Uniform Expected Utility Criteria for Decision Making under Ignorance or Objective Ambiguity*

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Abstract

We provide an axiomatic characterization of a family of criteria for ranking *completely uncertain* and/or *ambiguous* decisions. A completely uncertain decision is described by the set of all its consequences (assumed to be finite). An ambiguous decisions is described as a finite set of possible probabilities distributions over a finite set of prices. Every criterion in the family characterized can be thought of as assigning to every consequence - or to every probability distribution - of a decision an *equal probability* of occurence and as comparing decisions on the basis of the *expected utility* of their consequences - or their probability distributions - for some utility function.

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1 Introduction

It is common to categorize decision problems by the structure of the environment that is assumed to be known to the decision maker. In situations

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of *certainty*, the decision maker is assumed to know the *unique consequence* of every decision which can, therefore, be usefully identified by this unique consequence. In situations of *risk*, studied along the lines of Neumann and Morgenstern (1947), the decision maker knows the *probability distribution* over all consequences that decisions can have so that the problem of choosing the "right" decision amounts to that of choosing the "right" probability distribution over the set of consequences. In situations of *uncertainty*, decisions are described as functions - *acts* in Savage (1954) terminology - from a set of states of nature to a set of consequences. Finally, in situations of *complete uncertainty*, or *ignorance* as these are sometimes called, a decision is described even more parsimoniously by the *set* of all its (foreseeable) consequences. The problem of ranking decisions amounts therefore to a problem of ranking sets of these consequences. A somewhat hybrid category of decision problems is provided by the recent interesting literature on objective ambiguity without state space, illustrated by Ahn (2008) and Olszewski (2007), in which a decision is depicted as a set of probability distributions over a set of consequences.

In the last twenty years or so, a sizeable literature, surveyed by Barberà, Bossert, and Pattanaik (2004), has developed on the problem of ranking sets of consequences in the context of choice under ignorance. With the noticeable exception of Baigent and Xu (2004) and Nitzan and Pattanaik (1984), all rankings of decisions that have emerged in this literature are based on the *best* and the *worst* consequences of the decisions or on some lexicographic extension thereof. The limitation of these "extremist" rankings for understanding actual decision making under ignorance is clear enough. Suppose we consider an investor facing two alternative investment strategies in some completely uncertain environment. If strategy A is adopted, the investor gains (net of the cost of investing) either one or one million dollars. If strategy B is adopted, then the investor's gain is either nothing, or any (integer) amount between \$900000 and \$999999. Hence, the two investment strategies can be described by:

$A = \{1, 1000000\}$

$B = \{0, 900000, 900001, ..., 999999\}$

Under the assumption that the ranking of certain (singletons) decisions is increasing in money, most rules studied in the literature that are "monotonically increasing" with respect to the worst and the best elements would rank A above B. Yet it is not clear that an actual investor placed in that circumstance would make the same ranking. For instance an investor who would be somehow capable of assigning probabilities of occurrence to consequences - even without being able to identify clearly the states of nature and the mapping that associates consequences to states of nature - could very plausibly rank B above A on the basis that the "expected utility" of the consequences is higher in B than in A. The *median-based* ranking of sets characterized in Nitzan and Pattanaik (1984), and which compares sets in terms of their median consequence with respect to the underlying ranking of certain outcomes, would also consider B to be a better decision than A in a situation like this. So would the *average Borda rule* characterized in Baigent and Xu (2004) which ranks sets according to the average Borda score of their elements.

Another limitation of many rankings considered in the literature, which applies also to the median-based and the average Borda rule criteria, is that they do not allow for a diversity of attitudes toward ignorance across individuals. Consider again the case of decisions with pecuniary consequences. If all decision makers prefer more money received for sure to less and follow any particular positional rule such as the maximin, the maximax, the median or some lexicographic extension thereof, they will all rank uncertain decisions in the same fashion. This feature of positional rankings is clearly restrictive. After all, the fact that two individuals prefer more money to less and have a choice behavior that obeys the same axioms should not imply that they have the same attitude with respect to uncertainty.

The relative scarcity of criteria for comparing sets of consequences in the context of decision making under ignorance is particularly striking when compared with what is observed in classical (Savagian) situations of uncertainty. In the later case one finds, along with "extremist" criteria that compare acts on the basis of their worst or best consequence, as characterized in Arrow and Hurwicz (1972) and Maskin (1979), the well-known Expected Utility (EU) criterion characterized in Savage (1954) as well as many other "non-additive" criteria such as "Maximin Expected Utility over a Set of Priors" (characterized in Gilboa and Schmeidler (1989) and Casadesus-Masanell, Klibanoff, and Ozdenoren (2000)) or the "Choquet Expected Utility" criterion characterized in Schmeidler (1989). Contrary to their "extremist" or positional counterparts, individuals whose behavior satisfies a particular additive or non-additive EU criterion and who have the same preferences for the consequences do not need to have the same attitude toward uncertainty.

In this paper, we provide an axiomatic characterization of a family of criteria of choice under ignorance that is quite close in spirit to the classical EU family. Any criterion in this family can be viewed as ranking decisions (sets) on the basis of the expected utility of their consequences for some utility function, under the assumption that the decision maker assigns to every consequence of a decision an *equal probability* of occurrence. For this reason we refer to a criterion in this family as to a *Uniform Expected Utility* (UEU) criterion. Beside the framework of analysis, the main difference between UEU criteria and standard EU ones lies in the uniform assumption made on probabilities. In our view, the uniform assumption is not unreasonable in the context of choice under complete uncertainty. A decision maker who ignores the mechanism by which consequences are produced as a function of the states of nature, and who is only capable of identifying the set of possible consequences of a decision has *a priori* no reason to believe one consequence to be more likely than another. This principle of *insufficient reason*, renamed "principle of indifference" by Keynes (1921), has been, after all, the main justification given by early probability theorists such as Bernouilli and Laplace, to their assumption of uniform probabilities as applying to "games of chance" (see also Jaynes (2003) for a recent justification of this principle).

The framework used to characterize the family of UEU criteria is similar to that assumed in the literature on choices under ignorance in the sense that we describe decisions as *finite* sets of consequences and we propose axioms that apply to the ranking of these sets. We depart however from most of the literature by assuming that the universe of all conceivable consequences has a rich and *Archimedean structure*, as defined in Krantz, Luce, Suppes, and Tversky (1971). While we do not, for the main result, endow our universe of consequences with topological properties that would enable one to define appropriate continuity conditions on the ranking of decisions, our framework is compatible with such a topological setting. We actually illustrate this by characterizing, in our theorem 4 below, the UEU family of criteria for decisions having their consequences in $\mathbb{R}^k$. Assuming such an environment enables us to replace the richness and Archimedean properties by a mild continuity condition imposed on the ranking of decisions.

To that extent, our framework can be usefully compared to that of Nehring and Puppe (1996) in which the universe of consequences is endowed with a topology and a continuity property is imposed on the ranking of all finite subsets of the universe. Yet continuity is not a straightforward notion when applied to rankings of sets of objects (as opposed to a ranking of objects). For instance, a widely used notion of continuity for sets rankings, adopted by Nehring and Puppe (1996), is continuity with respect to the Hausdorff topology. However this notion of continuity fails to recognize as continuous a UEU ranking, even though such a ranking is continuous when characterized in a Savagian (uncertainty) framework. This remark explains the difference between our results and those of Nehring and Puppe (1996). These authors characterize rankings that compare sets on the basis of their maximal and minimal elements only using Hausdorff continuity and a mild independence condition (satisfied by UEU criteria). In contrast, we consider an abstract setting that is compatible (as demonstrated by our theorem 4) with many topological structures. We then characterize a family of rankings that are continuous in a very natural sense, albeit not Hausdorff continuous, and that are not based only on the maximal and minimal elements of the sets.

To the best of our knowledge, there have been two other papers that have provided axiomatic characterizations of UEU criteria for ranking sets of objects. The first of them is Fishburn (1972) who characterizes the UEU family of rankings of all non-empty subsets of a finite universe without

Archimedean structure. Yet the characterization provided by Fishburn is a direct adaptation of the *additivity axiom* of Scott (1964), Adams (1965) and Fishburn (1970). The unappealing nature of this axiom is well known and is especially striking when adapted to the problem of comparing sets on the basis of their average utility. This axiom involves the construction of arbitrarily long sequences of set comparisons which are both difficult to motivate as primitive axioms as well as hard to verify in practice. By contrast, the structure of our model enables us to characterize the UEU family of rankings of sets by means of axioms that are, in our opinion, considerably easier to interpret and verify. We note that one of our axioms, "averaging", is identified by Fishburn (1972) as being satisfied by any ranking in the UEU family . We show in this paper that, along with another axiom - "restricted independence" - averaging actually characterize the UEU family of rankings of sets if an Archimedean structure is assumed. The other paper that contains a characterization of a UEU criterion (but not of the whole family of such criteria) for ranking finite sets is the unpublished piece of Baigent and Xu (2004). In this paper, the authors characterize, again without an Archimedean structure, a ranking of finite sets based on the average *Borda score* of their elements. This ranking is clearly a member of the UEU family for which the utility of a consequence is defined by its Borda score. It is, here again, interesting to notice that Baigent and Xu (2004) uses, along with other axioms, the averaging axiom in their characterization.

While the main interpretation given to our results is framed in terms of choice under ignorance, it is clear that this interpretation is not necessary. What we provide in this paper is an axiomatic characterization of a family of rankings of all finite subsets of some universe of objects that have the property that each of these rankings *can* be interpreted *as if* it was assigning utility to every object in the universe and *as if* it was comparing sets on the basis of the (symmetric) average utility of these objects. There are at least two other contexts where such an axiomatic characterization could be useful.

The first is *mechanism design*, where several papers have used UEU criteria to model preferences of individual agents over subsets of some fundamental set of alternatives, with the subsets interpreted as possible results of a *social choice correspondence*. For example Barberà, Dutta, and Sen (2001) have characterized *strategy-proof* social choice correspondences when agents preferences are assumed to belong to the UEU family. Benoît (2002) studies a similar problem and recently, Ozyürt and Sanver (2006) have refined and extended this analysis. UEU criteria have also been considered by Peleg and Peters (2005) in their analysis of Nash consistent representation of effectivity functions

The second is the literature on *objective ambiguity* alluded to above, in which decision makers rank sets of lotteries. These sets of lotteries are interpreted as describing "ambiguous" decisions like those arising in the well-known Ellsberg paradox where the decision maker is uncertain about the probability distribution associated with a particular uncertain decision. In this setting, Ahn (2008) characterizes a family of criteria that contains the UEU one. Any criterion in the family characterized by Ahn (2008) can be thought of as resulting from the comparison of the expected utility of their consequences conditional on the fact of being in the set, but with expectation taken with respect to a probability measure that needs not be uniform. Ahn (2008) analysis bears many formal similarities with the somewhat non-standard Bolker-Jeffrey approach to decision making under uncertainty (see e.g. Bolker (1966), Bolker (1967), Jeffrey (1965) or, for a good discussion of this approach, Broome (1990)). While conceptually different, the frameworks used by Ahn (2008) and Bolker-Jeffrey consider sets that, except for singletons (considered in Ahn but excluded in Bolker-Jeffrey), contain a continuum of elements. These frameworks differ therefore substantially from ours in which attention is limited to finite sets of objects, which can be interpreted as lotteries with finite supports. Another important paper in the literature on objective ambiguity is that of Olszewski (2007) who considers a framework in which sets of lotteries can be of any size: finite, countably infinite or uncountably infinite but where lotteries are restricted to have a finite and given support. He characterizes in that framework the family of ranking of sets that can be expressed as a weighted average of the utility of their best and their worst element. This family can be viewed as an important subclass of the family of rankings characterized in Nehring and Puppe (1996).

The plan of the rest of this paper is as follows. In the next section, we present the formal framework and the definition of the axioms and the family of rankings. The third section presents the main results and the fourth section shows how the results can be obtained if topological properties are imposed on the universe. The fifth section comments on the independence of the axioms and the sixth section concludes.

2 Notation and basic concepts

2.1 Notation

The sets of integers, non-negative integers, real numbers and non-negative real numbers are denoted respectively by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$ and $\mathbb{R}_+$. The cardinality of any set A is denoted by $\#A$ and the k-fold Cartesian product of a set A with itself is denoted by A^k. If v is a vector in $\mathbb{R}^k$ for some strictly positive integer k and α is a real number, we denote by $\alpha.v$ the scalar product of α and v. Given a vector v in $\mathbb{R}^k$ and a positive real number ε, we denote by $N_\varepsilon(v)$ an *ε-neighborhood around* v defined by $N_\varepsilon(v) = \{x \in \mathbb{R}^k :\mid x_h - v_h \mid < \varepsilon$ for all $h = 1, ..., k\}$. Our notation for vectors inequalities is $\geqq$, $\geq$ and $>$. By a *binary relation* $\succsim$ on a set Ω, we mean a subset of $\Omega \times \Omega$. Following the convention in economics, we write $x \succsim y$ instead of $(x, y) \in R$. Given a binary relation $\succsim$, we define its *symmetric factor* $\sim$ by $x \sim y \Longleftrightarrow x \succsim y$ and $y \succsim x$ and its *asymmetric* factor $\succ$ by $x \succ y \Longleftrightarrow x \succsim y$ and not $(y \succsim x)$. A binary relation $\succsim$ on Ω is *reflexive* if the statement $x \succsim x$ holds for every x in Ω, is *transitive* if $x \succsim z$ always follows $x \succsim y$ and $y \succsim z$ for any x, y, $z \in \Omega$ and is *complete* if $x \succsim y$ or $y \succsim x$ holds for every distinct x and y in Ω. A reflexive, transitive and complete binary relation is called an *ordering*.

2.2 Basic concepts

Let X be the set of consequences. While we do not make any specific assumptions on X, it will be clear subsequently that the axioms that we impose makes it natural to regard this set as infinite. As an example, further considered in section 4 below, one could think of X as being $\mathbb{R}$, interpreted as the set of all conceivable financial returns (either negative or positive) of some investment decision in a highly uncertain environment. As another example, one could think of X as the set of all conceivable probability distributions on a basic set of k different prices.

We denote by $\mathcal{P}(X)$ the set of all non-empty *finite* subsets of X (with generic elements A, B, C, D, etc.). Any such a subset is interpreted as a description of all consequences of an *uncertain* decision or, for short, as a *decision*. A *certain* decision with consequence $x \in X$ is identified by the singleton $\{x\}$.

Let $\succsim$ (with asymmetric and symmetric factors $\succ$ and $\sim$ respectively) be an *ordering* on $\mathcal{P}(X)$. We interpret the statement $A \succsim B$ as meaning "decision with consequences in A is weakly preferred to decision with consequences in B". A similar interpretation is given to the statements $A \succ B$ ("strictly preferred to") and $A \sim B$ ("indifference").

We want to identify the properties (axioms) of the ordering $\succsim$ that are necessary and sufficient for the existence of a function $u : X \to \mathbb{R}$ such that, for every A and B in $\mathcal{P}(X)$:

$$A \succsim B \Longleftrightarrow \sum_{a \in A} \frac{u(a)}{\#A} \geq \sum_{b \in B} \frac{u(b)}{\#B} \tag{1}$$

An ordering satisfying this property *could* therefore be thought of as resulting from the comparisons of the *expected utility* of the consequences of the decision for some utility function under the assumption that the decision maker assigns to every consequence of a decision an *equal probability* of occurrence. There are obviously many criteria like that, as many as there are logically conceivable utility functions (up to an affine transform) defined on X. We refer to any ranking that satisfies (1) for some function u as to a *Uniform Expected Utility* (UEU) criterion.

We now introduce the two main axioms of our analysis which, as can be easily seen, are satisfied by any UEU criterion. The first of these axioms, that we call averaging, is stated as follows.

