psi)$.

Proposition 2.12 *Let A be a nonunital C^*-algebra. Let s be the extension of the map $(a, z) \mapsto (0, z)$ (from $A^\dagger$ to $A^\dagger$) to $\cup_n M_n(A^\dagger)$. Then*

$$K_0(A) = \{[p] - [s(p)] : p \in Proj(\cup_n M_n(A^\dagger))\}.$$

Proof: Let $p \in Proj(M_n(A^\dagger))$. Look at the element $[p] - [s(p)]$ in $K_0(A^\dagger)$. Since

$$K_0(\pi)([p] - [s(p)]) = [\pi(p)] - [\pi(s(p))] = [\pi(p)] - [\pi(p)] = 0,$$

we have $[p] - [s(p)] \in K_0(A)$.

Let us take an element $[p]-[q] \in K_0(A^\dagger)$ such that $[p]-[q] \in \ker K_0(\pi)$, $p, q \in Proj(\cup_n M_n(A^\dagger))$. Let $\lambda : \mathbb{C} \to A^\dagger$ be the map $z \mapsto (0, z)$. Then $s = \lambda \circ \pi$. Therefore $[p] - [q] \in \ker K_0(s)$. Let us write

$$\tilde{p} = \begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix}, \quad \tilde{q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

Then observe that

$$[p]-[q]=[\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}]-[\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}]=[\tilde{p}]-[\tilde{q}].$$

Therefore $K_0(s)([\tilde{p}]-[\tilde{q}])=0$. But clearly $s([\tilde{q}])=[\tilde{q}]$. Therefore $[s(\tilde{p})]-[\tilde{q}]=0$. Thus there exist $r,r' \in Proj(\cup_n M_n(A^\dagger))$ such that $[p]+[\tilde{q}]+[r]=[\tilde{p}]+[q]+[r]$ and $[s(\tilde{p})]+[r']=[\tilde{q}]+[r']$. Combining these, we get

$$[p]+[s(\tilde{p})]+[r\oplus r']=[\tilde{p}]+[q]+[r\oplus r'],$$

which means $[p]-[q]=[\tilde{p}]-[s(\tilde{p})]$. □

Lemma 2.13 *If $p \in Proj(M_k(A^\dagger))$ and $\pi(p) \sim_{MvN} 1_n$ in $M_k(\mathbb{C})$ $(n \le k)$, then there is an element $q \in Proj(M_k(A^\dagger))$ such that $p \sim_u q$ and $\pi(q)=1_n$.*

Proof: Since $\pi(p) \sim_{MvN} 1_n$ in $M_k(\mathbb{C})$, which is finite dimensional, we have $1_k-\pi(p) \sim_{MvN} 1_k-1_n$ and consequently $\pi(p) \sim_u 1_n$, i.e. there is a unitary $u \in M_k(\mathbb{C})$ such that $u\pi(p)u^*=1_n$. Then $q:=upu^*$ gives a required projection. □

Exercise 2.33 Elements of $K_0(A)$ can be written in the form $[p]-[1_n]$ where $p \in M_k(A^\dagger)$, $k \ge n$ and $p-1_n \in M_k(A)$.
Proof: First show that any element can be written as $[p']-[1_n]$. Next use the fact that this is in the kernel of $K_0(\pi)$ to conclude that $[\pi(p')]-[1_n]=0$. Since $V(\mathbb{C})=\mathbb{N}$ is cancellative, this implies $[\pi(p')]=[1_n]$, i.e. $\pi(p') \sim_{MvN} 1_n$. Now use lemma 2.13 to get a projection p such that $p \sim_u p'$ and $\pi(p)=1_n$.

Exercise 2.34 Let $p,q \in Proj(M_k(A^\dagger))$ and $[p]-[q]=0$ in $K_0(A)$. Then there exist $m,n \in \mathbb{N}$, $m \le n$ such that

$$\begin{pmatrix} p & 0 \\ 0 & 1_m \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 1_m \end{pmatrix} \qquad \text{in } M_{k+n}(A^\dagger).$$

Proof: Since $[p]-[q]=0$, there exists $r \in Proj(M_m(A^\dagger))$ for some $m \in \mathbb{N}$ such that $[p]+[r]=[q]+[r]$. Therefore

$$\begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & r \end{pmatrix} \qquad \text{in } M_{k+n}(A^\dagger)$$

for some $n \ge m$. The required homotopy now follows.

2.2.4 Properties

Theorem 2.14 *K_0 is half-exact, i.e. if we have a short exact sequence*

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\pi} A/J \longrightarrow 0,$$

then the sequence

$$K_0(J) \xrightarrow{K_0(\phi)} K_0(A) \xrightarrow{K_0(\pi)} K_0(A/J)$$

is exact in the middle.

Proof: Since $\pi \circ \phi = 0$, it follows that the range of $K_0(\phi)$ is contained in $\ker K_0(\pi)$. Now take an element x in $\ker K_0(\pi)$. By exercise 2.33, $x = [p] - [1_n]$, $p \in M_k(A^\dagger)$. Since this is in the kernel of $K_0(\pi)$, we have $[\pi(p)] - [1_n] = 0$ in $K_0(A/J)$. Hence it follows from exercise 2.34 that,

$$\begin{pmatrix} \pi(p) & 0 \\ 0 & 1_m \end{pmatrix} \sim_u \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix} \qquad \text{in } M_{k+j}((A/J)^\dagger).$$

Let u be a unitary in $M_{k+j}((A/J)^\dagger)$ such that

$$u \begin{pmatrix} \pi(p) & 0 \\ 0 & 1_m \end{pmatrix} u^* = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}.$$

Let w be a unitary in $M_{2k+2j}(A^\dagger)$ such that $\pi(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$ and $w \sim_h 1_{2k+2j}$. (Assume that such a w would exist; this is a fact from C^*-algebras that we will prove later) Now let $q = w \begin{pmatrix} p & 0 \\ 0 & 1_m \end{pmatrix} w^*$. Then

$$\pi(q) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \pi(p) & 0 \\ 0 & 1_m \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 1_k & 0 \\ 0 & 1_m \end{pmatrix}.$$

Therefore $q \in M_k(J^\dagger)$. Since $[q] = [\begin{pmatrix} p & 0 \\ 0 & 1_m \end{pmatrix}]$, we have

$$[p] - [1_n] = [\begin{pmatrix} p & 0 \\ 0 & 1_m \end{pmatrix}] - [\begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}] = [q] - [1_{n+m}].$$

But the right hand side is clearly in the range of $K_0(\phi)$. □

Theorem 2.15 *K_0 takes split exact sequences to split exact sequences, i.e. if the short exact sequence*

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\pi} A/J \longrightarrow 0,$$

splits with a splitting homomorphism $\lambda : A/J \to A$, then the sequence

$$0 \longrightarrow K_0(J) \xrightarrow{K_0(\phi)} K_0(A) \xrightarrow{K_0(\pi)} K_0(A/J) \longrightarrow 0$$

is exact and splits with splitting map $K_0(\lambda)$.

Proof: Since $\pi \circ \lambda = id_{A/J}$, it follows that

$$K_0(\pi) \circ K_0(\lambda) = K_0(id_{A/J}) = id_{K_0(A/J)}.$$

So $K_0(\pi)$ is onto.

Take an element in $K_0(J)$. By exercise 2.33, it is of the form $[p] - [1_n]$ where $p \in Proj(M_k(J^\dagger))$ for some $k \in \mathbb{N}$, $k \geq n$ and $p - 1_n \in M_k(J)$. If it is an element of $\ker K_0(\phi)$ then it follows that $[\phi_k^\dagger(p)] - [1_n] = 0$. From exercise 2.34, we conclude that there exist $m, j \in \mathbb{N}$, $m \le j$ such that

$$\begin{pmatrix} \phi_k^\dagger(p) & 0 \\ 0 & 1_m \end{pmatrix} \sim_h \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix} \qquad \text{in } M_{k+j}(A^\dagger),$$

i.e. there is a unitary $u \in M_{k+j}(A^\dagger)$ such that

$$u \begin{pmatrix} \phi_k^\dagger(p) & 0 \\ 0 & 1_m \end{pmatrix} u^* = \begin{pmatrix} 1_n & 0 \\ 0 & 1_m \end{pmatrix}.$$

Write

$$p' = \begin{pmatrix} p & 0 \\ 0 & 1_m \end{pmatrix} \in M_{k+j}(J^\dagger).$$

Then $[p] - [1_n] = [p'] - [1_{n+m}]$, $u\phi_{k+j}^\dagger(p')u^* = 1_{n+m}$ and $p' - 1_{n+m} \in M_{k+j}(J)$.

Exercise 2.35 Now complete the proof.

