Theory of Games - Problem Set 3

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1. Consider the following game played between players 1 and 2. Player 1 moves first and plays either L or R. This is is observed by both the players. If L is played, players 1 and 2 simultaneously choose from the sets $\{l, r\}$ and $\{t, d\}$ respectively and the game ends with payoffs $(3, 1)$, $(0, 0)$, $(0, 0)$ and $(1, 3)$ if (l, t), (l, d), (r, t) and (r, d) are played respectively. If Player 1 plays R, players 1 and 2 simultaneously choose from the sets $\{m, n\}$ and $\{q, s\}$ respectively and the game ends with payoffs $(2, 1)$, $(1, 2)$, $(1, 2)$ and $(2, 1)$ if (m, q), (m, s), (n, q) and (n, s) are played respectively.

(i) Represent this game in extensive form.

(ii) Compute all subgame perfect Nash equilibria (SPNE) in the game.

2. Let $c : [0, 1] \to R$ and $p : [0, 1] \to R$ be two functions such that $c(a) < p(a)$ for all $a \in [0, 1]$. Consider the following game played between two players "Child" (C) and "Parent" (P). Child moves first and picks $a \in [0, 1]$. Parent observes the choice a made by C and picks a real number T. The game now ends with the payoffs $(c(a) + T$, min $(p(a) - T, c(a) + T)$. Show that in SPNE, C chooses a to maximize $c(a) + p(a)$.

3. (Removing Stones) Two people take turns in removing stones from a pile of n stones with player 1 moving first. Each player may remove either one or two stones in each of his turns. The player who takes the last stone is the winner and receives Rs 1 from the loser. What is the SPNE outcome for $n = 4$? Generalize to arbitrary n.

4. Consider the infinite horizon alternating offer bargaining model where player 1 moves first. If agreement $x = (x_1, x_2)$ $((0 \leq x_1, x_2 \leq 1)$ and $x_1 + x_2 = 1)$ is reached in period t, the payoffs for players 1 and 2 are $\delta^{t-1}(x_1 + \beta x_2)$ and $\delta^{t-1}(x_2 + \beta x_1)$ respectively where δ is the discount rate and $0 < \beta < 1$. (Here a player "cares" about the share received by the other player.) What is the SPNE of this game?

5. Consider the infinite horizon alternating offer bargaining model where player 1 moves first with the following special feature. Whenever it is player

2's turn to respond to an offer of player 1, other than accepting or rejecting, he also has the option of "quitting". If 2 quits in period t, then he gets $\delta^{t-1}b$ and 1 gets zero. Here δ is the discount rate and $b > 0$. Prove the following:

(a) If $b < \frac{\delta}{1+\delta}$, then the game has a unique SPNE which is SPNE of the game where 2 does not have the possibility of quitting.

(b) If $b > \frac{\delta}{1+\delta}$, then the game has a unique SPNE where player 1 always proposes $(1-b, b)$ and accepts any offer which gives him at least $\delta(1-b)$; player 2 always proposes $(\delta(1-b), 1-\delta(1-b)$ and accepts any offer which gives him at least b. The outcome is therefore $(1-b, b)$ in period zero.

6. There are two firms a and b in an industry. If firm a is given a license to undertake a particular project (say, to install a new polluting technology), it will increase firm a's profits by $V_a \geq 0$ but will decrease firm b's profit by $V_b \geq 0$ (because of the pollution). The numbers V_a or V_b are common knowledge to a and b but are not known to the Government agency which has to decide on whether or not to give the license to a. The following procedure (or mechanism) is devised. Note that payoffs are specified as vectors where the first and second components denote payoffs to a and b respectively.

- Stage 1: Firm a decides whether or not to apply for the license. If it does not apply, the game ends with payoffs $(0, 0)$. Otherwise it can apply for the license and offer to pay T_b as compensation to b. The game then moves to Stage 2.
- Stage 2: Firm b can accept a's offer. In that case firm a can decide whether it wants to carry out the project or not. If it carries out the project, the payoffs are $(V_a - T_b, T_b - V_b)$. If a does not carry out the project, it has to compensate b anyway, so that payoffs are $(-T_b, T_b)$. If b does not accept a's offer, then it can propose an alternative compensation T_b' and the game moves to Stage 3.
- Stage 3: Firm a can decide either not to carry out the project in which case the game ends with payoffs (δ, δ) for some $\delta > 0$ (i.e. a pays b an amount δ) or decides to send the game to Stage 4.
- Stage 4: Firm b makes a final call on the project. If it now approves the project, a can execute the project without paying any compensation to b, i.e. the game ends with payoffs $(V_a, -V_b)$. If it rejects the project the game ends with payoffs $(-2\delta, -T_b')$ (both firms have to pay fines to the Government).

(i) Write down an extensive-form game corresponding to the procedure described above (note that this is a game of perfect information). (ii) Suppose $V_a - V_b < 0$. Prove that there is a unique SPNE where a does not apply for a license in Stage 1.

(iii) Suppose $V_a - V_b > 0$. Prove that there is a unique SPNE where a applies for the license in Stage 1 and proposes a compensation $T_b = V_b$.

(iv) Suppose that instead of the complicated procedure outlined above, the following "naive" procedure was followed. Firm a could either choose to apply for the license or not. If it did not apply, the game ends with payoffs $(0, 0)$. If it applies, it proposes a compensation T_b. Firm b can then either accept or reject the proposal. If it rejects, payoffs are $(0, 0)$. If it accepts, payoffs are $(V_a - T_b, T_b - V_b)$. What is the SPNE of this game in the cases $V_a - V_b > 0$ and $V_a - V_b < 0$?