Axiom 1 *(averaging) For all disjoint sets A and $B \in \mathcal{P}(X)$, $A \succsim B \Leftrightarrow A \succsim A \cup B \Leftrightarrow A \cup B \succsim B$.*

In words, this axiom asserts that enlarging a set A with a (disjoint) set B is worth doing (viz. not worth doing) if and only if the set B of added consequence is better (viz worse) than the set A to which it is added. We call this axiom "averaging" because it captures an intuitive property satisfied by calculations of "average" in various settings (e.g. adding a student to a class will increase the average of the class if and only if the grade of the added student is larger than the average of the class). The "only if" part of the axiom is obviously very strong since it asserts that the *only* reason for ranking a set B above (below) a set A is when the addition of B to A is considered a good (bad) thing. The averaging axiom is a compact version of the four averaging conditions AC1-AC4 discussed in Fishburn (1972) and shown by him to be implied by the UEU family of criteria (as well as by a variant of the additivity axiom of Scott (1964), Adams (1965) and Fishburn (1970)). The averaging axiom has been used also by Baigent and Xu (2004) in their characterization of the average Borda ranking of sets. This axiom is also used by Ahn (2008) and Bolker (1966) (see also Bolker (1967)) in their characterization of an important family of criteria, containing UEU ones, for ranking *atomless* subsets of a universe. A weaker version of averaging (that only requires the "if" part in its statement) is used by Olszewski (2007) in his characterization of a ranking of sets based on the weighted average of the utility of their best and their worst alternative, and by Gul and Pesendorfer (2001) in their ranking of sets of objects, interpreted as menus of alternatives, in a way that reflects temptation and self-control. It is also worth mentioning that the averaging axiom *implies* the Gärdenfors (1976) principle discussed at length in the literature on ignorance as surveyed in . Barberà, Bossert, and Pattanaik (2004). This Gärdenfors principle can be stated formally as follows.

Condition 1 *(Gärdenfors Principle) for all $A \in \mathcal{P}(X)$, ($x \in X \backslash A$ and $\{x\} \succ \{a\}$ for all $a \in A$) $\Rightarrow A \cup \{x\} \succ A$ and ($y \in X \backslash A$ and $\{a\} \succ \{y\}$ for all $a \in A$) $\Rightarrow A \succ A \cup \{y\}$.*

This principle says that is always (never) worth adding to a set a consequence which, if certain, is better (worse) than all consequences in the set. For further reference, we record in the following proposition, whose proof is left the reader, the fact that the averaging axiom implies the Gärdenfors principle when it is applied to a transitive ranking of $\mathcal{P}(X)$.

Proposition 1 *Let $\succsim$ be a transitive binary relation on $\mathcal{P}(X)$ that satisfies averaging. Then $\succsim$ satisfies the Gärdenfors principle*

The second axiom that enters into the characterization of the family of UEU rankings is the following *restricted independence* axiom.

Axiom 2 *(restricted independence) For all A, B and $C \in \mathcal{P}(X)$ satisfying $\#A = \#B$ and $A \cap C = B \cap C = \emptyset$, $A \succsim B \Leftrightarrow A \cup C \succsim B \cup C$.*

This axiom requires that the ranking of sets with the same number of elements be independent of any elements that they have in common. Adding or subtracting these common elements from the two sets should not affect their ranking. A weak form of the restricted independence condition, applied only to the case where A and B are singletons, plays an important role in Nehring and Puppe (1996) and Puppe (1995). It is worth noticing that the scope of this *independence* axiom is indeed significantly *restricted* by the fact that it applies only to sets that have the same number of elements.

We now formally state that these two axioms are satisfied by any UEU criterion. The straightforward proof of this proposition is left to the reader.

Proposition 2 *Any UEU criterion satisfies averaging and restricted independence.*

As shall be seen the axioms of averaging and restricted independence actually characterize the family of UEU rankings of sets if some structure is imposed on the environment. We provide this structure by imposing *two* other axioms on the pair $\langle X, \succsim \rangle$. These axioms, which we shall refer to as *structural,* impose smoothness and richness on both the set X of alternatives and on the ordering $\succsim$. Yet these axioms are not specifically tailored to UEU criteria, and one of them may even be violated by these criteria if the set X of alternatives is too "sparse". Theorem 4 below establishes that these structural axioms can be dispensed with if X is taken to be an arc-connected subset of $\mathbb{R}^k$, provided that a mild continuity condition is imposed on $\succsim$.

The first of these structural axioms is the following *richness* one.

Axiom 3 *(richness) For every set $B \in \mathcal{P}(X) \subset X$, and every finite, but possibly empty, subset A of X, if there are consequences c^* and c_* in X such that $A \cup \{c^*\} \succsim B \succsim A \cup \{c_*\}$, then there exists a consequence $c \in X$ such that $A \cup \{c\} \sim B$.*

As its name suggests, this axiom reflects the idea that the universe is sufficiently rich to enable, by the addition of *single* consequences to sets, various kinds of comparisons with the ordering $\succsim$. Suppose that, starting with two decisions A and B, it is possible to add consequences c^* and c_* to

A in such a way that A enlarged with c^* is ranked above B and A enlarged with c_* is ranked below B. Then it must also be possible to add to A a consequence c such that the resulting set of consequences is indifferent to A. In a sense, this axiom is weak since the asserted existence of the consequence c is contingent upon the existence of consequences c^* and c_* that have the required properties. Yet, the axiom applies also if the set A to which the consequences c^*, c_* and c are added is empty. Because of this, the richness axiom has the somewhat strong implication, at least when combined with the Gardenförs principle, that every uncertain decision has a "certainty equivalent". Put differently if a decision maker ranks uncertain decisions by an ordering that satisfies averaging and richness, then for any uncertain decision, there must exist a certain decision that the decision maker considers equivalent to it. For further reference, we state formally this "certainty equivalence" condition and the fact that it is implied by richness if the ranking satisfies averaging as follows.

Condition 2 *(certainty equivalence) For every $B \in \mathcal{P}(X)$, there exists a consequence $b \in X$ such that $\{b\} \sim B$.*

Proposition 3 *Let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying averaging and richness. Then $\succsim$ satisfies the certainty equivalence condition.*

Proof. *Let B be any set in $\mathcal{P}(X)$. Because B is non-empty and $\succsim$ is an ordering on $\mathcal{P}(X)$, there exists a consequence $c^* \in B$ such that $\{c^*\} \succsim \{b\}$ for all $b \in B$ and there exists a consequence c_* (not necessarily distinct from c^*) such that $\{b\} \succsim \{c_*\}$ for all $b \in B$. By averaging one has $\{c^*\} \succsim B \succsim \{c_*\}$ which can be written equivalently as $\varnothing \cup \{c^*\} \succsim B \succsim \varnothing \cup \{c_*\}$. By richness (for $A = \varnothing$), there exists c such that $\{c\} \cup \varnothing \sim B$, which proves certainty equivalence.* ■

It is also worth mentioning that the combination of the richness and averaging axioms implies either that the ranking $\succsim$ be trivial or that there are infinitely many consequences in X. Specifically, if X is finite, then a decision maker who compares decisions in $\mathcal{P}(X)$ with an ordering satisfying averaging and richness (and therefore certainty equivalence thanks to proposition 3) must be indifferent between all such decisions. We state this formally as follows..

Proposition 4 *Suppose $\#X < \infty$ and let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying averaging. Then $\succsim$ satisfies certainty equivalence if and only if $A \sim B$ for all $A, B \in \mathcal{P}(X)$.*

Proof. *It is clear that the trivial ordering defined by $A \sim B$ for all A, $B \in \mathcal{P}(X)$ satisfies certainty equivalence (as well as averaging). To prove the* *reverse implication, write the finite set X as $X = \{x_1, ..., x_{\#X}\}$ and assume without loss of generality (since $\succsim$ is an ordering) that $\{x_i\} \succsim \{x_{i+1}\}$ for $i = 1, ..., \#X - 1$. By averaging, we must have, for every $i = 1, ..., x_{\#X}$:*

$$\{x_i\} \succsim \{x_i, x_{i+1}\} \succsim \{x_{i+1}\}$$

Certainty equivalence implies therefore that, for every i, either $\{x_i\} \sim \{x_i, x_{i+1}\}$ or $\{x_{i+1}\} \sim \{x_i, x_{i+1}\}$. In either case, averaging implies that $\{x_i\} \sim \{x_i, x_{i+1}\} \sim \{x_{i+1}\}$. Hence all pairs and singletons must be indifferent. Repeated application of averaging (adding first indifferent singletons to pairs and then indifferent singletons to triples etc.) then immediately leads to the conclusion of universal indifference. ■

We note in passing that the trivial ordering that considers all sets to be indifferent is a member of the UEU family (any constant function u having X as domain could serve as a representation as per (1)). Hence, in the rest of the paper, we shall be interested in characterizing the UEU family of orderings of $\mathcal{P}(X)$ in the **non trivial** case where there are at least two sets A and B such $A \succ B$.

Beside forcing X to be infinite (at least when combined with averaging), the richness axiom precludes from consideration some "discontinuous" rankings such as the "Leximin" or the "Leximax" rules studied in Pattanaik and Peleg (1984). For instance, the Leximin rule compares sets on the basis of their worst consequences. If a tie in the worst consequence arises, then the second worst consequence is considered and so on until either a strict ranking is obtained or the consequences in at least one of the sets are exhausted. In the latter case the set which contains the largest number of elements is ranked above. It is clear that such a Leximin rule violates richness. Indeed if we take $X = \mathbb{R}_+$ one has that $\{1, 3\} \prec \{2\} \prec \{2, 3\}$ for the Leximin criterion. Yet, contrary to what is required by richness, it is impossible to find a non-negative real number x such that $\{x\} \cup \{3\} \sim \{2\}$.

It should be also noted that the richness axiom is not specifically related to the UEU family of ranking of decisions and may even be violated by a UEU criterion if the set X is not sufficiently rich. if $X = \mathbb{N}$ and the u function of (1) is the identity function, we notice that, since $\frac{2+6+7}{3} = 5 < \frac{5+6}{2} = 5.5 < \frac{2+6+10}{3} = 6$, we have $\{2, 6, 7\} \prec \{5, 6\} \prec \{2, 6, 10\}$. Yet, contrary to what would be required by richness, there does not exist any integer c such that $\frac{2+6+c}{3} = \frac{5+6}{2}$ and, therefore, such that $\{2, 6, c\} \sim \{5, 6\}$.

The other structural axiom, called Archimedean by Krantz, Luce, Suppes, and Tversky (1971) is the following.

Axiom 4 *(Archimedean) If a sequence $\{c_i\}$, for $i = 1, 2, ...$ of consequences $c_i \in X$ is such that one has either $\{c_i, a\} \sim \{c_{i+1}, b\}$ for all i, $i + 1$ with $i = 1, 2, ...$ or $\{c_{i+1}, a\} \sim \{c_i, b\}$ for all i, $i + 1$ with $i = 1, 2, ...$ for some consequences a and b, distinct from any element of the sequence, and satisfying*

$\{a\} \succ \{b\}$, then, if the sequence is strictly bounded by x and $y \in X$ in the sense that $\{x\} \succ \{c_i\} \succ \{y\}$ for every i, the sequence must be finite.

This axiom can be considered to be mild since it "bites" only when there exist sequences of the type described by the antecedent clause of this axiom (such sequences are called "standard sequences" in the measurement theory literature). It is trivially satisfied if X is finite since all sequences of sets must be finite. Yet, contrary to richness, the Archimedean axiom is always verified by a UEU criterion, no matter what is the universe X. We complete this section by stating this formally in the following proposition.

Proposition 5 *If a ranking $\succsim$ on $\mathcal{P}(X)$ is a UEU criterion, then it satisfies the Archimedean axiom.*

Proof. *Let the ranking $\succsim$ on $\mathcal{P}(X)$ be a UEU criterion and consider a sequence of consequences $\{c_i\}$ for $i = 1, 2, ...$ such that, for some consequences a and b distinct from every element in the sequence satisfying $\{a\} \succ \{b\}$, one has, say, $\{c_i, a\} \sim \{c_{i+1}, b\}$ for all $i = 1, 2, ...$(the argument is similar if $\{c_{i+1}, a\} \sim \{c_i, b\}$ for all $i = 1, 2, ...$). Since $\succsim$ is a UEU criterion, there exists a function $u : X \to \mathbb{R}$ such that $u(a) > u(b)$ and $u(c_i)+u(a) = u(c_{i+1})+u(b) \Leftrightarrow u(a)-u(b) = u(c_{i+1})-u(c_i)$ for all i. Assume that the sequence is strictly bounded by x and $y \in X$ so that $\{x\} \succ \{c_i\} \succ \{y\}$ for all i. Since $\succsim$ is a UEU criterion, one has $u(x) > u(c_i) > u(y)$. Define the numbers u_i by $u_i = u(c_i)$ and d by $d = u(a)-u(b) > 0$. We therefore have a sequence of numbers u_i (for $i = 1, ..$) such that $u_i = (i-1)d + u_1 > u(y)$ for every i for some strictly positive real number d. Clearly one can only have $u(x) > (i-1)d + u_1$ for all element i of the increasing sequence of numbers $\{(i-1)d + u_1\}$ if this sequence is finite.* ■

3 Main results

In order to prove the main result that the family of UEU rankings of all finite subsets of X is characterized, given certainty equivalence, richness and the Archimedean axiom, by averaging and restricted independence, we proceed as follows. We first consider the sets $m(X)$ and $M(X)$ of *minimal* and *maximal* (respectively) elements of X with respect to the restriction of $\succsim$ to singletons defined by:

$$m(X) = \{x \in X : \{x\} \precsim \{y\} \ \forall\ y \in X\} \text{ and}$$
$$M(X) = \{z \in X : \{z\} \succsim \{y\} \ \forall\ y \in X\}$$

The possibility that either (or both) of these two sets be empty is of course not ruled out. Let X' be defined by $X' = X \backslash (m(X) \cup M(X))$. Hence, X' is the set of all conceivable consequences that remain after one has removed the worst and the best certain consequences (if any) from X. It is easy to see that, if $\succsim$ is an ordering satisfying averaging and certainty equivalent, then the set X' is "unbounded" in the sense that, for any consequence $x \in X'$, one can find two consequences w and z in X' such that $\{w\} \succ \{x\} \succ \{z\}$. For later reference, we state formally this fact as follows.

Proposition 6 *If $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying richness and averaging, then, for all consequence $x \in X'$, there are consequences w and z in X' such that $\{w\} \prec \{x\} \prec \{z\}$.*

Proof. *Consider any $x \in X'$. Such a x exists if $\succsim$ is non-trivial. Let us first find a consequence w such that $\{w\} \prec \{x\}$. Suppose first that $m(X) = \varnothing$. This means that $x \notin m(X)$ and there exists some $t \in X$ such that $\{t\} \prec \{x\}$. Because, thanks to proposition 3, the ordering $\succsim$ satisfies certainty equivalence, there exists a consequence w such that $\{w\} \sim \{t, x\}$. By averaging and transitivity, $\{x\} \succ \{w\} \succ \{t\}$. Hence $w \notin m(X) \cup M(x)$ so that $w \in X'$. Suppose now that $m(X) \neq \varnothing$ and let $t \in m(X)$. By definition of $m(X)$, we have $\{t\} \prec \{x\}$ so that, by certainty equivalence again, there exists a consequence w such that $\{w\} \sim \{t, x\}$. As before, we can conclude from the averaging axiom that $\{t\} \prec \{w\} \prec \{x\}$ so that $w \in X \backslash (m(X) \cup M(X))$, as required. The argument for finding a consequence $z \in X'$ such that $\{z\} \succ \{x\}$ is similar.* $\blacksquare$

We proceed by first proving the result on $\mathcal{P}(X')$ defined as the set of all finite subsets of X'. Once we have obtained that any ordering on $\mathcal{P}(X')$ satisfying averaging and restricted independence as well as the structural axioms can be represented as per (1) for some function $u : X' \to \mathbb{R}$, we then show that this numerical representation can be "extended" to the whole set X.

The proof that any ordering on $\mathcal{P}(X')$ satisfying averaging, restricted independence and the structural axioms can be represented as per (1) for some function $u : X' \to \mathbb{R}$ proceeds itself in two steps.

First, we show that averaging and restricted independence characterize the family of UEU criteria in an environment where the structural axioms are satisfied if one restricts attention to subsets of X' that have **at most** two consequences. The proof of this first theorem, provided in the appendix, rides heavily on the important theorem 2 of Krantz, Luce, Suppes, and Tversky (1971) (p. 257) that enables an additively separable the numerical representation of an ordering over a Cartesian product $X \times X$ (see also Debreu (1960) or Adams and Fagot (1959) for an earlier statement framed in a topological setting). An important ingredient of the proof of this theorem is the demonstration that, given our axiomatic structure, the ranking of $X \times X$ induced by $\succsim$ satisfies the so-called "Thomsen condition" (see Krantz, Luce, Suppes, and Tversky (1971), definition 3, p. 250). This demonstration is made in lemma 3 of the appendix.

The statement of the theorem that establishes the validity of the representation for all subsets of X' containing at most two consequences is as follows.