□

Proposition 2.16 *Let A be a C^*-algebra, and let $n \in M_n(A)$. Then $K_0(M_n(A)) = K_0(A)$.*

Proof: Let $\phi : A \to M_n(A)$ be the map $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and let ψ be the corresponding map from $\mathbb{C}$ to $M_n(\mathbb{C})$. Then the following diagram commutes and have split exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & A^\dagger & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi^\dagger} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & M_n(A) & \longrightarrow & M_n(A^\dagger) & \longrightarrow & M_n(\mathbb{C}) & \longrightarrow & 0 \end{array}$$

It follows from the properties of K_0 that the following diagram also commutes and have split exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A^\dagger) & \longrightarrow & K_0(\mathbb{C}) = \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle K_0(\phi)} & & \downarrow{\scriptstyle K_0(\phi^\dagger)} & & \downarrow{\scriptstyle K_0(\psi)} & & \\ 0 & \longrightarrow & K_0(M_n(A)) & \longrightarrow & K_0(M_n(A^\dagger)) = K_0(A^\dagger) & \longrightarrow & K_0(M_n(\mathbb{C})) = \mathbb{Z} & \longrightarrow & 0 \end{array}$$

Exercise 2.36 Show that if $K_0(\phi^\dagger)$ and $K_0(\psi)$ are isomorphisms, then $K_0(\phi)$ is also an isomorphism.

Therefore the proof follows from the result for the unital case. □

Proposition 2.17 *Let A and B be two C^*-algebras. Let ι_A and ι_B be the natural inclusions of A and B into $A \oplus B$. Then $K_0(\iota_A) \oplus K_0(\iota_B) : K_0(A) \oplus K_0(B) \to K_0(A \oplus B)$ is an isomorphism.*

Proof: Let π be the projection $A \oplus B \to B$. Then the following sequence is split exact:

$$0 \longrightarrow A \xrightarrow{\iota_A} A \oplus B \underset{\iota_B}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

By split exactness, we have the split exact sequence of abelian groups

$$0 \longrightarrow K_0(A) \xrightarrow{K_0(\iota_A)} K_0(A \oplus B) \underset{K_0(\iota_B)}{\overset{K_0(\pi)}{\rightleftarrows}} K_0(B) \longrightarrow 0$$

Therefore the result follows. □

Proposition 2.18 *Let (A_i, ϕ_{jk}) be an inductive system of C^*-algebras. Then $K_0(\lim(A_i, \phi_{jk})) = \lim(K_0(A_i), K_0(\phi_{jk}))$.*

Proof: Since

$$\begin{array}{ccc} A_j & \xrightarrow{\phi_j} & A \\ {\scriptstyle \phi_{kj}}\downarrow & \nearrow {\scriptstyle \phi_k} & \\ A_k & & \end{array}$$

we have

$$\begin{array}{ccc} K_0(A_j) & \xrightarrow{K_0(\phi_j)} & K_0(A) \\ {\scriptstyle K_0(\phi_{kj})}\downarrow & \nearrow {\scriptstyle K_0(\phi_k)} & \\ K_0(A_k) & & \end{array}$$

By universality of $\lim(K_0(A_i), K_0(\phi_{jk}))$, there is a unique morphism $\psi_* : \lim K_0(A_i) \to K_0(A)$ such that

$$\begin{array}{ccc} K_0(A_j) & \xrightarrow{\xi_j} & \lim K_0(A_i) \\ & {\scriptstyle K_0(\psi_j)} \searrow & \downarrow {\scriptstyle \psi_*} \\ & & K_0(A) \end{array}$$

where ξ_j's are the maps corresponding to the inductive system $(K_0(A_i), K_0(\phi_{jk}))$.

We need to show that ψ_* is one-one and onto.

Since $\lim K_0(A_i) = \cup_j \xi_j(K_0(A_j))$, for injectivity it is enough to show that ψ_* is injective on $\xi_j(K_0(A_j))$. So take an element $x \in K_0(A_j)$ and assume $\psi_* \xi_j(x) = 0$. We have to show that $\xi_j(x) = 0$. We will use the facts that $\psi_* \xi_j = K_0(\psi_j)$ and $\xi_k K_0(\phi_{kj}) = \xi_j$ for $k \geq j$.

Exercise 2.37 Complete the proof of injectivity of ψ_*.

Next, take $[p] - [s(p)]$, $p \in Proj(M_k(A^\dagger))$. In order to show that this is in the range of ψ_*, complete the following steps:

Approximate p with $\phi_n(a_n)$ for some self adjoint element $a_n \in M_k(A_n^\dagger)$; write $a_m = \phi_{mn}(a_n)$ for $m \geq n$.

Now show:

1. $\|a_m - a_m^2\| < 1/4$ for large m,
2. there is a projection $q \in M_k(A_m^\dagger)$ such that $\|a_m - q\| < 1/2$,
3. $\|\phi_m(q) - p\| < 1$,
4. $[p] - [s(p)] = [\phi_m(q)] - [s(\phi_m(q))] = K_0(\phi_m)([q] - [s(q)])$.

$\square$

Exercise 2.38 Show that $K_0(\mathcal{K} \otimes A) = K_0(A)$.

2.2.5 Computations of K_0

A C^*-algebra is called properly infinite if there are projections p, q with $pq = 0$ and $1 \sim_{MvN} p \sim_{MvN} q$.

Exercise 2.39 If a C^*-algebra is properly infinite, then its quotients are also properly infinite. Show that $\mathcal{L}(\mathcal{H})$ for infinite dimensional $\mathcal{H}$ and the Cuntz algebras $\mathscr{O}_n$ are properly infinite.

Let A be properly infinite, p and q being projections with $pq = 0$ and $1 \sim_{MvN} p \sim_{MvN} q$. Let $v, w \in A$ such that $v^*v = 1 = w^*w$ and $p = vv^*$, $q = ww^*$. Since $pq = 0$, it follows that $v^*w = 0$. Let $s_k = v^k w$, $k \in \mathbb{N}$. Then $s_k^* s_j = \delta_{kj}$, i.e. s_k's are isometries with orthogonal range. Let $v_n = (s_1, \ldots, s_n)$. Then it is easy to see that $b_n p b_n^* \sim p$ in $Proj(\cup_n M_n(A))$.

Exercise 2.40 Let $p, q \in Proj(A)$. Write $r = s_1 p s_1^* + s_2(1-q)s_2^* + s_3(1 - s_1 s_1^* - s_2 s_2^*)s_3^*$. Show that

1. $r \in Proj(A)$,
2. $r \sim \begin{pmatrix} p & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 1-q & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$,
3. $[r] = [\begin{pmatrix} p & & \\ & 0 & \\ & & 0 \end{pmatrix}] - [\begin{pmatrix} 0 & & \\ & q & \\ & & 0 \end{pmatrix}] = [p] - [q]$.

2.3 K_1 group

2.3.1 Higher K-groups

Let A be a C^*-algebra. Then the C^*-algebra

$$\{f \in C([0,1], A) : f(0) = f(1)\}$$

is called the **suspension of** A and is denoted by SA.

Exercise 2.41 Show that $SA \cong C_0(\mathbb{R}) \otimes A$.

Exercise 2.42 $(SA)^\dagger = \{f \in C([0,1], A^\dagger) : f(0) = f(1) = \lambda \in \mathbb{C}, s(f(t)) = \lambda \in \mathbb{C} \text{ for } t \in [0,1]\}$.

Exercise 2.43 Let $\phi : [0,1] \times [0,1] \to A$ be a continuous map with $\phi(t,1) = \phi(t,0) = 0$ for all $t \in [0,1]$. Then $t \mapsto \phi_t$ where $\phi_t(s) = \phi(t,s)$ gives a homotopy in SA.

Conversely, any homotopy in SA arises in this way.

Exercise 2.44 Let $p_0, p_1 \in Proj(\cup_n M_n(A^\dagger))$. Then $p_0 \sim_{MvN} p_1$ if and only if there are projections $p_t \in \cup_n M_n((SA)^\dagger)$ such that for each $s \in [0,1]$, $t \mapsto p_t(s)$ is a homotopy between $p_0 = p_0(s)$ and $p_1 = p_1(s)$.

Exercise 2.45 Let $u_0, u_1 \in \mathscr{U}(\cup_n \mathscr{U}_n(A^\dagger))$. Then $u_0 \sim u_1$ if and only if there are unitaries $u_t \in \cup_n M_n((SA)^\dagger)$ such that for each $s \in [0,1]$, $t \mapsto u_t(s)$ is a homotopy between $u_0 = u_0(s)$ and $u_1 = u_1(s)$.

Definition 2.19 Let $n \in \mathbb{N}$. Define the **nth K-group of** A by $K_n(A) := K_0(S^n A)$. In particular $K_1(A) := K_0(SA)$.

Exercise 2.46 Show that $SM_n(A) \cong M_n(SA)$.