Theorem 1 *Let X be a set of consequences and let $\succsim$ be an ordering on $\mathcal{P}(X')$ satisfying richness and the Archimedean axiom. Then if $\succsim$ satisfies averaging and restricted independence, its restriction to the sets in $\mathcal{P}(X')$ of cardinality no greater than 2 can be represented as per 1 for some utility function $u : X \to \mathbb{R}$. Furthermore, the utility function u is unique up to a positive affine transformation.*

Our main result extends Theorem 1 to subsets of X' with an arbitrary (but finite) number of consequences using the same axioms. Specifically, we prove that the unique utility function whose expectation (under uniform probabilities) represents the ranking of sets containing no more than two elements exhibited in Theorem 1 also represents the ranking of sets of larger cardinality. While the full proof of this extension is done in the appendix using various auxiliary results, a key step in the argument, provided by the following lemma 1 proved also in the appendix, is the ability to approximate the arithmetic mean of a set of n numbers *recursively* from the arithmetic means of *pairs* of those numbers.

Lemma 1 *Let $U = \{u_1, ..., u_n\}$ be a set of n numbers such that $u_1 \leq u_2 \leq ... \leq u_n$ with arithmetic mean $\overline{u}$. Define the $n-1$ sequences $\{b_h^i\}$, $i = 1, 2, ...$ and $h = 1, ..., n-1$ by:*

$$b_{n-1}^0 = (u_n + u_{n-1})/2,$$

$$b_h^0 = (u_h + b_{h+1})/2$$

for $h = 1, ..., n-2$ and for $i = 1, 2, ...$:

$$\begin{aligned}
b_1^{2i-1} &= b_1^{2i-2}, \\
b_h^{2i-1} &= \frac{b_{h-1}^{2i-1} + b_h^{2i-2}}{2} \text{ for } h = 2, ..., n-1, \\
b_{n-1}^{2i} &= b_{n-1}^{2i-1} \text{ and} \\
b_h^{2i} &= \frac{b_h^{2i-1} + b_{h+1}^{2i}}{2} \text{ for } h = 1, ..., n-2.
\end{aligned}$$

Then:

$$\lim_{i\to\infty} b_h^i = \overline{u} \text{ for all } h = 1, ..., n-1$$

Endowed with this lemma and the other auxiliary results stated and proved in the appendix, we prove - also in the appendix - the following theorem.

Theorem 2 *Let $\succsim$ be an ordering on $\mathcal{P}(X')$ satisfying richness and the Archimedean axiom. Then $\succsim$ satisfies averaging and restricted independence if and only if it is a UEU criterion. Furthermore, the u function in the definition of a UEU criterion is unique up to a positive affine transformation.*

The last step in the proof consists in showing that the numerical representation of $\succsim$ restricted to $\mathcal{P}(X')$ can be extended to the whole set $\mathcal{P}(X)$. This step is provided by the proof, in the appendix, of the following theorem.

Theorem 3 *Let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying richness and the Archimedean axiom. Then $\succsim$ satisfies averaging and restricted independence if and only if it is a UEU criterion. Furthermore, the u function in the definition of a UEU criterion is unique up to a positive affine transformation.*

4 Interpretation of the structural environment

We show in this section that if one imposes a natural structure on the set X and the ordering $\succsim$ from the outset, then the richness and the Archimedean axioms can be replaced by a mild continuity condition in our characterization of the UEU family of orderings.

Assume specifically that X is a closed and arc-connected[1] subset of $\mathbb{R}^k$ for some integer $k \geq 1$. At least two interpretations could be given to X in that context. First, X could be thought of as the set of all bundles of k goods that could result from any uncertain decision (taking $k = 1$ would obviously cover the case, discussed earlier, of decisions with pecuniary consequences). In that case, it would be natural to take $X = \mathbb{R}^k$ (all bundles of goods or amounts of money - possibly negative - are *a priori* conceivable).

The second interpretation, developed along the line of a recent literature on objective ambiguity (see e.g. Ahn (2008) or Olszewski (2007)) would be to view X as the set of all *lotteries* yielding k different prices. A typical element $p \in X$ would then be though of as a probability vector assigning to every price i its probability of realization $p_i \in [0, 1]$. A finite set $A \subset X$ of such probability vectors would then be interpreted as an ambiguous decision in which the precise probability distribution over the set of k prices is not known with precision to the decision maker. A classical instance of decision making under this kind of ambiguity is provided by the so-called Ellsberg paradox in which a decision maker does not know how a certain number of balls of two different colors are split between the two colors (see Olszewski

[1] A subset A of a topological space is arc-connected if, for any two elements x and y of A, there exists a continuous function f from $[0, 1]$ to A such that $f(0) = x$ and $f(1) = y$.

(2007) for further discussion). If this interpretation is favoured, then $X = S^{k-1} = \{p \in [0,1]^k : \sum_{j=1}^{k} p_j = 1\}$.

For either of these interpretations, we shall assume that the ordering $\succsim$ satisfies the following *continuity* axiom.

Axiom 5 *For every set $A \in \mathcal{P}(X)$, and consequences y and $z \in X$, the sets $B(A) = \{x \in X : \{x\} \succsim A\}$ and $W(A) = \{x \in X : A \succeq \{x\}\}$ are both closed in X.*

This axiom says that a small change in a consequence should not have drastic effect on the ranking of this consequence obtained for sure *vis-à-vis* any set. Notice that this continuity axiom, which only concerns comparisons of sets vis-à-vis singletons is much weaker than the (Vietoris) continuity condition examined in Nehring and Puppe (1996) which restricts the comparisons of any two sets in a way that is not even compatible with the UEU family of set rankings.

We now establish, in the following theorem, that in this environment, the UEU family of rankings of $\mathcal{P}(X)$ is characterized by averaging and restricted independence. In order to prove this theorem, we only need to prove that, if X is an arc-connected subset of $\mathbb{R}^k$, then an ordering of $\mathcal{P}(X)$ that satisfies the continuity axiom as well as averaging and restricted independence satisfies the richness and the Archimedean axiom. As for the other theorems, its proof is relegated to the appendix.

Theorem 4 *Let X be an arc-connected subset of $\mathbb{R}^k$ and let $\succsim$ be an ordering of $\mathcal{P}(X)$ that satisfies the continuity condition. Then $\succsim$ satisfies averaging, and restricted independence if and only if it is a UEU criterion.*

Interpreted in the spirit of decision making under objective ambiguity, theorem 4 could serve as an alternative standpoint from the criteria characterized by Olszewski (2007). In the later paper, the author characterizes the family of rankings of finite sets of lotteries based on the comparison of a weighted average of the utility of the best and of the worst lottery. Olszewski's framework shares an important similarity with ours in that he assumes, as we do, that lotteries are defined on a given exogenous set of k prices. On the other hand, Olszewski does not make any assumption on the cardinality of the sets of lotteries that are compared while our approach limits attention to finite sets. As compared to the family of criteria characterized by Olszewski, the UEU family has the merit of enabling other lotteries than the worst and the best to play a role in the ranking of ambiguous decisions. It suffers however from the limitation that it forces the probabilities attributed subjectively to the various lotteries by the decision maker to be the same.

A generalization of the UEU family of criteria has been characterized, in a very different conceptual setting, by Ahn (2008), building on Bolker-Jeffrey theory. The generalization of the UEU family characterized by Ahn (2008) contains all orderings $\succsim$ of $P(X)$ that can be defined, for every sets A and B in $P(X)$, by:

$$A \succsim B \Longleftrightarrow \frac{\sum_{a \in A} p(a)u(a)}{\sum_{a \in A} p(a)} \geq \frac{\sum_{b \in B} p(b)u(b)}{\sum_{b \in B} p(b)} \tag{2}$$

for some real-valued functions u and p both having X as domain. Any UEU criterion is a member of this family that satisfies the additional property that, for all consequences $x \in X$, $p(x) = c$ for some real number c. Orderings that can be represented as per (2) for some real-valued functions u and p can be thought of as comparing sets on the basis of the expected utility of their consequence, but without imposing the requirement on the probability of all consequences to be the same. This interpretation obviously requires that we can interpret $p(x)$ as a probability, which in turns requires that some measure-theoretic structure be imposed on X. But if we can provide this interpretation, any ordering of $P(X)$ that can be represented as per (2) can be viewed as comparing sets on the basis of their expected utility *conditional of being in the sets*. It can be checked easily that any ordering that can be represented as per (2) satisfies averaging but may violate restricted independence.

Ahn (2008) (and in a somewhat different paradigm Bolker (1966), Bolker (1967)), has characterized the family of orderings of $P(X)$ that can be represented as per (2) in a setting in which sets (decisions) are atomless and contain, therefore, uncountably many elements. Moreover the elements of the sets are lotteries with possibly uncountable supports while our approach, as that of Olszewski, restricts attention to lotteries with finite support. These differences make difficult the comparisons of Ahn formal analysis with ours. In addition to using regularity and continuity conditions that can not be defined in our framework, Ahn and Bolker have characterized the family of rankings represented as per (2) by means of the averaging axiom and a weakening of the restricted independence axiom, called "balancedness" by Ahn.

As a family of criteria for decision making under objective ambiguity, the UEU family is rather abstract in the sense that it does not impose any specific structure on the functions u that appears in the representation as per (1). Yet, if the elements of the sets are interpreted as lotteries, it could make sense to impose some further properties on the utility function (for instance, that it be linear in probabilities). While we do not do this herein, it is clear that it could be done easily by means of additional axioms imposed on the ranking of singletons, as done in Ahn (2008) and Olszewski (2007).

5 Independence of the axioms

As it turns out, averaging, restricted independence and the Archimedean axioms are logically independent when they are imposed on an ordering that satisfies richness in the environment in which it is used. Since any UEU criterion satisfies averaging, restricted independence and the Archimedean axiom, it can therefore be said that these three axioms provides a minimal characterization of the UEU family of orderings on any environment on which these orderings satisfied richness. As the reader can appreciate through these examples, they can also be used to show that, if X is taken to be an arc-connected closed subset of $\mathbb{R}^k$, then they show also that the axioms of averaging, restricted independence and continuity are logically independent.

Example 1 *Let $X = \mathbb{R}$ and, for all A, $B \in \mathcal{P}(X)$, $A \succsim B \iff \sum_{a \in A} a \geq \sum_{b \in B} b$. The reader can check easily that this ranking satisfies restricted independence, the Archimedean axiom, and richness but violates averaging. Indeed, $\{3\} \succsim \{1, 2\}$ but $\{3\} \prec \{1, 2, 3\}$. The reader can also verify that $\succsim$ satisfies continuity.*

Example 2 *Let $X = \mathbb{R}_{++}$ and define $\succsim$ by:*

$$A \succsim B \Leftrightarrow \frac{\displaystyle\sum_{a \in A} a}{\displaystyle\sum_{a \in A} \frac{1}{a}} \geq \frac{\displaystyle\sum_{b \in B} b}{\displaystyle\sum_{b \in B} \frac{1}{b}}. \tag{3}$$

This ordering is clearly a member of the family represented as per (2) where p is defined by $p(x) = \frac{1}{x}$ and u by $u(x) = x^2$. It satisfies for this reason the averaging axiom. It is a continuous ordering of X because the set $B(A) = \{x \in X : x^2 \geq \frac{\sum_{a \in A} a}{\sum_{a \in A} \frac{1}{a}}\}$ and $W(A) = \{x \in X : \frac{\sum_{a \in A} a}{\sum_{a \in A} \frac{1}{a}} \geq x^2\}$ are closed for any set $A \in \mathcal{P}(X)$. Moreover it can be seen that it satisfies the Archimedean axiom using an argument that parallels that of proposition 5 Yet $\succsim$ violates restricted independence because if we take

$$A = \{1, 7\},\ B = \{2, 3\} \text{ and } C = \{4, 12\}$$

we have $A \succsim B$ since, using (3):

$$\frac{1 + 7}{1 + \frac{1}{7}} = 7 \geq \frac{2 + 3}{\frac{1}{2} + \frac{1}{3}} = 6$$

but, contrary to what is required by restricted independence, one has $A \cup C \prec B \cup C$ *since:*

$$\frac{1+4+7+12}{1+\frac{1}{4}+\frac{1}{7}+\frac{1}{12}} = \frac{24\times 84}{84+21+12+7} = \frac{6\times 84}{31} < \frac{2+3+4+12}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{12}} = 6\times 3$$

Example 3 *Let* $X = \mathbb{R}\times\mathbb{R}$ *and let, for any element* $a \in X$, a_i *denote the* i*-th component of* a*, for* $i \in \{1,2\}$*. Define the ordering* $\succsim$ *by:*
$A \sim B$ *iff* $\frac{1}{\#A}\sum\limits_{a\in A} a_1 = \frac{1}{\#B}\sum\limits_{b\in B} b_1$ *and* $\frac{1}{\#A}\sum\limits_{a\in A} a_2 = \frac{1}{\#B}\sum\limits_{b\in B} b_2$*;*
$A \succ B$ *iff either:*
(i) $\frac{1}{\#A}\sum\limits_{a\in A} a_1 > \frac{1}{\#B}\sum\limits_{b\in B} b_1$ *or:*
(ii) $\frac{1}{\#A}\sum\limits_{a\in A} a_1 = \sum\limits_{b\in B} b_1$ *and* $\frac{1}{\#A}\sum\limits_{a\in A} a_2 > \frac{1}{\#B}\sum\limits_{b\in B} b_2$*.*
We first prove that $\succsim$ *violates the Archimedean axiom. Indeed, the set* $\{(2,i) : i \in \mathbb{Z}\}$ *is a standard sequence because* $\{(2,i),(1,2)\} \sim \{(2,i+1),(1,1)\}$ *for all* $i \in \mathbb{Z}$*. This standard sequence is infinite but is bounded. Indeed, for any* $i \in N$*, we have* $\{(3,1)\} \succ \{(2,i)\} \succ \{(1,1)\}$*. We leave to the reader the (easy) task of verifying that this ordering, which is a lexicographic combination of two UEU orderings defined on each dimension, is not continuous either. Let us show that* $\succsim$ *satisfies averaging. Indeed,* $A \succ B \iff$ *either:*
$\frac{1}{\#A}\sum\limits_{a\in A} a_1 > \frac{1}{\#B}\sum\limits_{b\in B} b_1 \Leftrightarrow \frac{1}{\#A}\sum\limits_{a\in A} a_1 > \frac{1}{\#A+\#B}\sum\limits_{\alpha\in A\cup B} \alpha_1$
$\Leftrightarrow A \succ A\cup B$ *or*
$\frac{1}{\#A}\sum\limits_{a\in A} a_1 = \frac{1}{\#B}\sum\limits_{b\in B} b_1$ *and* $\frac{1}{\#A}\sum\limits_{a\in A} a_2 > \frac{1}{\#B}\sum\limits_{b\in B} b_2$
$\iff \frac{1}{\#A}\sum\limits_{a\in A} a_1 = \frac{1}{\#A+\#B}\sum\limits_{\alpha\in A\cup B} \alpha_1$ *and* $\frac{1}{\#A}\sum\limits_{a\in A} a_2 > \frac{1}{\#A+\#B}\sum\limits_{\alpha\in A\cup B} \alpha_2$
$\iff A \succ A\cup B$*. A similar reasoning holds when* $A \sim B$*. To show that* $\succsim$ *satisfies richness on* $X = \mathbb{R}\times\mathbb{R}$*, consider any finite subsets* A *and* B *of* X *(with* A *possibly empty) and define* c *by means of the following two equations:*

$$\frac{c_1 + \sum\limits_{a\in A} a_1}{1+\#A} = \frac{1}{\#B}\sum_{b\in B} b_1 \quad \textit{and} \quad \frac{c_2 + \sum\limits_{a\in A} a_2}{1+\#A} = \frac{1}{\#B}\sum_{b\in B} b_2.$$

We then have $A \cup \{c\} \sim B$*. We notice that this conclusion holds no matter what is assumed on the ranking of* A *vis-à-vis* B*. Hence this conclusion can also be obtained for sets* B *and* A *that satisfy the requirements of the richness axiom. Finally, to show that* $\succsim$ *satisfies restricted independence, consider finite and non-empty subsets* A *and* B *of* X *such that* $\#A = \#B$ *and* $A\cap C = \emptyset = B\cap C$*. We have* $A \succ B$ *if and only if either:*

$\frac{1}{\#A}\sum\limits_{a\in A} a_1 > \frac{1}{\#B}\sum\limits_{b\in B} b_1 \Leftrightarrow \frac{1}{\#A+\#C}\sum\limits_{\alpha\in A\cup C}\alpha_1 > \frac{1}{\#B+\#C}\sum\limits_{\beta\in B\cup C}\beta_1$
$\Leftrightarrow A\cup C \succ B\cup C$ *or*
$\frac{1}{\#A}\sum\limits_{a\in A} a_1 = \frac{1}{\#B}\sum\limits_{b\in B}$ *and* $\frac{1}{\#A}\sum\limits_{a\in A} a_2 > \frac{1}{\#B}\sum\limits_{b\in B} b_2$
$\Leftrightarrow \frac{1}{\#A+\#C}\sum\limits_{\alpha\in A\cup C}\alpha_1 = \frac{1}{\#B+\#C}\sum\limits_{\beta\in B\cup C}\beta_1$ *and* $\frac{1}{\#A+\#C}\sum\limits_{\alpha\in A\cup C}\alpha_2 > \frac{1}{\#B+\#C}\sum\limits_{\beta\in B\cup C}\beta_2$
$\Longleftrightarrow A\cup C \succ B\cup C$. *A similar reasoning holds when* $A\sim B$.

6 Conclusion

This paper characterizes by two axioms the UEU ranking of completely uncertain decisions, under the assumption that the ranking of uncertain decision is used in an Archimedean and rich environment. The axioms used in the characterization are finite and, therefore, verifiable from the mere observation of a choice behavior. We have also shown that UEU ranking can be used to rank ambiguous decisions or decisions with financial consequences and is characterized in that setting under the same axioms, but with the Archimedean and richness axioms replaced by a mild continuity one.

A limitation of UEU criteria is that they assign to every consequence of a decision the same probability of occurrence. A next step in the research agenda is therefore to identify the properties of a more general EU criterion that does not impose this uniform assumption on the probabilities assigned to the consequences of a decision. The family of orderings that can be represented as per (2) for some functions p and u is an obvious first step into that direction. We have seen that any ordering in this family satisfies averaging and continuity (or the Archimedean axiom) but may violate restricted independence. It would be nice to know the axioms which, along with averaging and continuity, characterize this large family of rankings of completely uncertain decisions. While Ahn (2008) has characterized this family in the somewhat specific context where decisions have continuously many consequences that are taken to be probabilities measures over a fundamental set of consequences, we think that obtaining a characterization of this family in a finite context is a high priority for future research.

7 Appendix

Proof of theorem 1

Before proving theorems 1 and 2 on the subdomain $P(X')$, we must be sure that all our axioms - formulated for the domain $P(X)$- are also valid for the subdomain $P(X')$. While this is clear for averaging, restricted independence, certainty equivalence and the Archimedean axiom, it may not be so clear for richness which, given any two sets in $P(X)$- and therefore in the subdomain $P(X')$- asserts the existence, in X, of a specific consequence c having some property. Yet we must prove that this consequence c can actually be shown to belong to X'. Specifically, we must prove that the following lemma is true.

Lemma 2 *Let $\succsim$ be an ordering on $P(X)$ satisfying averaging, restricted independence, richness and the Archimedean axiom. Then the restriction of $\succsim$ to $P(X')$ satisfies the same axioms.*

Proof.

We leave to the reader the task of verifying that this is indeed the case for averaging, restricted independence and the Archimedean axiom. For richness, let A and B be two finite subsets of X' (with A possibly empty) and assume that there are consequences c^* and $c_* \in X'$ such that $A \cup \{c^*\} \succsim B \succsim A \cup \{c_*\}$. By richness (applied to $P(X)$), there exists a consequence $c \in X$ such that $A \cup \{c\} \sim B$. We need to show that $c \in X'$. By contradiction, assume $c \in X \backslash X' = m(X) \cup M(X)$. If $c \in m(X)$, then $\{c\} \prec \{x\}$ for all $x \in X'$ so that, in particular, $\{c\} \prec \{c_*\}$ and $\{c\} \prec \{a\}$ for all $a \in A$. Hence $c \notin A$. One has therefore, by restricted independence (if $c_* \notin A$) or by averaging (if $c_* \in A$), that $B \succsim A \cup \{c_*\} \succ A \cup \{c\}$, a contradiction (if $\succsim$ is transitive). The argument is symmetric if $c \in M(X)$. QED

An important ingredient in the proof of theorem 1 is the following lemma, which states that if an ordering $\succsim$ on $P(X')$ satisfies averaging, restricted independence and the structural axioms, then it satisfies, when restricted to pairs and singletons, the following important condition that is closely related to the so-called "Thomsen condition" in the theory of conjoint measurement (using Krantz, Luce, Suppes, and Tversky (1971) terminology).

Lemma 3 *Let $\succsim$ be an ordering on $P(X')$ satisfying averaging, restricted independence, richness and the Archimedean axiom. Then for every (not necessarily distinct) consequences a, b, c, d, e and $f \in X'$, $\{a\} \cup \{f\} \sim \{c\} \cup \{e\}$ and $\{c\} \cup \{d\} \sim \{b\} \cup \{f\}$ must imply $\{a\} \cup \{d\} \sim \{b\} \cup \{e\}$.*

Proof.

We consider several cases.

1. $\{a\} \sim \{b\}$ **and** $\{d\} \sim \{e\}$**.** In this case, we conclude that $\{a\} \cup \{d\} \sim \{b\} \cup \{d\}$ by restricted independence if both $a \neq p$ and $b \neq p$. If either $a = d$ and $b \neq d$ or $a \neq d$ and $b = d$, the conclusion $\{a\} \cup \{d\} \sim \{b\} \cup \{d\}$ follows from averaging. Finally, if $a = d$ and $b = d$, the conclusion that $\{a\} \cup \{d\} \sim \{b\} \cup \{d\}$ follows trivially from the assumption that $\{a\} \sim \{b\}$. By an analogous reasoning we can obtain the conclusion that $\{b\} \cup \{d\} \sim \{b\} \cup \{e\}$. The conclusion that $\{a\} \cup \{d\} \sim \{b\} \cup \{e\}$ follows then at once from transitivity.

2. $\{a\} \succ \{b\}$ **and** $\{d\} \succsim \{e\}$**.** In this case, it follows from restricted independence (if $a \neq f \neq b$) or averaging (if $a = f$ or $b = f$) that $\{a\} \cup \{f\} \succ \{b\} \cup \{f\}$. Analogously, we can conclude from the premises of this case that $\{c\} \cup \{d\} \succsim \{c\} \cup \{e\}$ (using restricted independence if $d \neq c \neq e$ and averaging if $d = c$ or $e = c$). It then follows from transitivity that $\{c\} \cup \{d\} \succsim \{c\} \cup \{e\} \sim \{a\} \cup \{f\} \succ \{b\} \cup \{f\} \sim \{c\} \cup \{d\}$. Since this is a contradiction, we conclude that this case is impossible.

3. $\{a\} \prec \{b\}$ **and** $\{d\} \precsim \{e\}$**.** This case can also shown to be impossible, following a similar reasoning as for case 2.

4. $\{a\} \prec \{b\}$ **and** $\{d\} \succsim \{e\}$**.** We then consider several subcases.

(i) $c = f$. Since $\{a\} \cup \{f\} \sim \{c\} \cup \{e\}$, we conclude that $\{a\} \sim \{e\}$ using restricted independence (if $a \neq f$ and $c \neq e$), averaging (if $a = f$ and $c \neq e$ or $a \neq f$ and $c = e$) or trivially (if $a = f$ and $c = e$). By an analogous reasoning, we conclude from $\{c\} \cup \{d\} \sim \{b\} \cup \{f\}$ that $\{b\} \sim \{d\}$. Hence, we have $\{a\} \sim \{e\}$ and $\{b\} \sim \{d\}$. This implies that $\{a\} \cup \{d\} \sim \{b\} \cup \{e\}$ by restricted independence (if $a \neq d$ and $b \neq e$) or by averaging (in all other cases).

(ii) Suppose $f \neq c$, $e \neq c \neq d$, $d \neq a \neq f$and $e \neq b \neq f$. We first establish that there are consequences u and $v \in X'$such that $\{u\} \cup \{v\} \cap \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\} = \emptyset$ and $\{a, u\} \sim \{c, v\}$. Suppose first $\{a\} \sim \{c\}$. Take then any $u \in X' \backslash (\{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\})$ (the existence of such a u is secured by proposition 6) and define $v = u$. We then immediately obtain $\{a, u\} \sim \{f, v\}$ by restricted independence. Suppose now that $\{a\} \prec \{c\}$. By proposition 6, there is a consequence $v \in X'$ such that $\{v\} \prec \{a\}$. By restricted independence, one has $\{a, v\} \prec \{c, v\}$ and $\{c, v\} \prec \{c, a\}$ and, by transitivity, $\{a, v\} \prec \{c, v\} \prec \{a, c\}$. It follows from richness that there is a consequence $u \in X'$ such that $\{a\} \cup \{u\} \sim \{c, v\}$. If $\{u, v\} \cap \{a, b, f, p, q, x\} \neq \emptyset$, one can repeat this procedure, starting with another v. The repetition of the procedure will be finite because the set $\{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ is finite. Hence one is sure to find a consequence $u \in X' \backslash \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ such that $\{a, u\} \sim \{c, v\}$. Using an analogous argument, one can obtain a similar conclusion if $\{a\} \succ \{c\}$ is assumed. Now, by restricted independence, one has $\{c, f, v\} \sim \{a, f, u\}$ and $\{a, f, u\} \sim \{c, e, u\}$. It follows from transitivity that $\{c, f, v\} \sim \{c, e, u\}$ so that $\{f, v\} \sim \{e, u\}$ must hold by restricted independence. Using restricted independence again, we obtain from $\{a, u\} \sim \{c, v\}$ that $\{a, d, u\} \sim \{c, d, v\}$ and from $\{c, d\} \sim \{b, f\}$ that $\{c, d, v\} \sim \{b, f, v\} \sim \{b, e, u\}$. It follows from transitivity that $\{a, d, u\} \sim \{b, e, u\}$ and, by restricted independence, $\{a, d\} \sim \{b, e\}$.

(iii) Suppose $c \neq f$, $e \neq c \neq d$, and $e \neq b \neq f$. The only difference with subcase (ii) is that we relax the constraint '$d \neq a \neq f$'. Hence this case is more general than (ii). Suppose, contrary to the asserted implication of the lemma, that $\{a\} \cup \{d\} \not\sim \{b\} \cup \{d\}$. Since $\succsim$ is complete, two symmetric cases can arise: $\{a\} \cup \{d\} \prec \{b\} \cup \{e\}$ or $\{a\} \cup \{d\} \succ \{b\} \cup \{e\}$. We only handle the first one. We first show that we can find distinct consequences a' and $f' \in X' \backslash \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ such that $\{a'\} \prec \{a\}$, $\{f\} \prec \{f'\}$ and $\{a', f'\} \sim \{a\} \cup \{f\}$. The existence of consequences $a' \in X'$ such that $\{a'\} \prec \{a\}$ is secured by proposition 6. Assume first $a = f$. Either $\{a', g\} \prec \{a\}$ for all $g \in X'$ or there exists some g' such that $\{a', g'\} \succsim \{a\}$. In the second case, the existence of a consequence f' such that $\{a', f'\} \sim \{a\}$ follows from richness. In the first case, choose by proposition 6 some $\widehat{g} \in X'$ such that $\{\widehat{g}\} \succ \{a\}$ and, by certainty equivalence, some $\widetilde{g} \in X'$ such that $\{\widetilde{g}\} \sim \{a, \widehat{g}\}$. By averaging and transitivity, one has $\{a\} \prec \{\widetilde{g}\} \sim \{a, \widehat{g}\} \prec \{\widetilde{g}, \widehat{g}\} \prec \{\widehat{g}\}$. Hence one has $\{\widetilde{g}, \widehat{g}\} \succ \{a\} \succ \{a', \widetilde{g}\}$ so that, by richness, there exists a consequence $\widetilde{a} \in X'$ such that $\{\widetilde{a}, \widetilde{g}\} \sim \{a\}$. Choosing then $a' = \widetilde{a}$ and $f' = \widetilde{g}$ gives the result. If $a \neq f$, we can do the previous reasoning for the certainty equivalent of $\{a, f\}$ which exists by certainty equivalence. If either a' or $f' \in \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$, we can redo the procedure while starting with another a'. Since the set $\{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ is finite, we will redo the procedure at most a finite number of times. Hence we are sure to find consequences a' and $f' \in X' \backslash \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ such that $\{a'\} \prec \{a\}$, $\{f\} \prec \{f'\}$ and $\{a', f'\} \sim \{a\} \cup \{f\}$ and, by redoing the above procedure as many times as required, we can choose as many distinct pairs of such $a'$and f'as we want. Choose now a consequence b'such that $\{b'\} \cup \{f'\} \sim \{b, f\}$. This is possible because $\{b, f'\} \succ \{b, f\}$ thanks to restricted independence. Moreover it is impossible that $\{\widetilde{b}\} \cup \{f'\} \succ \{b, f\}$ for all consequences $\widetilde{b}$ in X'. Indeed, since $\{a', f'\} \sim \{a\} \cup \{f\}$, assuming $\{\widetilde{b}\} \cup \{f'\} \succ \{b, f\}$ for all $\widetilde{b}$ would imply, given transitivity, that $\{a\} \cup \{f\} \succ \{b, f\}$. Yet using averaging (if $a = f$) or restricted independence (if $a \neq f$), this would imply in turn that $\{a\} \succ \{b\}$, contradicting our assumption that $\{a\} \prec \{b\}$. Hence, there are consequences $\widetilde{b}$ such that $\{b, f\} \succsim \{\widetilde{b}, f'\}$ so that, by richness, one can find a consequence b' such that $\{b', f'\} \sim \{b, f\}$. Given the flexibility we have for choosing a' and f' it is clear that b' can be chosen so that it does not belong to $\{a\} \cup \{a'\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\} \cup \{f'\}$. Thanks to case (*ii*), we know that we can obtain $\{a', d\} \sim \{b', e\}$ if we replace a by a', f by $f'$and b by b' in the antecedent clause of the lemma. Since $\{a'\} \prec \{a\}$, we know that $\{a', d\} \prec \{a\} \cup \{d\}$ by restricted independence (if $a \neq d$) or by averaging (if $a = d$). Hence, it follows from transitivity that $\{b', e\} \sim \{a', d\} \prec \{a\} \cup \{d\} \prec \{b\} \cup \{e\}$. Combine now as before the axioms of averaging, certainty equivalent and richness to find a consequence b'' such that $\{a\} \cup \{d\} \prec \{b''\} \cup \{e\} \prec \{b\} \cup \{e\}$. Using richness, one can also find a consequence f'' such that $\{c\} \cup \{d\} \sim \{b''\} \cup \{f''\}$ and a consequence a'' such that $\{a''\} \cup \{f''\} \sim \{f\} \cup \{e\}$. As before, we have the flexibility to find these consequences b'', f'' or a'' in such a way that they not belong to $\{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$. Thanks to case ($ii$) again, we can obtain the conclusion that $\{a'', d\} \sim \{b'', e\}$. Since $\{a''\} \prec \{a\}$, we have $\{a''\} \cup \{d\} \prec \{a\} \cup \{d\}$ by restricted independence (if $a \neq d$) or by averaging (if $a = d$). We then obtain from transitivity that $\{a\} \cup \{d\} \prec \{b'', e\} \sim \{a'', d\} \prec \{a\} \cup \{d\}$. A contradiction.