Exercise 2.47 Let A and B be two C^*-algebras and let $\phi : A \to B$ be a $*$-homomorphism. Define a map $\tilde{\phi} : SA \to SB$ by

$$\tilde{\phi} f(t) = \phi(f(t)), \quad t \in [0,1].$$

Show that $\tilde{\phi}$ is a $*$-homomorphism from SA to SB.
(we will normally denote this map $\tilde{\phi}$ by $S(\phi)$ or ϕ_s)

Exercise 2.48 Let

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\pi} A/J \longrightarrow 0$$

be a short exact sequence of C^*-algebras. Then the sequence

$$0 \longrightarrow SJ \xrightarrow{S(\phi)} SA \xrightarrow{S(\pi)} S(A/J) \longrightarrow 0$$

is exact.

If the sequence

$$0 \longrightarrow J \longrightarrow A \underset{\lambda}{\overset{\pi}{\rightleftarrows}} A/J \longrightarrow 0$$

is split exact, then so is the sequence

$$0 \longrightarrow SJ \longrightarrow SA \underset{\lambda_s}{\overset{\pi_s}{\rightleftarrows}} S(A/J) \longrightarrow 0.$$

Exercise 2.49 Let A and B be C^*-algebras. Show that $S(A \oplus B) = SA \oplus SB$.

Exercise 2.50 Let B be a C^*-algebra. Show that $S(\mathcal{K} \otimes B) \cong \mathcal{K} \otimes SB$.

Proposition 2.20 *Let A and B be two C^*-algebras. Then*

1. $K_1(M_n(A)) = K_1(A)$,
2. $K_1(A \oplus B) = K_1(A) \oplus K_1(B)$,
3. $K_1(\mathcal{K} \otimes A) = K_1(A)$,
4. *K_1 is half exact and carries split exact sequences to split exact sequences.*

We have the following split exact sequences:

$$0 \longrightarrow A \longrightarrow A^\dagger \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

$$0 \longrightarrow SA \longrightarrow (SA)^\dagger \underset{\lambda_s}{\overset{\pi_s}{\rightleftarrows}} \mathbb{C} \longrightarrow 0.$$

Recall that

$$\begin{aligned}
(SA)^\dagger &= \{f \in C([0,1], A^\dagger) : f(0) = f(1) = \lambda \in \mathbb{C}, \pi(f(t)) = \lambda \text{ for all } t\},\\
M_n((SA)^\dagger) &= \{f \in C([0,1], M_n(A^\dagger)) : f(0) = f(1) = \lambda \in M_n(\mathbb{C}), \pi(f(t)) = \lambda \text{ for all } t\},\\
Proj(M_n((SA)^\dagger)) &= \{f \in C([0,1], M_n(A^\dagger)) : f(0) = f(1) = \lambda \in M_n(\mathbb{C}), \pi(f(t)) = \lambda \text{ for all } t,\\
&\quad \text{each } f(t) \text{ is a projection}\},\\
K_0((SA)^\dagger) &= \{[p] - [q] : p, q \in \cup_n Proj(M_n((SA)^\dagger))\}\\
&= \cup_n \{[p] - [q] : p, q \in Proj(M_n((SA)^\dagger))\}.
\end{aligned}$$

If $[p] - [q] \in \ker K_0(\pi_s)$, then $[\pi_s(p)] - [\pi_s(q)] = 0$, i.e. $[p(0)] - [q(0)] = 0$. But this equality takes place in the Grothendieck group of $V(\mathbb{C}) = \mathbb{N}$ where cancellation holds. So $p(0) \sim q(0)$. So there is a unitary $u \in M_{2n}(\mathbb{C})$ such that

$$u \begin{pmatrix} p(0) & 0 \\ 0 & 0 \end{pmatrix} u^* = \begin{pmatrix} q(0) & 0 \\ 0 & 0 \end{pmatrix}.$$

Define

$$p'(t) = \begin{pmatrix} p(t) & 0 \\ 0 & 0 \end{pmatrix}, \qquad q'(t) = \begin{pmatrix} q(t) & 0 \\ 0 & 0 \end{pmatrix}, \qquad u(t) = u.$$

Then $p', q' \in Proj(M_{2n}((SA)^\dagger))$, u is unitary in $M_{2n}((SA)^\dagger)$. So $p \sim p' \sim up'u^*$ and $q \sim q'$. Therefore $[p] - [q] = [up'u^*] - [q']$ and

$$up'u^*(0) = u \begin{pmatrix} p(0) & 0 \\ 0 & 0 \end{pmatrix} u^* = q'(0).$$

Thus

$$K_0(SA) \subseteq \cup_n \{[p] - [q] : p, q \in Proj(M_n((SA)^\dagger)), p(0) = q(0)\}.$$

The opposite inclusion is clear. So we have

$$K_0(SA) = \cup_n \{[p] - [q] : p, q \in Proj(M_n((SA)^\dagger)), p(0) = q(0)\}.$$

2.3.2 Homotopies of unitaries and invertibles

Define

$$GL_n^\dagger(A) = \{a \in GL_n(A^\dagger) : \pi(a) = 1_n\}, \qquad U_n^\dagger(A) = \{a \in U_n(A^\dagger) : \pi(a) = 1_n\}$$

Exercise 2.51 Show that

1. if A is unital, then $GL_n^\dagger(A) = \{a \oplus 1_n : a \in GL_n(A)\}$,
2. $z \in GL_n(A^\dagger)$ implies $z\pi(z^{-1}) \in GL_n^\dagger(A)$,
3. $u \in U_n(A^\dagger)$ implies $u\pi(u^*) \in U_n^\dagger(A)$.

Exercise 2.52 Let A be unital and let $x \in GL_n(A)$, $y \in M_n(A)$ satisfy

$$\|x - y\| < \frac{1}{\|x^{-1}\|}.$$

Then the path $t \mapsto tx + (1-t)y$, $t \in [0,1]$ lies in $GL_n(A)$.

Show that every path component of $GL_n(A)$ is open, so that every connected component coincides with a path component.

Exercise 2.53 Let A be a unital C^*-algebra and u be a unitary in A with $\sigma(u) \neq S^1$. Then there a continuous path of unitaries in A connecting u to the identity. (Hint: Get a self-adjoint element $a \in A$ such that $u = \exp(ia)$)

Exercise 2.54 Show that any unitary in $M_n(\mathbb{C})$ can be connected to the identity through a continuous path of unitaries.

Lemma 2.21 *Let $z \in GL_1(A)$. Then $u = z|z|^{-1} \in \mathscr{U}(A)$ and $u \sim_h z$.*

Proof: Let $z_t = u\exp(t\log|z|)$. This gives a homotopy between u and z. □

Lemma 2.22 *Let $u, v \in \mathscr{U}(A)$ with $\|u - v\| < 2$. Then $u \sim_h v$.*

Proof: Since $\|u - v\| < 2$, we have $\|uv^* - 1\| < 2$, so that $\sigma(uv^*) \subseteq S^1 - \{-1\}$. Therefore $uv^* \sim_h 1$, which implies that $u \sim_h v$. □

Proposition 2.23 *Let A be a unital C^*-algebra and let $u \in A$ be a unitary. Then*

1. $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$,
2. $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix}$,
3. $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof: Define

$$V(t) = \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}, \qquad t \in [0,1].$$

Then $u_t : t \mapsto V(t)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}V(t)^*$ gives a homotopy from $\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$.

The other two parts are immediate corollaries of part 1. □

Remark 2.24 If u and v are in $A^\dagger$ with $s(u) = 1 = s(v)$ in the above proposition, then the homotopies u_t etc constructed are such that $u_t \in M_2(A^\dagger)$ and $s(u_t) = 1$ for all t.

Proposition 2.25 $GL_n^\dagger(A)/GL_n^\dagger(A)_0 \cong U_n^\dagger(A)/U_n^\dagger(A)_0$.

Proof: Let π_1 and π_2 be the quotient maps from $GL_n^\dagger(A)$ to $GL_n^\dagger(A)/GL_n^\dagger(A)_0$ and from $U_n^\dagger(A)$ to $U_n^\dagger(A)/U_n^\dagger(A)_0$ respectively. Define $\phi : GL_n^\dagger(A) \to U_n^\dagger(A)/U_n^\dagger(A)_0$ by

$$\phi(z) = \pi_2(z|z|^{-1}).$$

Clearly ϕ is surjective.

Exercise 2.55 If x_t is a homotopy between z and w, then $t \mapsto x_t|x_t|^{-1}$ gives a homotopy between $z|z|^{-1}$ and $w|w|^{-1}$.

Thus ϕ lifts to a map $\tilde{\phi}$ from $GL_n^\dagger(A)/GL_n^\dagger(A)_0$ to $U_n^\dagger(A)/U_n^\dagger(A)_0$.

Exercise 2.56 Show that $\tilde{\phi}$ is injective and is a group homomorphism.

This completes the proof! □

Proposition 2.26 $GL_n(A^\dagger)/GL_n(A^\dagger)_0 \cong GL_n^\dagger(A)/GL_n^\dagger(A)_0$.

Proof: Use the map $z \mapsto z\pi(z^{-1})$ (π is the projection $GL_n(A^\dagger) \to GL_n(\mathbb{C})$. □

Proposition 2.27 $U_n(A^\dagger)/U_n(A^\dagger)_0 \cong U_n^\dagger(A)/U_n^\dagger(A)_0$.