(iv) Suppose $c \neq f$ and $e \neq b \neq f$. The difference with case (iii) is that we relax the constraint '$e \neq c \neq d$'. Hence, as before, this case is more general than case (iii). Suppose by contradiction that the lemma is false and that $\{a\} \cup \{d\} \not\sim \{b\} \cup \{e\}$. As before, the completeness of $\succsim$ implies either $\{a\} \cup \{d\} \prec \{b\} \cup \{e\}$ or $\{a\} \cup \{d\} \succ \{b\} \cup \{e\}$. Since these two cases are symmetric, we only provide the argument for the first one. Using analogous argument than in case (iii), one can find consequences c' and $d' \in X' \backslash \{a\} \cup \{b\} \cup \{c\} \cup \{d\} \cup \{e\} \cup \{f\}$ such that $\{d'\} \prec \{d\}$, $\{c\} \prec \{c'\}$ and $\{c', d'\} \sim \{c\} \cup \{d\}$. As in case ($iii$) also, one can find a consequence e' $\in X' \backslash \{a\} \cup \{b\} \cup \{c, c'\} \cup \{d, d'\} \cup \{e\} \cup \{f\}$ such that $\{c', e'\} \sim \{c\} \cup \{e\}$. Thanks to case ($iii$), we know that we can obtain $\{a\} \cup \{d'\} \sim \{b\} \cup \{e'\}$ out of the assumption that $\{a\} \cup \{f\} \sim \{c'\} \cup \{e'\}$ and $\{c'\} \cup \{d'\} \sim \{b\} \cup \{f\}$. Since $\{d'\} \prec \{d\}$, we have $\{a, d'\} \prec \{a\} \cup \{d\}$ by restricted independence (if $a \neq d$) or averaging (if $a = d$). Hence by transitivity we have: $\{b\} \cup \{e'\} \sim \{a\} \cup \{d'\} \prec \{a\} \cup \{d\} \prec \{b\} \cup \{e\}$. Using analogous argument as for the consequences a'', b'' and f'' of case (iii), find now consequences c'', d'' and $e'' \in X' \backslash \{a\} \cup \{b\} \cup \{c, c'\} \cup \{d, d'\} \cup \{e, e'\} \cup \{f\}$ such that 1) $\{a\} \cup \{d\} \prec \{b, e''\} \prec \{b\} \cup \{e\}$, 2) $\{c'', e''\} \sim \{a\} \cup \{f\}$ and 3) $\{c'', d''\} \sim \{b\} \cup \{f\}$. We know from (iii) that $\{a\} \cup \{f\} \sim \{c, e''\}$ and $\{c, d''\} \sim \{b\} \cup \{f\}$ implies $\{a\} \cup \{d''\} \sim \{b\} \cup \{e''\}$. Since $\{d''\} \prec \{d\}$, we have $\{a\} \cup \{d''\} \prec \{a\} \cup \{d\}$ by restricted independence (if $a \neq d$) or by averaging (if $a = d$). Transitivity then yields $\{a\} \cup \{d\} \prec \{b, e''\} \sim \{a, d''\} \prec \{a\} \cup \{d\}$, a contradiction.

(v) Suppose $c \neq f$. The difference with case (iv) is that we relax the constraint '$e \neq b \neq f$. Hence this case is more general than case (iv) and we handle it in an analogous fashion (conditional on (iv)) to what was done for case (iv) conditional to case (iii).

We notice that subcases (i) and (v) are exhaustive, conditional on case 4.

5. $\{a\} \succ \{b\}$ **and** $\{p\} \precsim \{q\}$**.** This case is handled in the same way as case 4

Another result used in the proof of theorem 1 is the following.

Lemma 4 *Let $\succsim$ be an ordering on $P(X')$ satisfying averaging and restricted independence. Then for every (not necessarily distinct) consequences a, b, c, and $d \in X'$, $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\} \Leftrightarrow \{a\} \cup \{d\} \succsim \{c\} \cup \{d\}$.*

Proof.
We consider several cases.

1) $a \neq b$**,** $c \neq b$**,** $a \neq d$ **and** $c \neq d$**.** The result then follows immediately from restricted independence.

2) $a = b$**,** $c \neq b$**,** $a \neq d$ **and** $c \neq d$**.** Assume $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\}$ or, equivalently under our assumption, that $\{a\} \succsim \{a, c\}$. By averaging this statement is equivalent to $\{a\} \succsim \{c\}$ which is itself equivalent, using restricted independence, to $\{a, d\} = \{a\} \cup \{d\} \succsim \{c, d\} = \{c\} \cup \{d\}$.

3) $a \neq b$**,** $c = b$**,** $a \neq d$ **and** $c \neq d$**.** Assume $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\}$ or, equivalently under our assumption, that $\{a, b\} \succsim \{b\}$. By averaging, this statement is equivalent to $\{a\} \succsim \{b\}$ and, by restricted independence, to $\{a, d\} = \{a\} \cup \{d\} \succsim \{b, d\} = \{c\} \cup \{d\}$.

4) $a \neq b$**,** $c \neq b$**,** $a = d$ **and** $c \neq d$**.** Assume $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\}$ or, equivalently under our assumption, that $\{a, b\} \succsim \{c, b\}$. Using restricted independence, this is equivalent to $\{a\} = \{d\} \succsim \{c\}$ which, by averaging, is equivalent to $\{d\} = \{a\} \cup \{d\} \succsim \{d, c\} = \{c\} \cup \{d\}$.

5) $a \neq b$**,** $c \neq b$**,** $a \neq d$ **and** $c = d$**.** Assume $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\}$ or, equivalently under our assumption, that $\{a, b\} \succsim \{c, b\} = \{d, b\}$. Using restricted independence, this is equivalent to $\{a\} \succsim \{d\}$ which, by averaging, is equivalent to $\{a, d\} = \{a\} \cup \{d\} \succsim \{d\} = \{c\} \cup \{d\}$.

6) $a = b = c \neq d$**.** In that case reflexivity ensures that $\{a\} \cup \{b\} \succsim \{c\} \cup \{b\} \Leftrightarrow \{a\} \succsim \{a\} \Leftrightarrow \{a\} \cup \{d\} = \{a, d\} \succsim \{a, d\} = \{c, d\} = \{c\} \cup \{d\}$.

All other cases are handled trivially using reflexivity.

Proof of theorem 1.

Proposition 1 establishes that any UEU criterion satisfies averaging and restricted independence. To prove the converse implication, consider the restriction of the ordering $\succsim$ to the set of all subsets of X'containing at most two consequences. Define the binary relation $\widehat{\succsim}$ on $X' \times X'$ by $(a, b) \mathrel{\widehat{\succsim}} (c, d) \Leftrightarrow \{a\} \cup \{b\} \succsim \{c\} \cup \{d\}$. The binary relation $\widehat{\succsim}$ is well-defined and is clearly an ordering of $X' \times X'$ if $\succsim$ is an ordering of $P(X')$. We also notice that, thanks to lemma 4, $\widehat{\succsim}$ satisfies the property that if $(a, b) \mathrel{\widehat{\succsim}} (c, b)$ holds for some consequence a, b and c, then $(a, d) \mathrel{\widehat{\succsim}} (c, d)$ holds for all consequences $d \in X'$. This property is called "independence" by Krantz, Luce, Suppes, and Tversky (1971) (KLST for short) (p. 249, definition 1). We similarly obtain that $\widehat{\succsim}$ satisfies both Thomsen's condition (see KLST, p. 250, definition 3) and the "restricted solvability" condition (KLST, p. 250, definition 5) using, respectively lemma 3, and the richness axiom. Finally, we note that our Archimedean axiom implies the property of the same name of KLST (p. 253, definition 4) while our assumption, made in the text, that $\succsim$ is non trivial implies, thanks to averaging (and specifically the Gardenförs condition), that each component of $X' \times X'$is essential as per KLST definition 6 (p. 256). Hence the triple

$(X', X', \widehat{\succsim})$ is an additive conjoint structure in the sense of KLST (definition 7, p. 256). By virtue of theorem 2 of KLST (p. 257), there are real-valued functions Φ_i (for $i = 1, 2$) having both X' as domain such that:

$$(a, b) \widehat{\succsim} (c, d) \Leftrightarrow \Phi_1(a) + \Phi_2(b) \geq \Phi_1(c) + \Phi_2(d)$$

for all consequences a, b, c and $d \in X'$. Since $(a, b) \widehat{\succsim} (c, d) \Leftrightarrow \{a\} \cup \{b\} \succsim \{c\} \cup \{d\} \Leftrightarrow \{b\} \cup \{a\} \succsim \{d\} \cup \{d\} \Leftrightarrow (b, a) \widehat{\succsim} (d, c)$, the ordering $\widehat{\succsim}$ is symmetric so that $\Phi_1(x) = \Phi_2(x) = u(x)$ must hold for every consequence $x \in X'$ for some function $u : X' \to \mathbb{R}$. By virtue of the second part of theorem 2 in KLST, the function u is unique up to an affine transform. Let us now show that, for all subsets A and B of X' containing at most two consequences, one has $A \succsim B \Leftrightarrow \sum\limits_{a \in A} \frac{u(a)}{\#A} \geq \sum\limits_{a \in A} \frac{u(a)}{\#A}$ so that $\succsim$ can be represented as per (1). If $\#A = \#B = 1$, then one has, for all consequences $x$and $y \in X'$, $\{x\} \succsim \{y\} \Leftrightarrow \{x\} \cup \{x\} \succsim \{y\} \cup \{y\} \Leftrightarrow 2u(x) \geq 2u(y) \Leftrightarrow u(x) \geq u(y)$ so that the numerical representation holds for that case. The argument clearly works just as well if $\#A = \#B = 2$. Suppose now that $\#A = 1$ and $\#B = 2$. Then, for all consequences x, y and $z \in X'$such that $y \neq z$, one has:

$$\begin{aligned}
\{x\} \succsim \{y, z\} &\Longleftrightarrow \{x\} \cup \{x\} \succsim \{y\} \cup \{z\} \\
&\Longleftrightarrow (x, x) \widehat{\succsim} (y, z) \\
&\Longleftrightarrow u(x) + u(x) \geq u(y) + u(z) \Longleftrightarrow u(x) \geq \frac{u(y) + u(z)}{2}
\end{aligned}$$

so that the numerical representation holds for that case as well. QED

Proof of lemma 1.

We find useful to represent the sequence defined in this lemma in the following array, with $n - 1$ columns and an infinite number of rows:

	1	2	...	$n-2$	$n-1$
$\mathbf{b}^0$	$(u_1 + b_2^0)/2$	$(u_2 + b_3^0)/2$	...$\leftarrow$	$(u_{n-2} + b_{n-1}^0)/2$	$(u_{n-1} + u_n)/2$
$\mathbf{b}^1$	$(u_1 + b_2^0)/2$	$(b_1^1 + b_2^0)/2$	$\rightarrow$...	$(b_{n-3}^1 + b_{n-2}^0)/2$	$(b_{n-2}^1 + b_{n-1}^0)/2$
$\mathbf{b}^2$	$(b_2^2 + b_1^1)/2$	$(b_3^2 + b_2^1)/2$	...$\leftarrow$	$(b_{n-1}^2 + b_{n-2}^1)/2$	$(b_{n-2}^1 + b_{n-1}^0)/2$
$\mathbf{b}^3$	$(b_2^2 + b_1^1)/2$	$(b_1^3 + b_2^2)/2$	$\rightarrow$...	$(b_{n-3}^3 + b_{n-2}^2)/2$	$(b_{n-2}^3 + b_{n-1}^2)/2$
...	...	...	...	...	...

We are going to show that the "grand" sequence that starts from the "north-east" of the array and follows the arrows up to infinity, converges to $\overline{u}$. Since the sequence $\{b_h^i\}$ is the h^{th}column of this array and therefore, a subsequence of the grand sequence, the conclusion of the lemma would follow immediately. Define accordingly the grand sequence $\{\widehat{b}^t\}$, for $t = i(n-1)+1, ..., (i+1)(n-1)$, and $i = 0, 1, 2, ...$by:

$$\widehat{b}^t = b_{n-(t+1-(i(n-1)+1))}^i \text{ if } i \text{ is even and}$$

$$\widehat{b}^t = b_{t+1-(i(n-1)+1)}^i \text{ if } i \text{ is odd}$$

Any element of the grand sequence can be written as a weighted average of $\{u_1, ..., u_n\}$. In particular, for all $t = 1, ...,$ there exists $n-1$ real numbers $\beta_1^t, ..., \beta_{n-1}^t$ such that:

$$\widehat{b}^t = \beta_1^t u_1 + \beta_2^t u_2 + ... + \beta_{n-1}^t u_{n-1} + \beta_{n-1}^t u_n$$

Moreover inspection reveals that β_h^t is defined by the following recursive formula:

$$\begin{aligned} \beta_h^t &= 0 \text{ if } t \in \{1, ..., n-h-1\} \\ \beta_h^{n-h} &= \frac{1}{2} \text{ and} \\ \beta_h^t &= \frac{1}{2}(\beta_h^{t-1} + \beta_h^{2m(t)-t+1} \text{if } t \geq n-h+1 \end{aligned} \tag{4}$$

where $m(t)$ is defined as the largest integer strictly smaller than t that is divisible by $n-1$. In order to prove the lemma, it suffices to prove that $\lim\limits_{t\to\infty} \beta_h^t = \frac{1}{n}$for all h. In what follows we will fix $h \in \{1, .., n-1\}$ and drop the subscript h from the sequence $\{\beta_h^t\}$ for notational convenience.

Once again, it is convenient to refer to the aforementioned representation of the sequence $\{\beta^t\}$, $t = 1, ..$as an array with $n-1$ columns and an infinite number of rows. We start from the first row with β^1 and move left until we reach β^{n-1}. We then move down to the second row where the first element from the left is β^n. The sequence then increases from left right and the right-most element in the this row is β^{2n-2}. The right-most element in the third row is then β^{2n-1} and the sequence increases as it moves left (like in the first row) so that the left-most element is β^{3n-3} and so on. Let t be an arbitrary integer. If we write $t = m(t) + s$, it follows that β^t lies in the $(m(t) + 1)^{th}$ row of this array. If $m(t)$ is even then, the $(m(t) + 1)^{th}$ row is increasing from right to left so that β^t is the $(s + 1)^{th}$ element from the right in this row. If $m(t)$ is odd, then β^t is the $(s+1)^{th}$ element from the left in the $(m(t) + 1)^{th}$ row which increases from left to right. It follows that in this array, β^t for $t > n-1$ is the arithmetic mean of the element which immediately precedes it and the element directly in the row above.

The proof proceeds in two steps. The first is to show that the sequence $\{\beta^t\}$, $t = 1, ...$is convergent and the second is to show that the limit of the sequence is, in fact $\frac{1}{n}$. In order to establish the first step, we first record the two following properties $P1$ and $P2$ of the sequence which can be easily verified.

$P1$. Let $r > 1$ be an odd integer. The sequence strictly increases from $\beta^{(r-1)(n-1)+1}$ to $\beta^{(r-1)(n-1)+h}$ and then strictly decreases from $\beta^{(r-1)(n-1)+h}$ to $\beta^{r(n-1)}$. If r is an even integer, then the sequence strictly increases from $\beta^{(r-1)(n-1)+1}$ to $\beta^{(r-1)(n-1)+n-h}$ and strictly decreases from $\beta^{(r-1)(n-1)+n-h}$ to $\beta^{r(n-1)}$. Thus for every row r in the array, the sequence increases from the right as we move left for h terms and then decreases for the remaining $n-h-1$ terms. Clearly $\beta^{(r-1)(n-1)+h}$ is the largest element of the r^{th} row if r is odd and $\beta^{(r-1)(n-1)+n-h}$

if r is even. Note that the maximal element of any row is in the h^{th}column from the right.

$P2$. Let $t = (n-1)r + s$ where $m(t) = r$ (note that $1 \leq s \leq n-1$). Then $\beta^t = \frac{1}{2}\beta^{(n-1)(r-1)+(n-s)} + \frac{1}{2^2}\beta^{(n-1)(r-1)+(n-s-1)} + ... + \frac{1}{2^{s-1}}\beta^{(n-1)(r-1)+1} + \frac{1}{2^{s-1}}\beta^{(n-1)(r-1)}$. Thus each term of the sequence can be expressed as the weighted sum of the terms of the sequence in the row above.

CLAIM: Let $r > 1$ be an integer. Then:
(i) $\beta^{(r-1)(n-1)+h} - \beta^{r(n-1)} < \gamma_1(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-2)(n-1)+1})$ and
(ii) $\beta^{(r-1)(n-1)+h} - \beta^{(r-1)(n-1)+1} < \gamma_2(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)})$
where $\gamma_1 = \frac{2^{n-h-1}-1}{2^{n-h-1}}$ and $\gamma_2 = \frac{2^{h-1}-1}{2^{h-1}}$ if r is odd and:
(iii) $\beta^{(r-1)(n-1)+n-h} - \beta^{(r-1)(n-1)+1} < \gamma_1(\beta^{(r-2)(n-1)+h} - \beta^{(r-1)(n-1)})$ and,
(iv) $\beta^{(r-1)(n-1)+n-h} - \beta^{r(n-1)} < \gamma_2(\beta^{(r-2)(n-1)+h} - \beta^{(r-2)(n-1)+1})$
if r is even.