Proof: Use the map $u \mapsto u\pi(u^*)$ from $U_n(A^\dagger)$ to $U_n^\dagger(A)$ (π is the projection $U_n(A^\dagger) \to U_n(\mathbb{C})$). □

Let us now define the group $\tilde{K}_1(A)$. Take the disjoint union $\sqcup_n U_n^\dagger(A)$. Suppose $u \in U_n^\dagger(A)$ and $v \in U_k^\dagger(A)$. Declare them to be equivalent ($u \sim v$) if there are integers $r, s \in \mathbb{N}$ such that $n + r = k + s$ and

$$\begin{pmatrix} u & 0 \\ 0 & 1_r \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & 1_s \end{pmatrix}$$

in $U_{n+r}^\dagger(A)$. On the quotient $\sqcup_n U_n^\dagger(A)/\sim$, define

$$[u] + [v] := [\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}].$$

This turns it into an abelian group which we denote by $\tilde{K}_1(A)$.

Proposition 2.28 $\tilde{K}_1(A) = \lim U_n^\dagger(A)/U_n^\dagger(A)_0 = \lim GL_n^\dagger(A)/GL_n^\dagger(A)_0 = \lim U_n(A^\dagger)/U_n(A^\dagger)_0 = \lim GL_n(A^\dagger)/GL_n(A^\dagger)_0$.

2.3.3 Equivalence of the two pictures

Theorem 2.29 *Let A be a C^*-algebra. Then $\tilde{K}_1(A) \cong K_0(SA)$.*

Proof: Let us first define a map $\phi : \tilde{K}_1(A) \to K_0(S(A))$.

Take $v \in \mathscr{U}(M_n(A^\dagger))$ with $s(v) = 1_n$. Let $u(t)$ be a path of unitaries such that

$$u(0) = \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}, \qquad u(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad s(u(t)) = 1_{2n},\ t \in [0,1].$$

Next let

$$p(t) = u(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u(t)^*, \qquad q(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Exercise 2.57 Show that $[p] - [q]$ gives an element of $K_0(SA)$.
($s(p(t)) = 1_n$, i.e. $p(t) - 1_n \in M_{2n}(A)$ for all t. This means $t \mapsto p_{ij}(t) \in (SA)^\dagger$, which in turn implies that $p \in M_{2n}((SA)^\dagger)$. Thus $[p] - [q] \in K_0((SA)^\dagger)$. Since $p(t) - 1_n \in M_{2n}(A)$ for all t and $p(0) - 1_n = 0 = p(1) - 1_n$, it follows that $p - q \in SM_{2n}(A) = M_{2n}(SA)$. Thus $\pi(p) = \pi(q)$ so that $[p] - [q] \in K_0(SA)$.)

Exercise 2.58 If v' is a unitary homotopic to v, u' is a homotopy of unitaries connecting $\begin{pmatrix} v' & 0 \\ 0 & v'^* \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $p' = u'(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u'(t)^*$, then show that $[p] - [q] = [p'] - [q]$.
(Let $t \mapsto w_t$ be a homotopy from v to v'. Define

$$z(t) = u(t) \begin{pmatrix} v^* w_t & 0 \\ 0 & v w_t^* \end{pmatrix} u'(t)^*.$$

Now show that $z \in \mathscr{U}_{2n}^\dagger(SA)$ and $zp'z^* = p$.)

Exercise 2.59 Let $v' = \begin{pmatrix} v & 0 \\ 0 & 1_m \end{pmatrix}$, u' is a homotopy of unitaries connecting $\begin{pmatrix} v' & 0 \\ 0 & v'^* \end{pmatrix}$ and 1_{2m+2n} and $p' = u'(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u'(t)^*$. Show that $[p] - [q] = [p'] - [q]$.

Define $\phi([v]) = [p] - [q]$. We will show that it is an isomorphism.

COMPLETE THE PROOF. □

Remark 2.30 The inverse map $\psi : K_0(S(A)) \to \tilde{K}_1(A)$ is given as follows.

Take a $p \in Proj(M_n(S(A)^\dagger))$. Then p can be viewed as a projection valued map on $[0,1]$ such that $p(0) = p(1) \in M_n(\mathbb{C})$. Assume that $p(0) = p(1) = \begin{pmatrix} 1_m & 0 \\ 0 & 0 \end{pmatrix}$. Now there is a path of unitaries $u(t)$ with $u(1) = 1$ such that $p(t) = u(t)p(1)u(t)^*$. Since $p(0) = p(1) = \begin{pmatrix} 1_m & 0 \\ 0 & 0 \end{pmatrix}$, it follows that $u(0)$ is of the form $\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$. Define $\psi([p]) = [v]$.

Exercise 2.60 Show the following:
1. $K_1(\mathbb{C}) = 0$, 2. $K_1(\mathcal{L}(L_2(\mathbb{N}))) = 0$, 3. $K_1(Q(L_2(\mathbb{N}))) = \mathbb{Z}$, 4. $K_0(C(S^1)) = \mathbb{Z}$.

Exercise 2.61 Let (A_i, ϕ_{jk}) be an inductive system of C^*-algebras. Show that

$$K_1(\lim(A_i, \phi_{jk})) = \lim(K_1(A_i), K_1(\phi_{jk})).$$

3 Computational tools

3.1 Six term exact sequence

3.1.1 Lifting of homotopies

Proposition 3.1 *Suppose we have a short exact sequence*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} A/J \longrightarrow 0$$

If u_t is a path of unitaries in A/J and v_0 is a unitary in A such that $\pi(v_0) = u_0$, then there is a continuous path of unitaries v_t in A such that $\pi(v_t) = u_t$ for $t \in [0,1]$.

Proof: For each $t \in [0,1]$, there is an open interval $N(t)$ around t such that $\|u_s - u_{s'}\| < 2$ for all $s, s' \in N(t)$. By compactness of $[0,1]$, there are $t_1, \ldots, t_k$ such that $[0,1] \subseteq \cup N(t_i)$. It is now enough to prove that a lifting exists on each $N(t_i)$. In other words, without loss in generality we can assume that $\|u_t - u_s\| < 2$ for all t, s.

Since $\|u_0^* u_t - 1\| < 2$, the spectrum $\sigma(u_0^* u_t)$ does not contain the point -1 for all t. So there is a continuous path of self adjoint elements x_t such that $\exp(ix_t) = u_0^* u_t$.

Exercise 3.1 Show that x_t admits a lift to a continuous path y_t of self adjoint elements in A.

Define $v_t = v_0 \exp(iy_t)$. Then v_t gives a required lifting. □

Exercise 3.2 If p_t is a path of projections in A/J and q_0 is a projection in A such that $\pi(q_0) = q_0$, then there is a continuous path of unitaries q_t in A such that $\pi(q_t) = p_t$ for $t \in [0,1]$.

3.1.2 Fredholm operators

Let π denote the projection map from $\mathcal{L}(\mathcal{H})$ onto $Q(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called **Fredholm** if $\ker T$ and $\operatorname{coker} T$ are finite dimensional. If T is a Fredholm operator, then the range of T is closed.

1. **Theorem (Atkinson)**: T is Fredholm if and only if $\pi(T)$ is invertible in $Q(\mathcal{H})$.

2. Define $\operatorname{index}(T) := \dim \ker T - \dim \ker T^*$. If S and T are both Fredholm, then ST and T^* are also Fredholm and one has $\operatorname{index}(ST) = \operatorname{index}(S) + \operatorname{index}(T)$ and $\operatorname{index}(T^*) = -\operatorname{index}(T)$.

3. If T is Fredholm and K is compact, then $T+K$ is Fredholm and $\operatorname{index}(T+K) = \operatorname{index}(T)$.

4. If T is Fredholm and $\operatorname{index}(T) = 0$, then there is a finite rank operator F such that $T+F$ is invertible.

5. The map $T \mapsto \operatorname{index}(T)$ is continuous.

6. index $(T) =$ index (S) if and only if S and T are homotopic.

Exercise 3.3 Use the fact that $\mathcal{L}(\mathcal{H})$ and $Q(\mathcal{H})$ are properly infinite C^*-algebras to show that $K_1(Q(\mathcal{H})) = \{[T] : T \text{ Fredholm}\}$, where $[T]$ stands for the homotopy class for T.

Use the above facts to show that index : $K_1(Q(\mathcal{H})) \to \mathbb{Z}$ is a group isomorphism.

We will next see that the above map ($T \mapsto$ index (T)) can be looked upon as a map from $K_1(Q(\mathcal{H}))$ to $K_0(\mathcal{K}(\mathcal{H}))$.