Proof of the Claim: We first prove (ii). We do that by first noting that, according to $P2$:

$$\beta^{(r-1)(n-1)+h} = \frac{1}{2}\beta^{(r-2)(n-1)+n-h} + ... + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)+1} + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)}$$

Since $\beta^{(r-1)(n-1)+1} = \beta^{(r-1)(n-1)}$ and $\beta^{(r-2)(n-1)+n-h}$ is the largest term in the $(r-1)^{th}$ row according to $P1$, we conclude that :

$$\begin{aligned}\beta^{(r-1)(n-1)+h} - \beta^{(r-1)(n-1)+1} &< \beta^{(r-2)(n-1)+h}(\frac{1}{2} + ... + \frac{1}{2^{h-1}}) \\ &\quad -(1 - \frac{1}{2^{h-1}})\beta^{(r-1)(n-1)} \\ &= (1 - \frac{1}{2^{h-1}})(\beta^{(r-2)(n-1)+n+h} - \beta^{(r-1)(n-1)})\end{aligned}$$

Since $\gamma_2 = (1 - \frac{1}{2^{h-1}})$, this establishes (ii).

We now prove (iii). According to $P2$:

$$\beta^{(r-1)(n-1)+n-h} = \frac{1}{2}\beta^{(r-2)(n-1)+h} + ... + \frac{1}{2^{n-h-1}}[\beta^{(r-1)(n-1)+1} + \beta^{(r-1)(n-1)}]$$

Since $\beta^{(r-1)(n-1)+1} = \beta^{(r-1)(n-1)}$ and since, from $P1$, we know that $\beta^{(r-2)(n-1)+h}$ is the largest term in the $(r-1)^{th}$ row, we obtain:

$$\begin{aligned}\beta^{(r-1)(n-1)+n-h} - \beta^{(r-1)(n-1)+1} &< (\frac{1}{2} + ... + \frac{1}{2^{n-h-1}})\beta^{(r-2)(n-1)+n-h} \\ &\quad - (1 - \frac{1}{2^{n-h-1}})\beta^{(r-1)(n-1)+1} \\ &= (1 - \frac{1}{2^{n-h-1}})(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)+1})\end{aligned}$$

Since $\gamma_2 = (1 - \frac{1}{2^{n-h-1}})$, this establishes (iii).

We now prove (i). Applying $P2$, we have:

$$\begin{aligned}\beta^{(r-1)(n-1)+h} &= \frac{1}{2}\beta^{(r-2)(n-1)+n-h} + ... + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)+1} \\ &+ \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)}\end{aligned}$$

and:

$$\begin{aligned}\beta^{r(n-1)} &= \frac{1}{2}\beta^{(r-2)(n-1)+1} + ... + \frac{1}{2^{n-h}}\beta^{(r-1)(n-1)+n-h} \\ &+ .... + \frac{1}{2^{n-2}}\beta^{(r-1)(n-1)+1} + \frac{1}{2^{n-2}}\beta^{(r-1)(n-1)}\end{aligned}$$

We thus have:

$$\Delta = \beta^{(r-1)(n-1)+h} - \beta^{r(n-1)}$$

$$\begin{aligned}&= (\frac{1}{2} - \frac{1}{2^{n-h}})\beta^{(r-1)(n-1)+n-h} + ... + (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}})\beta^{(r-1)(n-1)+1} \\ &+ (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}})\beta^{(r-1)(n-1)} \\ &- \frac{1}{2}\beta^{(r-1)(n-1)+1} ..... - \frac{1}{2^{n-h-1}}\beta^{(r-1)(n-1)+n-h-1}\end{aligned}$$

Note that, according to $P1$, $\beta^{(r-1)(n-1)+n-h}$ is the largest element in its row. This, combined to the fact that:

$$\beta^{(r-1)(n-1)+1} < ... < \beta^{(r-2)(n-1)+n-h-1}$$

implies:

$$\begin{aligned}\Delta &< ((\frac{1}{2} - \frac{1}{2^{n-h}}) + ... + (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}}) \\ &+ (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}}))\beta^{(r-1)(n-1)+n-h} \\ &- (\frac{1}{2} + ... + \frac{1}{2^{n-h-1}})\beta^{(r-2)(n-1)+1} \\ &= (\frac{1}{2} - \frac{1}{2^{n-h}})(1 + ... + \frac{1}{2^{h-2}}) + (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}}))\beta^{(r-2)(n-1)+n-h} \\ &+ (1 - \frac{1}{2^{n-h-1}})\beta^{(r-1)(n-1)+1} \\ &= (\frac{1}{2} - \frac{1}{2^{n-h}})(2 - \frac{1}{2^{n-h}}) + (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}}))\beta^{(r-2)(n-1)+1} \\ &+ (1 - \frac{1}{2^{n-h-1}})\beta^{(r-1)(n-1)+1} \\ &= (1 - \frac{1}{2^{n-h-1}})(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)+1}) \\ &= \gamma_1(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)+1})\end{aligned}$$

which proves (i).

The proof of (iv) is symmetric to that of (i) and we omit the details.

We will use the inequalities in the Claim to put an upper bound on the distance between terms in the same row of the array. Let $r > 1$ be an odd integer. Applying (i) in the Claim, we have:

$$\beta^{(r-1)(n-1)+h} - \beta^{r(n-1)} < \gamma_1(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-2)(n-1)+1})$$

Observe that $\beta^{(r-2)(n-1)+n-h} - \beta^{(r-2)(n-1)+1})$ can be written as $\beta^{(r'-1)(n-1)+n-h} - \beta^{(r'-1)(n-1)+1})$ where $r' = r-1$. Since r' is an even integer, we can apply (iii) to obtain:

$$\beta^{(r-1)(n-1)+h} - \beta^{r(n-1)} < \gamma_1^2(\beta^{(r-3)(n-1)+n-h} - \beta^{(r-2)(n-1)+1}).$$

Hence applying (i) and (iii) repeatedly, we conclude that:

$$\begin{aligned} \beta^{(r-1)(n-1)+h} - \beta^{r(n-1)} &< \gamma_1^{r-1}(\beta^h - \beta^{n-1}) \\ &= \gamma_1^{r-1}(\frac{1}{2} - \frac{1}{2^{n-h-1}}) \\ &< \gamma_1^{r-1}(\frac{1}{2}). \end{aligned}$$

By the same argument $\beta^{(r-1)(n-1)+n-h} - \beta^{(r-1)(n-1)+1} < \gamma_1^{r-1}(\frac{1}{2})$ when r is even. Moreover, from analogous arguments, we obtain that:

$$\beta^{(r-1)(n-1)+h} - \beta^{(r-1)(n-1)+1} < \gamma_2^{r-1}(\frac{1}{2})$$

when r is odd and:

$$\beta^{(r-1)(n-1)+n-h} - \beta^{r(n-1)} < \gamma_2^{r-1}(\frac{1}{2})$$

when r is even.

Let r be an odd integer. The left-most and right-most terms in row r are $\beta^{(r(n-1)}$and $\beta^{(r-1)(n-1)+1}$respectively. Using the triangle inequality and the bounds derived in the previous paragraph, it follows that:

$$\begin{aligned} ||\beta^{r(n-1)} - \beta^{(r-1)(n-1)+1}|| &\le ||\beta^{r(n-1)} - \beta^{r(n-1)+h}|| + ||\beta^{r(n-1)+h} - \beta^{(r-1)(n-1)+1}|| \\ &< \frac{1}{2}(\gamma_1^{r-1} + \gamma_2^{r-1}). \end{aligned}$$

If r is an even integer, and the left-most and right-most terms in row r are $\beta^{(r-1)(n-1)+1}$ and $\beta^{(r(n-1)}$ respectively, one has:

$$\begin{aligned} ||\beta^{r(n-1)} - \beta^{(r-1)(n-1)+1}|| &\le ||\beta^{r(n-1)} - \beta^{r(n-1)+n-h}|| + ||\beta^{(r(n-1)+n-h} - \beta^{(r-1)(n-1)+1}|| \\ &< \frac{1}{2}(\gamma_1^{r-1} + \gamma_2^{r-1}). \end{aligned}$$

Note that the maximal difference of terms in row r is strictly less than $\frac{1}{2}\max[\gamma_1, \gamma_2]^{r-1}$.

Pick an integer t such that $t = r(n-1)$ where r is an odd integer i.e. β^t is the left-most term in row r and $m(t) = r - 1$. Let $q = r'(n-1)$ where $r' > r$. Note that, by repeated application of the triangle inequality, it follows that $||\beta^t - \beta^q||$ is less than the sum of the differences between the left-most and right-most terms of all rows starting from $r + 1$. Hence:

$$\begin{aligned}
||\beta^t - \beta^q|| &< \frac{1}{2}(\gamma_1^r + \gamma_1^{r+1} + ... + \gamma_2^r + \gamma_2^{r+1} .....) \\
&= \frac{1}{2}(\frac{\gamma_1{}^r}{1-\gamma_1} + \frac{\gamma_2{}^r}{1-\gamma_2}) \\
&\equiv \lambda(r) \\
&\equiv \lambda(m(t))
\end{aligned}$$

(note that we critically use the fact that γ_1 and γ_2 are strictly less than 1). Now let β^q be a term in row r' where $r' > r$. Applying the triangle inequality again, we have:

$$\begin{aligned}
||\beta^t - \beta^q|| \ &< \lambda(m(t)) + \frac{1}{2}\max[\gamma_1, \gamma_2]^{r-1} \\
&< \lambda(m(t)) + \frac{1}{2}\max[\gamma_1, \gamma_2]^{m(t)} \\
&\equiv \hat{\lambda}(t).
\end{aligned}$$

Observe that $\hat{\lambda}(t) \to 0$ as $t \to \infty$. Pick $\varepsilon > 0$ and let T be such that $\hat{\lambda}(t) < \varepsilon$ for all $t > T$. We have shown that $||\beta^T - \beta^q|| < \varepsilon$ for all $q > T$. Hence the sequence β^t is a Cauchy sequence and is convergent.

We now show that the sequence converges to $\frac{1}{n}$. Suppose it converges to α. Let $t$and k be positive integers such that $t + 1 = k(n-1)$ and consider the following sequence of differences.

$$\beta^{t+1} - \beta^t = \frac{1}{2}(\beta^{(k-2)(n-1)+1} - \beta^t) \tag{5}$$

$$\beta^t - \beta^{t-1} = \frac{1}{2}(\beta^{(k-2)(n-1)+2} - \beta^{t-1}) \tag{6}$$

$$... = ...$$

$$\beta^{t-(n-3)} - \beta^{t-(n-2)} = \frac{1}{2}(\beta^{(k-1)(n-1)} - \beta^{(k-1)(n-1)}) \tag{7}$$

$$\beta^{t-(n-2)} - \beta^{t-(n-1)} = \frac{1}{2}(\beta^{(k-2)(n-1)+1} - \beta^{(k-1)(n-1)-1}) \tag{8}$$

$$... = ...$$

$$\beta^{n-h+1} - \beta^{n-h} = \frac{1}{2}(\beta^0 - \beta^{n-h}) \tag{9}$$

It is clear from these $t-n+h$ equalities that, except for the first $n-2$ negative terms of the right hand sides, every positive term of the first $n-1$ lines has an identical negative term in one of the lines n+1,...,2n. Hence, if we sum the equalities (5)-(9), we get:

$$\beta^{t+1} - \beta^{n-h} = \frac{1}{2}(\sum_{i=1}^{n-2} \beta^{k(n-1)+i})$$

Observe that $\beta^{n-h} = 1/2$. Also, $\{\beta^{k(n-1)+i}\}$, for $k = 1, ...,$ is a subsequence of the original sequence for all $i = 1, ..., n-2$. Since the original sequence converges to α, these subsequences must also converge to α. Therefore by taking limits on both sides of the equation above, we obtain $\alpha - 1/2 = -1/2(n-2)\alpha$, so that $\alpha = \frac{1}{n}$, as required. QED

An important preliminary step in the proof of theorem 2 is the proof that if the ordering $\succsim$ of $P(X')$ satisfies restricted independence and averaging, then it satisfies, given the structural axioms, the following property of *attenuation*.

Definition 1 *The ordering $\succsim$ of $P(X')$ satisfies attenuation if for all sets A, $B$and $C \in P(X)$ satisfying $A \sim B$, $A \cap C = B \cap C = \emptyset$and $\#A > \#B$, $C \succ A$ implies $A \cup C \prec B \cup C$ and $A \succ C$ implies $A \cup C \succ B \cup C$.*

Loosely speaking, attenuation states that the level of uncertainty of a decision, as measured by the number of its different consequences, "attenuates" the impact, positive or negative, of adding new consequences to it Specifically, if one adds, to two decisions that are, preference-wise, equivalent but that differ in terms of their uncertainty, consequences in C that are better than the existing consequences, then the positive impact of the addition should be larger for the more certain set than for the less certain one. Of course attenuation goes both ways so that if the added consequences are worse than the existing one, then adding them to the certain set will have a larger negative impact than adding them to the less certain one.

The next two lemmas establish that any ordering $\succsim$ of $\mathcal{P}(X')$ satisfying averaging, restricted independence and richness satisfies attenuation.

Lemma 5 *Let $\succsim$ be an ordering of $\mathcal{P}(X)$ satisfying averaging, restricted independence and richness. Then, for all finite sets A $B \in P(X')$, such that $\#A - \#B \geq 2$, and for all sets $C \in \mathcal{P}(X')$ such that $C \cap (A \cup B) = \varnothing$, there are consequences $x_1, \ldots, x_n \in X' \backslash (A \cup B \cup C)$ such that $B \sim B \cup \{x_1, \ldots, x_n\}$.*

Proof. Define $n = \#A - \#B$. We distinguish three cases.

$n = 2$. Using proposition 6 and certainty equivalence, choose a consequence a such that $B \prec \{a\}$. By averaging, $B \prec B \cup \{a\}$. Using again proposition 6 and certainty equivalence, find a consequence $e \in X'$ such that $\{e\} \succ B \cup \{a\}$. By averaging and transitivity, we have $B \prec B \cup \{a\} \prec B \cup \{a, e\}$.

- If there is a consequence $b \in X'$such that $B \cup \{a,b\} \precsim B$, then, by richness, there is a consequence $c \in X'$ such that $\{a,c\} \cup B \sim B$. If a or c belongs to $A \cup B$, we then repeat the same procedure starting with another a. Since $A \cup B \cup C$ is finite, we can do this repetition at most a finite number of times so that, at the end, we are sure to find consequences a and $c \in X'$such that $\{a,c\} \cup B \sim B$ and $\{a,c\} \cap (A \cup B) = \emptyset$.
- If $B \cup \{a,b\} \succ B$ for all consequences $b \in X'$, choose (thanks to proposition 6 and certainty equivalence), consequences b and $e \in X'$ such that $\{e\} \prec \{b\} \prec B$. By averaging, one has $B \cup \{b,e\} \prec B \cup \{b\} \prec B$ and, by assumption, $B \cup \{a,b\} \succ B$. Hence, by richness, there is a consequence $c \in X'$ such that $B \cup \{b,c\} \sim B$. If b or c belongs to $A \cup B$, we can repeat the same reasoning starting with another b. Again, the finiteness of $A \cup B \cup C$ guarantees that the repetition of the procedure will be finite and will lead, eventually, to b and c such that $\{b,c\} \cup B \sim B$ and $\{b,c\} \cap (A \cup B) = \emptyset$.

$n = 3$. We have just proved that we can find consequences a and $c \in X'$ such that $B \cup \{a,c\} \sim B$ and $\{a,c\} \cap (A \cup B) = \emptyset$. It can be noticed that $\{a\} \not\sim \{c\}$. Choose now (thanks again to proposition 6 and certainty equivalence), a consequence $d \in X'$such that $\{d\} \prec B \cup \{a,c\}$. By averaging and transitivity, one has $B \cup \{a,c,d\} \prec B \cup \{a,c\} \sim B$. Choose also (proposition 6 and certainty equivalence) a consequence $e \in X'$ such that $\{e\} \succ B \cup \{a,c\}$. By averaging and transitivity, $B \cup \{a,c,e\} \succ B \cup \{a,c\} \sim B$. By richness, there is a consequence $f \in X'$ such that $B \cup \{a,c,f\} \sim B \cup \{a,c\} \sim B$. By restricted independence and transitivity, we must have either $\{a\} \succ \{f\} \succ \{c\}$ or $\{a\} \prec \{f\} \prec \{c\}$. If $f \in A \cup B \cup C$, then we can redo the procedure as many (finite) times as required starting with another a or c.

$n > 3$. If $n = 2m$ for some integer $m > 1$, then we apply m times the reasoning of the case $n = 2$. If $n = 2m + 1$ for some integer $m > 1$, then we apply $(m - 1)$ times the reasoning of the case $n = 2$ and once the reasoning of case $n = 3$. QED

Lemma 6 *Let $\succsim$ be an ordering on $P(X')$ satisfying averaging, restricted independence and richness. Then $\succsim$ satisfies the property of attenuation.*

Proof.

Let A and B be sets in $P(X)$ such that $A \sim B$ and $\#A > \#B$ and let $n = \#A - \#B$ and let C be a set in $P(X)$ such that $A \nsim C$. Since the argument works symmetrically for $A \succ C$ or $A \prec C$, we only provide it for the later case. The argument requires that we distinguish 3 cases.

$n \geq 2$. By Lemma 5, there are consequences $x_1, \ldots, x_n$ such that $B \cup \{x_1, \ldots, x_n\} \sim B$ and $\{x_1, \ldots, x_n\} \cap (B \cup C) = \emptyset$. By restricted independence, $A \cup C \sim B \cup \{x_1, \ldots, x_n\} \cup C$. By averaging, $\{x_1, \ldots, x_n\} \sim B \sim A$. Suppose $A \prec C$. By Averaging, $B \prec B \cup C$. Hence $\{x_1, \ldots, x_n\} \prec B \cup C$. By averaging again, $B \cup \{x_1, \ldots, x_n\} \cup C \prec B \cup C$. By transitivity, $A \cup C \prec B \cup C$.

$n = 1$, $\#B \geq 2$. We first show that there exists a consequence $x \in X'$ and a set $B' \in P(X')$ such that $x \notin B'$, $\{x\} \sim B' \sim B$, $\#B' = \#B$ and $B' \cap C = \emptyset$. Indeed, use certainty equivalence to define x by $\{x\} \sim B$. If $x \notin B$, then define $B' = B$ and the proof is done. If $x \in B$, choose a consequence $c \in X'$ such that $\{c\} \prec \{x\}$ and $\{c\} \succsim \{y\}$ for all $y \in B \cup C$ (if any) such that $\{y\} \prec \{x\}$. The finiteness of $B \cup D$ as well as proposition 6 guarantees the existence of such a c. Using similar arguments, one can also find a consequence $d \in X'$ such that $\{x\} \prec \{d\}$ and $\{d\} \precsim \{z\}$ for all $z \in B \cup C$ (if any) such that $\{x\} \prec \{z\}$. Moreover, c and d can be chosen in such a way that $\{x\} \precsim \{c, d\}$. Indeed, if $\{x\} \succ \{c, d\}$ for some initial choice of c and d, then, we know from averaging that $\{x, d\} \succ \{x\} \succ \{c, d\}$. Hence by richness, there exists a c'such that $\{c', d\} \sim \{x\}$. Since $\{c', d\} \sim \{x\} \succ \{c, d\}$, we must have from restricted independence that $\{c'\} \succ \{c\}$ and, since $\{d\} \succ \{x\}$ and $\{c', d\} \sim \{x\}$, it follows from averaging and transitivity that $\{x\} \succ \{c'\}$. We then have $\{x\} \succ \{c'\} \succ \{c\} \succsim \{y\}$ for all $y \in B \cup C$ (if any) such that $\{y\} \prec \{x\}$. Hence replacing c by c' leads immediately to the statement that $\{x\} \precsim \{c', d\}$. Assuming therefore $\{x\} \precsim \{c, d\}$, we consider two cases.

1: $\#B = 2m$, for some strictly positive integer m. Choose m different consequences $z_1, \ldots, z_m \in X'$ such that$\{c\} \prec \{z_1\} \prec \ldots \prec \{z_m\} \prec \{x\}$. This is clearly possible thanks to certainty equivalence. By assumption, $z_i \notin B \cup C$. For $i = 1 \ldots m$, define z'_i by $\{x\} \sim \{z_i, z'_i\}$. This is possible thanks to richness and the fact that $\{x\} \precsim \{c, d\} \precsim \{z_i, d\}$ (by restricted independence and transitivity) and that $\{x\} \succ \{z_i\} \succ \{z_i, c\} \succ \{c\}$ (by averaging and transitivity). By averaging and transitivity, one has $\{z_i, z'_i\} \sim \{x\} \succ \{z_i, x\}$. It then follows from restricted independence that $\{z'_i\} \succ \{x\}$. We now prove that $\{z'_i\} \prec \{d\}$, for $i = 1 \ldots m$. Suppose by contradiction, using the completeness of $\succsim$, that $\{d\} \precsim \{z'_i\}$ for some i. By restricted independence and transitivity, we would then have $\{z_i, d\} \precsim \{z_i, z'_i\} \sim \{x\}$ Yet, since $\{c\} \prec \{z_i\}$, we have by restricted independence and transitivity that $\{c, d\} \prec \{z_i, d\} \precsim \{z_i, z'_i\} \sim \{x\}$, in violation of $\{x\} \precsim \{c, d\}$. Hence, since $\{x\} \prec \{z'_i\} \prec \{d\}$, we know that $z'_i \notin B \cup C$. Define then $B' = \{z_1, \ldots, z_m, z'_1, \ldots, z'_m\}$. By repeated application of averaging and transitivity, one obtains that $B' \sim \{x\} \sim B$ and, by construction, that $B' \cap C = \emptyset$ and $x \notin B'$.

2: $\#B = 2m + 1$, for some strictly positive integer m. Using certainty equivalence, define $c'$and c'' by $\{c'\} \sim \{c, x\}$and $\{c''\} \sim \{c', c\}$. We know that $\{x\} \precsim \{c, d\}$ and, by restricted independence and transitivity,

$\{x\} \precsim \{c, d\} \prec \{c'', d\} \prec \{c', d\}$. Moreover, by averaging and transitivity, we have that $\{x\} \succ \{c, x\} \sim \{c'\} \succ \{c', c\} \sim \{c''\} \succ \{c\}$. Hence, using richness, one can define $d'$and $d''$by $\{c', d'\} \sim \{x\}$ and $\{c'', d''\} \sim \{x\}$. As in the case '$\#B = 2m$', we can show that $\{d\} \succ \{d''\} \succ \{d'\} \succ \{x\}$. Hence, $\{c'', c', d', d''\} \cap (B \cup C) = \emptyset$. Since $\{d'\} \prec \{d''\}$, we have $\{x\} \sim \{c', d'\} \prec \{c', d''\}$ (by restricted independence and transitivity) and, by averaging and transitivity, that $\{x\} \prec \{c', d'', d\}$. Now, by averaging again, $\{c'', c', d', d''\} \sim \{x\}$. Hence, since $\{c\} \prec \{c''\}$, we have $\{c, c', d', d''\} \prec \{x\}$ (by restricted independence and transitivity). Moreover, since $\{x\} \prec \{d'\}$, averaging implies $\{c', c, d''\} \prec \{x\}$. Hence, $\{c', c, d''\} \prec \{x\} \prec \{c', d'', d\}$. By richness, there is a consequence e such that $\{x\} \sim \{c', d'', e\}$. One can not have $\{c\} \succsim \{e\}$ because this would imply, using restricted independence and transitivity, that $\{x\} \sim \{c', d'', e\} \precsim \{c', d'', c\}$, in contradiction of $\{x\} \succ \{c', d'', c\}$. Analogously $\{e\} \succsim \{x\}$ can not hold because, if it did, one would have, using restricted independence and transitivity, that $\{x\} \sim \{c', d'', e\} \succsim \{c', d'', x\}$ and, using averaging, that $\{x\} \succsim \{c', d''\}$, a contradiction. Hence, since $\succsim$ is complete $\{c\} \prec \{e\} \prec \{x\}$. We therefore conclude that $\{c', d'', e\} \cap B \cup C = \emptyset$. Choose now $(m-1)$ different consequences $z_1, \ldots, z_{m-1} \in X'$ in such a way that $\{c'\} \prec \{z_1\} \prec \ldots \prec \{z_{m-1}\} \prec \{x\}$. It is always possible to choose them different from $c', d''$and e. For $i = 1 \ldots m-1$, define as in the previous case z_i' by $\{x\} \sim \{z_i, z_i'\}$. As in the previous case also, we can show that $z_i'$is such that $\{x\} \prec \{z_i'\} \prec \{d\}$for $i = 1 \ldots m-1$. Define therefore $B'$by $B' = \{c', d'', e, z_1, \ldots, z_{m-1}, z_1', \ldots, z_{m-1}'\}$. By averaging, $B' \sim \{x\} \sim B$ and, by construction, $B' \cap B \cup C = \emptyset$ and $x \notin B'$.
Given the existence of the set $B'$and the consequence x with the required property, we consider two cases.

$x \notin C$. By averaging, $B' \cup \{x\} \sim A$. By restricted independence, $B \cup C \sim B' \cup C$and $B' \cup \{x\} \cup C \sim A \cup C$. Suppose $A \prec C$. Hence, $B' \prec C$and, by averaging, $B' \prec B' \cup C \prec C$. We also have $\{x\} \prec B' \cup C$and, by averaging, $\{x\} \prec B' \cup \{x\} \cup C \prec B' \cup C$. By transitivity, $A \cup C \prec B \cup C$.

$x \in C$. We must then have that $\#C > 1$, as assuming otherwise would imply that $C = \{x\} \sim B \sim A$). Using the same argument as above, there is a set $C' \in P(X)$ satisfying $C' \sim C$, $\#C' = \#C, x \notin C'$,$B \cap C' = \emptyset, B' \cap C' = \emptyset, A \cap C' = \emptyset$. By restricted independence, $B \cup C \sim B' \cup C$, $B' \cup C \sim B' \cup C'$,$A \cup C \sim A \cup C'$, and $B' \cup \{x\} \cup C' \sim A \cup C'$. Suppose $A \prec C$. Hence, $B' \prec C'$and, by averaging, $B' \prec B' \cup C' \prec C'$. We also have $\{x\} \prec B' \cup \{x\} \cup C' \prec B' \cup C'$. By transitivity, $A \cup C \prec B \cup C$.

$n = 1$, $\#B = 1$. Suppose first that $\#C = 1$. Write A, B and C as: $A = \{a, b\}$, $B = \{x\}$ and $C = \{c\}$ and assume that $\{x\} \sim \{a, b\} \prec \{c\}$ but, contrary to what is required by attenuation, that $\{a,b,c\} \succsim \{x,c\}$. By certainty equivalence, there exists a consequence $z \in X'$ such that $\{z\} \sim \{x,c\}$. Since $\{x\} \prec \{c\}$, $\{x\} \prec \{z\} \prec \{c\}$ by averaging so that z is distinct from either x or c. We therefore have (using averaging and transitivity) $\{a,b,c\} \succsim \{x,c\} \sim \{z\} \sim \{x,c,z\}$. It then follows from restricted independence and transitivity that $\{a,b\} \succsim \{x,z\} \succ \{x\}$, contradicting $\{x\} \sim \{a,b\}$. Suppose now that $\#C > 1$. Suppose $\{x\} \sim \{a,b\} \prec C$ but, contrary to what is required by attenuation, $\{a,b\} \cup C \succsim \{x\} \cup C$. By certainty equivalence, there is a consequence $z \in X'$such $\{z\} \sim \{x\} \cup C$. By averaging (since $x \notin C$), one has $\{x\} \prec \{z\} \prec C$. One has therefore $\{a,b\} \cup C \succsim \{x\} \cup C \sim \{z\}$. If $z \notin C$, then averaging and transitivity entails that $\{z\} \sim \{x,z\} \cup C$. Using then restricted independence and transitivity, one obtains that $\{a,b\} \succsim \{x,z\} \succ \{x\}$, a contradiction. If $z \in C$, then apply certainty equivalence recursively to find a sequence of z_t such that $\{z_t\} \sim \{z_{t-1}, x\}$ for $t = 1, ...$starting with $z_0 = z$. Since there are only finitely many elements in C, one will eventually find some t for which $z_t \notin C$and $\{x\} \prec \{z_t\} \prec ... \prec \{z\} \prec C$. By averaging $\{x\} \cup C \sim \{x\} \cup C \backslash \{z\} \sim \{z\}$. Since $\{z\} \succ \{z_t\}$, we have, by transitivity and averaging, that $\{z\} \sim \{x\} \cup C' \succ \{x, z_t\} \cup C \succ \{z_t\}$. We therefore have $\{a,b\} \cup C \succsim \{x\} \cup C \sim \{z\} \sim \{x\} \cup C \succ \{x, z_t\} \cup C$ which implies, thanks to restricted independence and transitivity, that $\{a,b\} \succ \{z_t, x\}$ and, by averaging and transitivity, that $\{a,b\} \succ \{x\}$, again a contradiction.

We next establish some further auxiliary lemmas.

Lemma 7 *Let $\succsim$ be an ordering on $\mathcal{P}(X')$ satisfying averaging, restricted independence, and richness. Then, if A and B are subsets of X' and c is a consequence in X' such that $A \prec B \cup \{c\}$ and $\{d\} \prec \{c\}$ for some $d \in X'$, there exists some $e \in X$ such that $\{e\} \prec \{c\}$ and $A \prec B \cup \{e\}$. Dually, if A and B are sets and c is a consequence in X' such that $A \succ B \cup \{c\}$ and $\{d\} \succ \{c\}$ for some $d \in X'$, then there exists $e \in X'$such that $\{e\} \succ \{c\}$ and $A \succ B \cup \{e\}$.*

Proof.

We only prove the first statement and distinguish three cases.

(a) $A \prec B \cup \{d\}$, in which case the proof is done.

(b) $A \sim B \cup \{d\}$. Then, by certainty equivalence, there exists e such that $\{e\} \sim \{d,c\}$. By averaging, $\{d\} \prec \{e\} \prec \{c\}$. By restricted independence, $B \cup \{d\} \prec B \cup \{e\}$ so that the statement $A \prec B \cup \{e\}$ follows.

(c) $A \succ B \cup \{d\}$. In that case the richness axiom applies and there is a consequence f such that $A \sim B \cup \{f\}$ and we proceed as in case (b).

We next establish that if $\succsim$ is an ordering of $P(X)$ satisfying averaging, restricted independence, richness and, by lemma 6, attenuation, then it satisfies the following condition.

Condition 3 *(C) For all distinct consequences* a*,* b*, cand* $d \in X$ *and every set* $B \in P(X)$ *such that* $\{b\} \sim \{c, d\}$ *and* $B \cap \{b, c, d\} = \emptyset$*, we must have:*
(i) $\{a\} \succsim B \cup \{b\}$*and* $\{b\} \succsim \{a\}$ *with at least one strict ranking imply* $\{a, b\} \succ B \cup \{c, d\}$*, and*
(ii)$\{a\} \precsim B \cup \{b\}$*and* $\{b\} \prec \{a\}$ *with at least one strict ranking imply* $\{a, b\} \prec B \cup \{c, d\}$*.*

Three auxiliary lemmas are needed in order to establish this. The first of them is the following.

Lemma 8 *Let* $\succsim$ *be an ordering on* $P(X')$ *satisfying averaging, restricted independence and richness. Let* A *and* B *be two finite subsets of* X' *and let* a*,* b*, cand* d *be consequences in* X' *satisfying* $A \cup \{a\} \sim B \cup \{b\}$*,* $\#A = \#B$*,* $\{b\} \sim \{c, d\}$*,* $a \neq b$*,* $c \neq d$*,* $\{a, b\} \cap A = \{c, d\} \cap B = \emptyset$ *and* $b \notin B$*. Then* $A \cup \{a, b\} \sim B \cup \{c, d\}$*.*

Proof.

Suppose first $\{c\} \sim \{d\}$. By averaging, $\{b\} \sim \{c\} \sim \{d\}$. Since $c \neq d$, we have $c \neq b$ or $d \neq b$. Assume without loss of generality that $c \neq b$. By restricted independence, $B \cup \{b\} \sim B \cup \{c\}$. Therefore $A \cup \{a\} \sim B \cup \{b\} \sim B \cup \{c\}$ and, by restricted independence, $A \cup \{a, b\} \sim B \cup \{c, b\}$. By restricted independence again, $B \cup \{c, b\} \sim B \cup \{c, d\}$. Finally, by transitivity, $A \cup \{a, b\} \sim B \cup \{c, d\}$.

Suppose now $\{c\} \not\sim \{d\}$ and assume, without loss of generality, that $\{c\} \prec \{d\}$. Two cases need to be considered.

1. Assume by contradiction that $A \cup \{a, b\} \prec B \cup \{c, d\}$. Let us show that there is a consequence $\underline{d}$such that $A \cup \{a, b\} \prec B \cup \{c, \underline{d}\} \prec B \cup \{c, d\}$. Choose a consequence u distinct from c such that $\{u\} \prec \{d\}$. The existence of such a consequence is guaranteed by the fact that $\{c\} \prec \{d\}$ and, using certainty equivalence, that one can always define u by $\{u\} \sim \{c, d\}$. By averaging, one must have $\{c\} \prec \{u\} \prec \{d\}$ which, given the reflexivity of $\succsim$, implies that u is distinct from both c and d. By restricted independence, one has $B \cup \{c, u\} \prec B \cup \{c, d\}$. Two mutually exclusive cases can occur.