Take an operator $T \in \mathcal{L}(\mathcal{H})$, T Fredholm. Then $z := \pi(T)$ is invertible in $Q(\mathcal{H})$. Let $T = V|T|$ and $\pi(T) = u|z|$ be the polar decompositions of T and z respectively. Then $\pi(V) = u$, i.e. V is a lift of u in $\mathcal{L}(\mathcal{H})$. Now

$$\operatorname{range} V = \operatorname{range} T = (\ker T^*)^\perp, \qquad \ker V = (\operatorname{range} |T|)^\perp = \ker |T| = \ker T.$$

Therefore $1 - VV^*$ is the projection onto $\ker T^*$ and $1 - V^*V$ is the projection onto $\ker T$. For $p \in Proj(\mathcal{K}(\mathcal{H}))$, $p \mapsto \dim p$ gives the natural inclusion of $V(\mathcal{K}(\mathcal{H}))$ in $K_0(\mathcal{K}(\mathcal{H})) = \mathbb{Z}$. Thus the number $\dim \ker T - \dim \ker T^*$ corresponds to the element $[1 - V^*V] - [1 - VV^*]$ in $K_0(\mathcal{K}(\mathcal{H}))$.

3.1.3 The index map

Suppose we have a short exact sequence

$$0 \longrightarrow J \xrightarrow{\phi} A \xrightarrow{\pi} A/J \longrightarrow 0.$$

We have already seen that in such a case one has the following two exact sequences:

$$K_0(J) \xrightarrow{K_0(\phi)} K_0(A) \xrightarrow{K_0(\pi)} K_0(A/J)$$

$$K_1(A/J) \xleftarrow[K_1(\pi)]{} K_1(A) \xleftarrow[K_1(\phi)]{} K_1(J)$$

We will now define a map $\partial : K_1(A/J) \to K_0(J)$ such that the following sequence is exact:

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{K_0(\phi)} & K_0(A) & \xrightarrow{K_0(\pi)} & K_0(A/J) \\ \partial \big\uparrow & & & & \\ K_1(A/J) & \xleftarrow[K_1(\pi)]{} & K_1(A) & \xleftarrow[K_1(\phi)]{} & K_1(J) \end{array} \tag{3.1}$$

For a C^*-algebra A, define the **cone over** A to be the C^*-algebra $Cone\,(A) := \{f \in C([0,1], A) : f(0) = 0\}$.

Exercise 3.4 Show that $Cone\,(A)$ is contractive and hence $K_0(Cone\,(A)) = 0$.

The **mapping cone** *Cone* $(A, A/J)$ of π is the C^*-algebra

$$\{(a, f) : a \in A, f \in C([0,1], A/J), f(1) = 0, f(0) = \pi(a)\}.$$

Theorem 3.2 *Let $\phi : J \to Cone\,(A, A/J)$ be given by $\phi(a) = (a, 0)$. Then $K_0(\phi)$ gives an isomorphism between $K_0(J)$ and $K_0(Cone\,(A, A/J))$.*

Proof: Let $\phi : J \to Cone\,(A, A/J)$ be given by $\phi(a) = (a, 0)$. Then we have a short exact sequence

$$0 \longrightarrow J \xrightarrow{\phi} Cone\,(A, A/J) \longrightarrow Cone\,(A/J) \longrightarrow 0.$$

Therefore

$$K_0(J) \xrightarrow{K_0(\phi)} K_0(Cone\,(A, A/J)) \longrightarrow K_0(Cone\,(A/J))$$

is exact in the middle. But *Cone* (A/J) is contractible, so that $K_0(Cone\,(A/J)) = 0$. So $K_0(\phi)$ is onto.

Next, let $B = \{f \in C([0,1], A) : f(1) \in J\}$.

Exercise 3.5 Let $\theta_1 : J \to B$ be given by $\theta_1(a) =$ the map $t \mapsto a$ and $\theta_2 : B \to J$ be given by $\theta_2(f) = f(1)$. Show that these give homotopy equivalence between J and B.

Exercise 3.6 Show that there is a short exact sequence

$$0 \longrightarrow C_0((0,1], J) \longrightarrow B \xrightarrow{\psi} Cone\,(A, A/J) \longrightarrow 0.$$

Since $K_0(C_0((0,1], J)) = 0$, by half exactness, $K_0(\psi)$ is injective. Since the diagram

$$\begin{array}{ccc} B & \xrightarrow{\psi} & Cone\,(A, A/J) \\ \uparrow{\scriptstyle \theta_1} & \nearrow{\scriptstyle \phi} & \\ A & & \end{array}$$

commutes, we have $K_0(\phi) = K_0(\psi) \circ K_0(\theta_1)$. But $K_0(\theta_1)$ is an isomorphism. So $K_0(\phi)$ is injective. $\square$

Exercise 3.7 Show that the map $(a, f) \mapsto a$ gives rise to a short exact sequence

$$0 \longrightarrow S(A/J) \longrightarrow Cone\,(A, A/J) \longrightarrow A \longrightarrow 0.$$

By half-exactness of K_0, the sequence

$$K_0(S(A/J)) \longrightarrow K_0(Cone\,(A, A/J)) \longrightarrow K_0(A)$$

is exact in the middle. View the map on the left as a map ∂ from $K_1(A/J)$ to $K_0(J)$. This is called the **index map** for the short exact sequence

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} A/J \longrightarrow 0.$$

Exercise 3.8 Show that the map $(a, f) \mapsto a$ gives rise to a short exact sequence

$$0 \longrightarrow SA \longrightarrow Cone\,(Cone\,(A, A/J), A) \longrightarrow Cone\,(A, A/J) \longrightarrow 0.$$

Again by half exactness,

$$K_0(SA) \longrightarrow K_0(Cone\,(Cone\,(A, A/J), A)) \longrightarrow K_0(Cone\,(A, A/J))$$

is exact in the middle. But $K_0(Cone\,(A, A/J)) = K_0(J)$ and $K_0(Cone\,(Cone\,(A, A/J), A)) = K_0(S(A/J)) = K_1(A/J)$. Thus we have a sequence

$$K_0(SA) \longrightarrow K_1(A/J) \longrightarrow K_0(J)$$

that is exact at $K_1(A/J)$.

Exercise 3.9 Verify that the map on the left is $K_1(\pi)$ and the one on the right is ∂.

Thus the sequence

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{K_0(\phi)} & K_0(A) & \xrightarrow{K_0(\pi)} & K_0(A/J) \\ \partial \uparrow & & & & \\ K_1(A/J) & \xleftarrow[K_1(\pi)]{} & K_1(A) & \xleftarrow[K_1(\phi)]{} & K_1(J) \end{array}$$

is exact and repeating the procedure we get the following long exact sequence

$$\longrightarrow K_{n+1}(J) \longrightarrow K_{n+1}(A) \longrightarrow K_{n+1}(A/J) \xrightarrow{\partial_{n+1}} K_n(J) \longrightarrow K_n(A) \longrightarrow K_n(A/J) \longrightarrow$$

where ∂_{n+1} is the index map for the exact sequence

$$0 \longrightarrow S^n J \xrightarrow{\phi} S^n A \xrightarrow{\pi} S^n A/S^n J \longrightarrow 0.$$

3.1.4 Computation of the index map

Assume that A is unital. We will derive a computable formula for the index map now.

Let $p \in Proj(M_n((S(A/J))^\dagger))$. Then $p(0) \in Proj M_n(\mathbb{C})$ so that it admits a lift to a projection P in $Proj M_n(A^\dagger)$. Since $\pi(P) = p(0) \in M_n(\mathbb{C})$, we have $s \circ \pi(P) = \pi(P)$. Therefore one has $P \in Proj M_n(J^\dagger) \subseteq Proj M_n(A^\dagger)$. By lifiting property of homotopy of projections, there is a path $P(t)$ of projections in $M_n(A^\dagger)$ with $P(0) = P$.

Lemma 3.3 *Suppose $P(t)$ and $P'(t)$ are two such liftings, so that $P(0) = P'(0) = P$. Then $P(1)$ and $P'(1)$ are unitarily equivalent in $M_n(J^\dagger)$.*

Proof: Exercise!

Exercise 3.10 Show that $[P(1)] - [P(0)] \in K_0(J)$.

Proposition 3.4 *Let $p, q \in ProjM_n((S(A/J))^\dagger)$ with $p(0) = q(0)$, let $P \in ProjM_n(A^\dagger)$ be a lifting of $p(0)$ and let $P(t)$ and $Q(t)$ be the liftings of p and q respectively with $P(0) = Q(0) = P$. Then*

$$\partial([p] - [q]) = ([P(1)] - [P(0)]) - ([Q(1)] - [Q(0)]). \tag{3.2}$$

Proof:

Exercise 3.11 Assuming A is unital, show that

$$Cone\,(A, A/J)^\dagger = \{(a, f) : a \in A, f \in C([0, 1], A/J), f(0) = \pi(a), f(1) \in \mathbb{C}\}.$$

Recall that we have an exact sequence

$$0 \longrightarrow S(A/J) \xrightarrow{\phi} Cone\,(A, A/J) \longrightarrow A \longrightarrow 0,$$

and the index map ∂ is the map $K_0(\phi)$. Therefore

$$\partial([p] - [q]) = K_0(\phi)([p] - [q]) = [(P(0), p)] - [(Q(0), q)].$$

On the other hand, we have the inclusion $\psi : J \to Cone\,(A, A/J)$ given by $\psi(a) = (a, 0)$. $K_0(\psi)$ gives an isomorphism from $K_0(J)$ to $K_0(Cone\,(A, A/J))$ and we have to check that the image under $K_0(\psi)$ of the right hand side coincides with the above.

Exercise 3.12 Show that the unique extension $\psi^\dagger : J^\dagger \to Cone\,(A, A/J)^\dagger$ of ψ is given by

$$\psi^\dagger(a) = (a, s(a)),$$

where $s(a)$ is the constant loop $t \mapsto s(a)$.

Now,

$$K_0(\psi)(([P(1)] - [P(0)]) - ([Q(1)] - [Q(0)])) = [(P(1), p(1))] - [(Q(1), q(1))].$$

Therefore it is enough to show that

$$(P(0), p) \sim_h (P(1), p(1)) \qquad \text{in } Cone\,(A, A/J)^\dagger.$$

For this, take the homotopy $\widetilde{P}(t) = (P(1 - t), p_t)$, where $p_t(s) = p(1 - t(1 - s))$. □

Exercise 3.13 Let u be a unitary element of $M_n((A/J)^\dagger)$. Show that there is an $a \in M_n(A^\dagger)$ such that $\|a\| = 1$ and $\pi(a) = u$.

Exercise 3.14 Show that $w := \begin{pmatrix} a & -(1 - aa^*)^{\frac{1}{2}} \\ (1 - a^*a)^{\frac{1}{2}} & a^* \end{pmatrix}$ is unitary and $\pi(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$.

Let $a(t) = ta + 1 - t$, $t \in [0,1]$. Then

$$w(t) := \begin{pmatrix} a(t) & -(1 - a(t)a(t)^*)^{\frac{1}{2}} \\ (1 - a(t)^*a(t))^{\frac{1}{2}} & a(t)^* \end{pmatrix}$$

is a path of unitaries that connect w to 1_{2n}. Write $v(t) = \pi(w(t))$. Define $p(t) = v(t)1_n v(t)^*$ and $q(t) = 1_n$. Then $p, q \in M_{2n}((S(A/J))^{\dagger})$ and $p(0) = q(0) = 1_n$. Therefore $[p] - [q]$ gives the element of $K_0(S(A/J))$ corresponding to the element $[u]$ in $K_1(A/J)$.

Since $w(t)1_n w(t)^*$ is a lifting of p with $w(0)1_n w(0)^* = 1_n$ and the constant loop $t \mapsto 1_n$ is a lifting of q, by the previous proposition, we have

$$\partial([u]) = \partial([p] - [q]) = [w(1)1_n w(1)^*] - [1_n] = [\begin{pmatrix} aa^* & a(1 - a^*a)^{\frac{1}{2}} \\ (1 - a^*a)^{\frac{1}{2}} a^* & 1 - a^*a \end{pmatrix}] - [1_n]. \quad (3.3)$$

If a happens to be a partial isometry so that $a(1 - a^*a)^{\frac{1}{2}} = 0$, then

$$\partial([u]) = [\begin{pmatrix} aa^* & 0 \\ 0 & 1 - a^*a \end{pmatrix}] - [1_n] = [1 - a^*a] - [1 - aa^*]. \quad (3.4)$$

3.1.5 Bott periodicity

Toeplitz algebra. Let S be the unilateral shift $e_n \mapsto e_{n+1}$ in $L_2(\mathbb{N})$. The C^*-subalgebra $\mathscr{T}$ of $\mathcal{L}(L_2(\mathbb{N}))$ generated by the operator S is called the Toeplitz algebra.

Exercise 3.15 Show that

1. $\mathcal{K} \subseteq \mathscr{T}$,
2. if π is the projection of $\mathscr{T}$ onto $\mathscr{T}/\mathcal{K}$, then the element $\pi(S)$ is a unitary in $\mathscr{T}/\mathcal{K}$ and has spectrum S^1, so that there is a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{T} \xrightarrow{\pi} C(S^1) \longrightarrow 0.$$

3. if $\phi : \mathscr{T} \to \mathbb{C}$ is the morphism given by $\phi = ev_1 \circ \pi$ (ev_1 is evaluation at 1)so that $\phi(S) = 1$, then $\mathscr{T}_0 := \ker \phi$ is the C^*-subalgebra of $\mathscr{T}$ generated by the operator $1 - S$, and one has the following split exact sequence

$$0 \longrightarrow \mathscr{T}_0 \longrightarrow \mathscr{T} \underset{j}{\overset{\pi_0}{\rightleftarrows}} \mathbb{C} \longrightarrow 0,$$

where j is the map $t \mapsto t.1$.

4. there is a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{T}_0 \xrightarrow{\pi} C_0(\mathbb{R}) \longrightarrow 0.$$

Theorem 3.5 *Let $\mathscr{T}$ be the Toeplitz algebra. Then there exists a canonical surjection $\pi_0 : \mathscr{T} \to \mathbb{C}$ such that $K_0(\pi_0)$ gives an isomorphism between $K_0(\mathscr{T})$ and $K_0(\mathbb{C}) = \mathbb{Z}$.*

Proof: From split exactnes of the sequence

$$0 \longrightarrow \mathscr{T}_0 \longrightarrow \mathscr{T} \underset{j}{\overset{\pi_0}{\rightleftarrows}} \mathbb{C} \longrightarrow 0.$$

we conclude that the sequence

$$0 \longrightarrow K_0(\mathscr{T}_0) \longrightarrow K_0(\mathscr{T}) \underset{K_0(j)}{\overset{K_0(\pi_0)}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0$$

is split exact, so that $K_0(\pi_0) \circ K_0(j) = id$. We will now show that $K_0(j) \circ K_0(\pi_0) = id_{K_0(\mathscr{T})}$.

Exercise 3.16 Let $\sigma : \mathscr{T} \to \mathcal{K} \otimes \mathscr{T}$ be the embedding $a \mapsto (I - SS^*) \otimes a$. Show that $K_0(\sigma)$ is an isomorphism.

since $K_0(\sigma)$ is an isomorphism, it is enough to show that

$$K_0(\sigma) \circ K_0(j) \circ K_0(\pi_0) = K_0(\sigma).$$

Let $\mathscr{T}'$ be the C^*-subalgebra of $\mathscr{T} \otimes \mathscr{T}$ generated by $\mathcal{K} \otimes \mathscr{T}$ and $\mathscr{T} \otimes 1$.

Exercise 3.17 Show that $\mathcal{K} \otimes \mathscr{T}$ is an ideal in $\mathscr{T}'$ and $\mathscr{T}'/(\mathcal{K} \otimes \mathscr{T}) \cong C(S^1)$.

Denote by π' the projection of $\mathscr{T}'$ onto $C(S^1)$. Let $\widetilde{\mathscr{T}}$ be the join of $\mathscr{T}'$ and $\mathscr{T}$ along $C(S^1)$, i.e.

$$\widetilde{\mathscr{T}} = \{a \oplus b \in \mathscr{T}' \oplus \mathscr{T} : \pi'(a) = \pi(b)\}.$$

Define maps $i : \mathcal{K} \otimes \mathscr{T} \to \widetilde{\mathscr{T}}$, $\tilde{\pi} : \widetilde{\mathscr{T}} \to \mathscr{T}$ and $\gamma : \mathscr{T} \to \widetilde{\mathscr{T}}$ by

$$i(a) = a \oplus 0, \qquad \tilde{\pi}(a \oplus b) = b, \qquad \gamma(b) = (b \otimes 1) \oplus b.$$

Then one has the split exact sequence

$$0 \longrightarrow \mathcal{K} \otimes \mathscr{T} \stackrel{i}{\longrightarrow} \widetilde{\mathscr{T}} \underset{\gamma}{\overset{\tilde{\pi}}{\rightleftarrows}} \mathscr{T} \longrightarrow 0.$$

Since K_0 is split exact, it follows that $K_0(i)$ is injective. Therefore it is now enough to show that

$$K_0(i) \circ K_0(\sigma) \circ K_0(j) \circ K_0(\pi_0) = K_0(i) \circ K_0(\sigma).$$

We have $i \circ \sigma(S) = (1 - SS^*) \otimes S \oplus 0$ and $i \circ \sigma \circ j \circ \pi_0(S) = (1 - SS^*) \otimes 1 \oplus 0$.

Exercise 3.18 Write

$$P = 1 - SS^*, \quad V = S \otimes 1, \quad Q = P \otimes 1, \quad W = P \otimes S, \quad R = P \otimes P.$$

Let

$$u_0 = V(1 - Q)V^* + WV^* + VW^* + R, \quad u_1 = V(1 - Q)V^* + QV^* + VQ.$$

Show that u_0 and u_1 are self-adjoint unitaries.

It follows that there is a homotopy of unitaries u_t connecting u_0 and u_1. Define $\phi_t : \mathscr{T} \to \mathscr{T}'$ by $\phi_t(S) = u_t(S \otimes 1)$. This gives a homotopy of morphisms. Next, define $\psi_t(S) = \phi_t(S) \oplus S$.

Exercise 3.19 Show that ψ_t is a homotopy of morphisms from $\mathscr{T}$ to $\tilde{\mathscr{T}}$.

Exercise 3.20 Define $\psi(S) = (S^2S^* \otimes 1) \oplus S$. Show that ψ extends to a morphism from $\mathscr{T}$ to $\tilde{\mathscr{T}}$.

Show that

$$\psi_0 - \psi = i \circ \sigma, \qquad \psi_1 - \psi = i \circ \sigma \circ j \circ \pi_0.$$

Show that $K_0(\psi_0) = K_0(\psi) + K_0(i \circ \sigma)$ and $K_0(\psi_1) = K_0(\psi) + K_0(i \circ \sigma \circ j \circ \pi_0)$.

The required equality follows. □

Theorem 3.6 *For any C^*-algebra A, one has a natural isomorphism between $K_0(A)$ and $K_0(S^2A)$.*

Proof: [**Cuntz**]

From the short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{T}_0 \longrightarrow C_0(\mathbb{R}) \longrightarrow 0.$$

we get, by tensoring with A,

$$0 \longrightarrow \mathcal{K} \otimes A \longrightarrow \mathscr{T}_0 \otimes A \longrightarrow SA \longrightarrow 0.$$

Therefore we now have the long exact sequence

$$\longrightarrow K_1(\mathcal{K} \otimes A) \longrightarrow K_1(\mathscr{T}_0 \otimes A) \longrightarrow K_1(SA) \longrightarrow K_0(\mathcal{K} \otimes A) \longrightarrow K_0(\mathscr{T}_0 \otimes A) \longrightarrow K_0(SA) \longrightarrow$$

Since $K_0(\mathcal{K} \otimes A) = K_0(A)$ and $K_1(SA) = K_0(S^2A)$, if we can now prove that $K_1(\mathscr{T}_0 \otimes A) = 0 = K_0(\mathscr{T}_0 \otimes A)$, then we are through. Since $K_1(\mathscr{T}_0 \otimes A) = K_0(\mathscr{T}_0 \otimes SA)$, it is enough to show that for any C^*-algebra B, we have $K_0(\mathscr{T}_0 \otimes B) = 0$.

Exercise 3.21 Show that one has the following split exact sequence:

$$0 \longrightarrow \mathscr{T}_0 \otimes B \longrightarrow \mathscr{T} \otimes B \longrightarrow B \longrightarrow 0.$$

Prove that $K_0(\mathscr{T}_0 \otimes B) = 0$.

The proof is thus complete. □

The Bott map. Assume A is unital. Denote by z the map $w \mapsto w$ from S^1 to $\mathbb{C}$. Let $p \in M_n(A)$ be a projection. Then $pz + 1 - p : w \mapsto pw + 1 - p$ is an element in $\mathscr{U}(M_n((SA)^\dagger))$.

Exercise 3.22 If $p \in Proj(M_n(A))$ and $q \in Proj(M_k(A))$ are homotopic, then $pz + 1 - p$ and $qz + 1 - q$ can be connected by a homotopy of unitaries.

The map $\beta : [p] \mapsto [pz + 1 - p]$ from $K_0(A)$ to $K_1(SA) \cong K_0(S^2A)$ is called **the Bott map**.

3.1.6 Computation of K-groups

Stable multiplier algebra.

Lemma 3.7 *For any C^*-algebra A, one has $K_0(M(\mathcal{K}\otimes A)) = 0$.*

Proof: Let $p \in Proj M(\mathcal{K}\otimes A)$. Choose isometries $v_1, v_2, \ldots$ in $\mathcal{L}(\mathcal{H})$ with $v_i^* v_j = 0$ for $i \neq j$. Define

$$q = \sum (v_i \otimes 1) p (v_i^* \otimes 1), \qquad a = \sum v_{i+1} v_i^* \otimes 1.$$

Exercise 3.23 Show that both the above series converge in the strict topology in $M(\mathcal{K}\otimes A)$.

Now define

$$w = \begin{pmatrix} 0 & 0 \\ v_1 \otimes 1 & \sum v_{i+1} v_i^* \otimes 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ (v_1 \otimes 1)p & (v_{i+1}\otimes 1) p (v_i^* \otimes 1) \end{pmatrix}.$$

Then

$$w^* w = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \qquad w w^* = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

Thus $[p] + [q] = [q]$. Since $M(\mathcal{K}\otimes A)$ is properly infinite, so that $K_0(M(\mathcal{K}\otimes A)) = \{[p] : p \in Proj M(\mathcal{K}\otimes A)\}$, it follows that $K_0(M(\mathcal{K}\otimes A)) = 0$. □

Exercise 3.24 Show that for any unital C^*-algebra B, $K_0(M(\mathcal{K}\otimes A)\otimes B) = 0$. Use this to prove that $K_1(M(\mathcal{K}\otimes A) = 0$ for any C^*-algebra A.

Proposition 3.8 $K_i(Q(\mathcal{K}\otimes A)) = K_{1-i}(A)$, $i = 0, 1$.

Proof: From the short exact sequence

$$0 \longrightarrow \mathcal{K}\otimes A \longrightarrow M(\mathcal{K}\otimes A) \longrightarrow Q(\mathcal{K}\otimes A) \longrightarrow 0,$$

we have the following six-term exact sequence:

$$\begin{array}{ccccc} K_0(\mathcal{K}\otimes A) & \longrightarrow & K_0(M(\mathcal{K}\otimes A)) & \longrightarrow & K_0(Q(\mathcal{K}\otimes A)) \\ \uparrow & & & & \downarrow \\ K_1(Q(\mathcal{K}\otimes A)) & \longleftarrow & K_1(M(\mathcal{K}\otimes A)) & \longleftarrow & K_1(\mathcal{K}\otimes A) \end{array}$$

Since $K_i(M(\mathcal{K}\otimes A)) = 0$, the result follows. □

Toeplitz algebra.

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{T} \longrightarrow C(S^1) \longrightarrow 0.$$

$$\begin{array}{ccccc}
K_0(\mathcal{K}) = \mathbb{Z} & \longrightarrow & K_0(\mathscr{T}) & \xrightarrow{K_0(\sigma)} & K_0(C(S^1)) = \mathbb{Z} \\
\uparrow \partial & & & & \downarrow \\
K_1(C(S^1)) = \mathbb{Z} & \xleftarrow[K_1(\sigma)]{} & K_1(\mathscr{T}) & \longleftarrow & K_1(\mathcal{K}) = 0
\end{array}$$

Quantum $SU(2)$. The C^*-algebra A associated with the quantum $SU(2)$ group is defined to be the universal C^*-algebra generated by two elements α and β satisfying the following relations:

$$\begin{aligned}
\alpha^*\alpha + \beta^*\beta = 1, &\qquad \alpha\alpha^* + q^2\beta\beta^* = 1, \\
\alpha\beta - q\beta\alpha = 0, &\qquad \alpha\beta^* - q\beta^*\alpha = 0, \\
\beta^*\beta \;\; &= \;\; \beta\beta^*.
\end{aligned}$$

The C^*-algebra A has two families of irreducible representations:

$$\left.\begin{array}{rcl}
\mathcal{H} & = & L_2(\mathbb{N}) \\
\alpha & \mapsto & S^*\sqrt{1-q^{2N}} \\
\beta & \mapsto & zq^N.
\end{array}\right\} z \in S^1, \qquad
\left.\begin{array}{rcl}
\mathcal{H} & = & \mathbb{C} \\
\alpha & \mapsto & z, \\
\beta & \mapsto & 0.
\end{array}\right\} z \in S^1.$$

The following representation gives a faithful representation of A:

$$\pi : \left\{\begin{array}{rcl}
\mathcal{H} & = & L_2(\mathbb{N}) \otimes L_2(\mathbb{Z}), \\
\alpha & \mapsto & S^*\sqrt{1-q^{2N}} \otimes 1, \\
\beta & \mapsto & q^{2N} \otimes \ell.
\end{array}\right.$$

One can thus identify A with the C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $\pi(\alpha)$ and $\pi(\beta)$.

Exercise 3.25 Show that the map given by $\alpha \mapsto 1$ and $\beta \mapsto 0$ gives rise to the following short exact sequence:

$$0 \longrightarrow \mathcal{K} \otimes C(S^1) \longrightarrow C(SU_q(2)) \xrightarrow{\sigma} C(S^1) \longrightarrow 0.$$

$$\begin{array}{ccccc}
K_0(\mathcal{K} \otimes C(S^1)) = \mathbb{Z} & \longrightarrow & K_0(C(SU_q(2))) & \xrightarrow{K_0(\sigma)} & K_0(C(S^1)) = \mathbb{Z} \\
\uparrow \partial & & & & \downarrow \\
K_1(C(S^1)) = \mathbb{Z} & \longleftarrow & K_1(C(SU_q(2))) & \longleftarrow & K_1(\mathcal{K} \otimes C(S^1)) = \mathbb{Z}
\end{array}$$

Exercise 3.26 Show that

1. ∂ is one-one and onto.
2. $K_0(\sigma)$ is onto.
3. $K_0(C(SU_q(2))) = \mathbb{Z} = K_1(C(SU_q(2)))$.

Podles spheres S^2_{qc}, $c > 0$. This is the universal C*-algebra, denoted by $C(S^2_{qc})$, generated by two elements A and B subject to the following relations:

$$A^* = A, \qquad B^*B = A - A^2 + cI,$$
$$BA = q^2 AB, \qquad BB^* = q^2 A - q^4 + cI.$$

Here the deformation parameters q and c satisfy $|q| < 1, c > 0$. Let $\mathcal{H}_+ = l^2(\mathbb{N}), \mathcal{H}_- = \mathcal{H}_+$. Define $\pi_\pm(A), \pi_\pm(B) : \mathcal{H}_\pm \to \mathcal{H}_\pm$ by

$$\pi_\pm(A)(e_n) = \lambda_\pm q^{2n} e_n \qquad \text{where} \qquad \lambda_\pm = \frac{1}{2} \pm (c + \frac{1}{4})^{1/2},$$
$$\pi_\pm(B)(e_n) = c_\pm(n)^{1/2} e_{n-1} \quad \text{where} \quad c_\pm(n) = \lambda_\pm q^{2n} - (\lambda_\pm q^{2n})^2 + c.$$

Exercise 3.27 $\pi_\pm$ are irreducible and the direct sum $\pi_+ \oplus \pi_-$ is faithful.

Since $\pi = \pi_+ \oplus \pi_-$ is a faithful representation, an immediate corollary follows.

Proposition 3.9 (Sheu) (i) *$C(S^2_{qc}) \cong \mathscr{T} \oplus_\sigma \mathscr{T} := \{(x, y) : x, y \in \mathscr{T}, \sigma(x) = \sigma(y)\}$ where $\mathscr{T}$ is the Toeplitz algebra and $\sigma : \mathscr{T} \to C(S^1)$ is the symbol homomorphism.*
(ii) *We have a short exact sequence*

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} C(S^2_{qc}) \xrightarrow{\alpha} \mathscr{T} \longrightarrow 0 \tag{3.5}$$

Proof: (i) An explicit isomorphism is given by $x \mapsto (\pi_+(x), \pi_-(x))$.
(ii) Define $\alpha((x, y)) = x$. Then $\ker \alpha = \mathcal{K}$. □

Exercise 3.28 Show that the sequence (3.5) above is split exact. Conclude that $K_0(C(S^2_{qc})) = \mathbb{Z} \oplus \mathbb{Z}$ and $K_1(C(S^2_{qc})) = 0$.

Another way to compute the K-groups for the Podlès sphere is to prove that one has the following short exact sequence:

$$0 \longrightarrow \mathcal{K} \oplus \mathcal{K} \longrightarrow C(S^2_{qc}) \xrightarrow{\sigma} C(S^1) \longrightarrow 0.$$

so that one has the six term sequence:

$$\begin{array}{ccccc} K_0(\mathcal{K} \oplus \mathcal{K}) = \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & K_0(C(S^2_{qc})) & \xrightarrow{K_0(\sigma)} & K_0(C(S^1)) = \mathbb{Z} \\ \partial \uparrow & & & & \downarrow \\ K_1(C(S^1)) = \mathbb{Z} & \xleftarrow[K_1(\sigma)]{} & K_1(C(S^2_{qc})) & \longleftarrow & K_1(\mathcal{K} \oplus \mathcal{K}) = 0 \end{array}$$

Exercise 3.29 Show that

1. ∂ is one-one.
2. $K_0(\sigma)$ is onto.
3. $K_0(C(S^2_{qc})) = \mathbb{Z} \oplus \mathbb{Z}$, $K_1(C(S^2_{qc})) = 0$.

3.2 K-groups of crossed products

3.2.1 Crossed products

3.2.2 Crossed products with $\mathbb{Z}$

Theorem 3.10 *Let A be a C^*-algebra and let τ be an action of $\mathbb{Z}$ on A. Then there is a six-term exact sequence*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-K_0(\tau)} & K_0(A) & \xrightarrow{K_0(\iota)} & K_0(A \rtimes_\tau \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\tau \mathbb{Z}) & \xleftarrow[K_1(\iota)]{} & K_1(A) & \xleftarrow[1-K_1(\tau)]{} & K_1(A) \end{array}$$

The irrational rotation algebra. Let $\theta \in [0,1]$ be an irrational. The irrational rotation algebra A_θ is the universal C^*-algebra generated by two unitaries u and v satisfying the relation $uv = \exp(2\pi i\theta)vu$. The C^*-algebra A_θ can be written as a crossed product as follows: let α be the automorphism of $C(S^1)$ induces by the map $z \mapsto \exp(2\pi i\theta)z$ on S^1. Then $A_\theta \cong C(S^1) \rtimes_\alpha \mathbb{Z}$. Therefore we have the following Pimsner-Voiculescu exact sequence:

$$\begin{array}{ccccc} K_0(C(S^1)) = \mathbb{Z} & \longrightarrow & K_0(A_\theta) & \longrightarrow & K_1(C(S^1)) = \mathbb{Z} \\ {\scriptstyle 1-K_0(\alpha)}\uparrow & & & & \downarrow{\scriptstyle 1-K_1(\alpha)} \\ K_0(C(S^1)) = \mathbb{Z} & \longleftarrow & K_1(A_\theta) & \longleftarrow & K_1(C(S^1)) = \mathbb{Z} \end{array}$$

The automorphism α is homotopic to the identity. Therefore both $K_0(\alpha)$ and $K_1(\alpha)$ are identity. Thus we have two short exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_0(A_\theta) \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow K_1(A_\theta) \longrightarrow \mathbb{Z} \longrightarrow 0$$

It follows that $K_0(A_\theta) = \mathbb{Z} \oplus \mathbb{Z} = K_1(A_\theta)$.

3.2.3 Crossed products with $\mathbb{R}$

Theorem 3.11 (Connes) *Let A be a C^*-algebra and let τ be an action of $\mathbb{R}$ on A. Then one has*

$$K_n(A \rtimes_\tau \mathbb{R}) \cong K_{1-n}(A), \quad n = 0, 1.$$

Exercise 3.30 Deduce Bott periodicity from the above theorem.

Pimsner-Voiculescu sequence from Connes' theorem. Let A be a C^*-algebra and let α be an automorphism of A. Define the mapping torus M_α by

$$M_\alpha = \{f \in C([0,1], A) : f(1) = \alpha(f(0))\}.$$

Define $\pi : M_\alpha \to A$ by $\pi(f) = f(0)$. It is easy to see that one has the following short exact sequence:

$$0 \longrightarrow SA \longrightarrow M_\alpha \longrightarrow A \longrightarrow 0$$

This gives rise to the following six-term exact sequence:

$$\begin{array}{ccccc} K_0(SA) & \longrightarrow & K_0(M_\alpha) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(M_\alpha) & \longleftarrow & K_1(SA) \end{array}$$

Next one shows that the connecting maps are $1 - K_0(\alpha)$ and $1 - K_1(\alpha)$ and using Connes-Thom isomorphism one shows that

$$K_0(M_\alpha) \cong K_1(A \rtimes_\alpha \mathbb{Z}), \qquad K_1(M_\alpha) \cong K_0(A \rtimes_\alpha \mathbb{Z}).$$

3.3 K-groups of tensor products

4 K-groups of some C^*-algebras

C^*-algebra	K_0	K_1
$C[0,1]$	$\mathbb{Z}$	0
$C_0(0,1]$	0	0
$C(S^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z}$
$C(S^{2n})$	$\mathbb{Z}^2$	0
$C_0(\mathbb{R}^{2n})$	$\mathbb{Z}$	0
$C_0(\mathbb{R}^{2n+1})$	0	$\mathbb{Z}$

C^*-algebra	K_0	K_1
$\mathbb{C}$	$\mathbb{Z}$	0
$M_n(\mathbb{C})$	$\mathbb{Z}$	0
$\mathcal{K}(\mathcal{H})$	$\mathbb{Z}$	0
$\mathcal{B}(\ell_2)$	0	0
$\mathcal{B}(\ell_2)/\mathcal{K}(\ell_2)$	0	$\mathbb{Z}$
$M(\mathcal{K} \otimes A)$	0	0

C^*-algebra	K_0	K_1
$\mathscr{T}$	$\mathbb{Z}$	0
A_θ	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$C(S_q^{2\ell+1})$	$\mathbb{Z}$	$\mathbb{Z}$
$C(S_q^{2\ell})$	$\mathbb{Z}^2$	0
$C(SU_q(\ell+1))$	???	???
$\mathscr{O}_n$	$\mathbb{Z}_{n-1}$	0

5 References

References

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