    - $B \cup \{c, u\} \precsim A \cup \{a, b\}$. By averaging and certainty equivalence, one can find a consequence e such that $A \cup \{a, b\} \prec e \prec B \cup \{c, d\}$. By richness, there is $\underline{d}$: $B \cup \{c, d\} \sim \{e\}$. Hence $A \cup \{a, b\} \prec B \cup \{c, \underline{d}\} \prec B \cup \{c, d\}$
    - $A \cup \{a, b\} \prec B \cup \{c, u\}$. In this case, let $\underline{d} = u$.

By certainty equivalence, there is a consequence $\underline{b}$ such that $\{\underline{b}\} \sim \{c, \underline{d}\}$. Notice that we can always choose $\underline{d}$ so that $\underline{d}$ and $\underline{b}$ do not belong to $B \cup \{c\} \cup A \cup \{a\}$. By restricted independence, $\{\underline{b}\} \prec \{b\}$. By averaging, $\{\underline{b}\} \sim \{c, \underline{d}\} \prec \{b, \underline{b}\} \prec \{b\}$. By restricted independence, $B \cup \{c, \underline{d}\} \prec B \cup \{b, \underline{b}\}$. By restricted independence, $A \cup \{a, \underline{b}\} \prec A \cup \{a, b\}$ and $A \cup \{a, \underline{b}\} \sim B \cup \{b, \underline{b}\}$. By transitivity, $B \cup \{c, \underline{d}\} \prec A \cup \{a, b\}$. But we have previously shown that $A \cup \{a, b\} \prec B \cup \{c, \underline{d}\}$. A contradiction.

2. Assume by contradiction that $A \cup \{a, b\} \succ B \cup \{c, d\}$. This case is treated like the previous one.

The next lemma provides the second step in the proof that averaging, restricted independence and richness imply Condition C.

Lemma 9 *Let $\succsim$ be an ordering on $\mathcal{P}(X')$ satisfying averaging, restricted independence and richness and let a, b, c and d be consequences in $X'$and B be a finite subset of X' such that $\{a\} \succsim B \cup \{b\}$, $\{b\} \sim \{c, d\}$, $\{b\} \succ \{a\}$, $b \notin B$ and $\{c, d\} \cap B = \emptyset$. Then there exists a finite subset $A'$of X' and a consequence $a' \in X'$ such that $A' \cup \{a'\} \sim B \cup \{b\}$, $a' \notin A'$and $\#A' = \#B$.*

Proof.

Start with $\{b\} \succ \{a\} \succsim B \cup \{b\}$. By averaging, $\{b\} \succ B$. Write B as $B = \{b_1, \ldots, b_r\}$ with $\{b_1\} \precsim \{b_2\} \precsim \ldots \precsim \{b_r\}$. Let b_j be such that $\{b_j\} \prec \{b\}$ and $\{b\} \precsim b_i$ for all $i > j$. The existence of such a b_j is guaranteed by averaging. By certainty equivalence, one can find a consequence b'_j in X' such that $b'_j \sim \{b, b_j\}$. By averaging, $b_j \prec b'_j \prec b$. Define A' by $A' = B \cup \{b'_j\} \setminus \{b_j\}$. By averaging and transitivity, one has $A' \succ B$. By restricted independence, $A' \cup \{b\} \succ B \cup \{b\}$. By construction, $A' \cup \{b_j\} = B \cup \{b'_j\}$. By restricted independence, $B \cup \{b\} \succ B \cup \{b'_j\}$. Hence $A' \cup \{b\} \succ B \cup \{b\} \succ B \cup \{b'_j\} = A' \cup \{b_j\}$. By richness, there exists a consequence a'such that $A' \cup \{a'\} \sim B \cup \{b\}$. By restricted independence, one has $b \succ a' \succ b'_j$, which, given the definition of A', establishes that $a' \notin A'$.

Combining these two lemmas, we can establish the following.

Lemma 10 *Let $\succsim$ be an ordering on $P(X')$ satisfying averaging, restricted independence and richness. Then $\succsim$ satisfies condition C.*

Proof.

We prove only part (i) of condition C, the proof of the other part being similar. Suppose that we have $\{a\} \succsim B \cup \{b\}$, $\{b\} \sim \{c, d\}$, $\{b\} \succ \{a\}$, $b \notin B$ and $\{c, d\} \cap B = \emptyset$ for consequences a, b, c, d in X' and some finite subset B of X'. By Lemma 9, there exists a finite set A' and a consequence a' such that $A' \cup \{a'\} \sim B \cup \{b\}$, $a' \notin A'$and $\#A' = \#B$. By Lemma 8, we must have

$A' \cup \{a', b\} \sim B \cup \{c, d\}$. By certainty equivalence, there exists a consequence a'' such that $a'' \sim A' \cup \{a'\}$. By transitivity, $\{b\} \succ \{a\} \succsim A' \cup \{a'\} \sim \{a''\}$. Since, by lemma 6, the ordering $\succsim$ satisfies attenuation, one has $A' \cup \{a', b\} \prec \{a'', b\}$. By transitivity, $\{a'', b\} \succ B \cup \{c, d\}$. Restricted independence and $\{a\} \succsim \{a''\}$ imply $\{b, a\} \succsim \{b, a''\}$. Transitivity finally yields $\{a, b\} \succ B \cup \{c, d\}$.

Endowed with this result, we are equipped to prove theorem 2.

Proof of theorem 2.

Using Theorem 1, we find a function u that uniquely represents (up to an affine transform) $\succsim$ as per (1) on the subset of $\mathcal{P}(X')$ containing sets of cardinality no greater than 2. We want to prove that the same function u can also be used to represent $\succsim$ on the whole set $\mathcal{P}(X')$. We must prove specifically that, for any $A \in \mathcal{P}(X')$ and $g \in X'$,

$$A \succsim \{g\} \iff \sum_{a \in A} \frac{u(a)}{\#A} \geq u(g).$$

where u is the (unique up to an affine transform) utility function identified in theorem 1. Since $\succsim$ is complete, it is sufficient to prove $\Rightarrow$. Suppose $\#A = m$ and write $A = \{a_1, a_2, \ldots, a_m\}$ with $\{a_1\} \precsim \ldots \precsim \{a_m\}$. By certainty equivalence, there exists $b^0_{m-1} \in X'$ such that $b^0_{m-1} \sim \{a_{m-1}, a_m\}$. Similarly, for $i = m-2, \ldots, 1$, we can find, by certainty equivalence, a consequence b^0_i such that $b^0_i \sim \{a_i, b^0_{i+1}\}$. Using certainty equivalence repeatedly, one can define this way for $j = 1, 2, 3, \ldots$the sequence of consequences b^j_i by:

$$b_1^{2j-1} = b_1^{2j-2},$$

$$b_i^{2j-1} \sim \{b_{i-1}^{2j-1}, b_i^{2j-2}\}$$

for $i = 2, \ldots, m-1$,

$$b_{m-1}^{2j} = b_{m-1}^{2j-1}$$

and

$$b_i^{2j} \sim \{b_i^{2j-1}, b_{i+1}^{2j}\}$$

for $i = m-2, \ldots, 1$. We first show that:
(i) $\{b^j_1\} \precsim \{b^j_2\} \precsim \cdots \precsim \{b^j_{m-1}\}$,
(ii) $\{a_1\} \precsim \{b^i_1\} \precsim \{b^{i+1}_1\} \precsim \{b^{i+1}_{m-1}\} \precsim \{b^i_{m-1}\} \precsim \{a_m\}$ and
(iii) $\{b^i_1\} \precsim A \precsim \{b^i_{m-1}\}$.
If $\{a_1\} \sim \{a_m\}$, then, by averaging, $\{a_1\} \sim A$, $\{b^i_j\} \sim \{a_1\} \sim A$ for all $i \in \mathbb{N}$ and $j \in \{1, \ldots, m-1\}$ and the implications (i)-(iii) are immediately established. If $\{a_1\} \prec \{a_m\}$, let k be the largest integer such that $\{a_k\} \prec \{a_{k+1}\}$. We first prove implications (i) and (ii). By averaging, $\{a_{m-1}\} \precsim \{b^0_{m-1}\} \precsim \{a_m\}$. By transitivity, $\{a_{m-2}\} \precsim \{b^0_{m-1}\}$. By averaging again, $\{a_{m-2}\} \precsim \{b^0_{m-2}\} \precsim$

$\{b^0_{m-1}\}$. By repeated use of transitivity and averaging, one is led to the conclusion that $\{a_{k+1}\} \precsim \{b^0_{k+1}\} \precsim \{b^0_{k+2}\}$. Now, by transitivity $\{a_k\} \prec \{b^0_{k+1}\}$ and, by averaging, $\{a_k\} \prec \{b^0_k\} \prec \{b^0_{k+1}\}$. Analogously, a repeated combination of averaging and transitivity leads to the conclusion that $\{a_1\} \prec \{b^0_1\} \prec \{b^0_2\}$. Hence, we have established that $\{a_1\} \prec \{b^0_1\} \prec \{b^0_{k+1}\} \precsim \{b^0_{k+2}\} \precsim \dots \precsim \{b^0_{m-1}\} \precsim \{a_m\}$. Now, by averaging, $\{b^0_1\} \prec \{b^1_2\} \prec \{b^0_2\}$ and, by transitivity, $\{b^1_2\} \prec \{b^0_3\}$. Combining in this way averaging and transitivity leads us to $\{b^1_{m-2}\} \prec \{b^1_{m-1}\} \prec \{b^0_{m-1}\}$ and, therefore, to $\{a_1\} \prec \{b^0_1\} \sim \{b^1_1\} \prec \{b^1_2\} \prec \dots \prec \{b^1_{m-1}\} \prec \{b^0_{m-1}\} \precsim \{a_m\}$. Repeatedly using the same reasoning, one finds that, for all $i \in N$, $\{b^i_1\} \prec \{b^i_2\} \prec \dots \prec \{b^i_{m-1}\}$ and $\{a_1\} \precsim \{b^i_1\} \precsim \{b^{i+1}_1\} \precsim \{b^{i+1}_{m-1}\} \precsim \{b^i_{m-1}\} \precsim \{a_m\}$. We now turn to implication (*iii*) that we prove in the following infinite number of steps.

Step 1. We notice that by virtue of the Gärdenfors principle, $\{b^0_{m-1}\} \succ A$.

Step 2. We prove that $\{b^0_1\} \prec A$. Since by assumption $a_l = a_{l+1}$for all $l = k+1, ..., m-1$, we have by averaging that $\{b^0_{m-1}\} \sim \{a_m\} \sim \{a_{m-1}, a_m\} \sim \{a_{m-2}\} \sim \{a_{m-2}, a_{m-1}, a_m\} \sim ... \sim \{a_{k+1}, \dots, a_{m-1}, a_m\}$. We therefore have $\{b^0_{k+1}\} \sim \{a_{k+1}, \dots, a_{m-1}, a_m\} \sim \{a_{k+1}\}$. Now, since $\{a_k\} \prec \{b^0_{k+1}\} \sim \{a_{k+1}, \dots, a_{m-1}, a_m\}$, it follows from attenuation property (satisfied thanks to lemma 6) that $\{a_k, b^0_{k+1}\} \prec \{a_k, a_{k+1}, \dots, a_{m-1}, a_m\}$ and, since $\{b^0_k\} \sim \{a_k, b^0_{k+1}\}$ and $\precsim$ is transitive, that $\{b^0_k\} \prec \{a_k, \dots, a_{m-1}, a_m\}$. Applying the same reasoning below k enables us to reach the conclusion that $\{b^0_1\} \prec \{a_1, \dots, a_2, a_m\} = A$.

Step 3. Since $b^1_1 = b^1_0$, we trivially have that $\{b^1_1\} \prec A$.

Step 4. We prove that $\{b^1_{m-1}\} \succ A$. Notice that $\{b^1_1\} \sim \{a_1, b^0_2\}$, $\{b^1_2\} \sim \{b^1_1, b^0_2\}$, $\{b^0_2\} \sim \{a_2, b^0_3\}$, $\{b^1_1\} \prec \{b^0_2\}$and clause (*i*) of condition C (satisfied thanks to lemma 10) imply that $\{b^1_2\} \succ \{a_1, a_2, b^0_3\}$. Similarly, $\{b^1_2\} \sim \{b^1_1, b^0_2\}$, $\{b^1_3\} \sim \{b^1_2, b^0_3\}$and clause (*i*) of the condition C imply that $\{b^1_3\} \succ \{a_1, a_2, a_3, b^0_4\}$. Repeating this reasoning, we obtain $\{b^1_{m-2}\} \succ \{a_1, \dots, a_{m-2}, b^0_{m-1}\}$and, finally, $\{b^1_{m-1}\} \succ \{a_1, \dots, a_m\} = A$.

Step 5. Trivially, $\{b^2_{m-1}\} = \{b^1_{m-1}\} \succ A$.

Step 6. We prove that $\{b^2_1\} \prec A$. We have $\{b^2_{m-1}\} \sim \{b^0_{m-1}, b^0_{m-2}\}$, $\{b^2_{m-2}\} \sim \{b^2_{m-1}, b^1_{m-2}\}$, $\{b^1_{m-2}\} \sim \{b^1_{m-3}, b^0_{m-2}\}$ and $\{b^1_{m-2}\} \prec \{b^2_{m-1}\}$. Hence, by clause (*ii*) of condition C, $\{b^2_{m-2}\} \prec \{b^1_{m-3}, b^0_{m-2}, b^0_{m-1}\}$. We have also $\{b^2_{m-2}\} \prec \{b^1_{m-3}, b^0_{m-2}, b^0_{m-1}\}$, $\{b^2_{m-3}\} \sim \{b^2_{m-2}, b^1_{m-3}\}$, $\{b^1_{m-3}\} \sim \{b^1_{m-4}, b^0_{m-3}\}$

and $\{b^1_{m-3}\} \prec \{b^2_{m-2}\}$. Hence, by clause (ii) of condition C, $\{b^2_{m-3}\} \prec \{b^1_{m-4}, b^0_{m-3}, b^0_{m-2}, b^0_{m-1}\}$. This process can be repeated until we obtain:

$$\{b^2_2\} \prec \{b^1_1, b^0_2, b^0_3, \ldots, b^0_{m-1}\} = \{b^0_1, b^0_2, b^0_3, \ldots, b^0_{m-1}\}.$$

By lemma 7, there exists $\{c^0_2\} \prec \{b^0_2\}$ such that $\{b^2_2\} \prec \{b^0_1, c^0_2, b^0_3, \ldots, b^0_{m-1}\}$. Repeatedly applying lemma 7, we find $\{c^0_i\} \prec \{b^0_i\}$, for $i = 3 \ldots m-1$ such that $\{b^2_2\} \prec \{b^0_1, c^0_2, c^0_3, \ldots, c^0_{m-1}\}$. This, combined with $\{b^2_1\} \sim \{b^2_2, b^1_1\}$, $\{b^1_1\} \sim \{a_1, b^0_2\}$, $\{b^1_1\} \prec \{b^2_2\}$ and clause (ii) of condition C, implies $\{b^2_1\} \prec \{a_1, c^0_2, b^0_2, c^0_3, \ldots, c^0_{m-1}\}$. By averaging, it follows that $\{c^0_2, b^0_2\} \prec \{b^0_2\} \sim \{a_2, b^0_3\}$. By restricted independence, one has $\{b^2_1\} \prec \{a_1, a_2, c^0_3, b^0_3, \ldots, c^0_{m-1}\}$. By averaging, $\{c^0_3, b^0_3\} \prec \{b^0_3\} \sim \{a_3, b^0_4\}$. By restricted independence:

$$\{b^2_1\} \prec \{a_1, a_2, a_3, c^0_4, b^0_4, \ldots, c^0_{m-1}\}.$$

Repeating this process leads us to the conclusion that:

$$\{b^2_1\} \prec \{a_1, a_2, \ldots, a_{m-2}, c^0_{m-1}, b^0_{m-1}\}.$$

By averaging, $\{c^0_{m-1}, b^0_{m-1}\} \prec \{b^0_{m-1}\} \sim \{a_{m-1}, a_m\}$. By restricted independence:

$$\{b^2_1\} \prec \{a_1, a_2, \ldots, a_{m-2}, a_{m-1}, a_m\} = A.$$

Step 7. Trivially, $\{b^3_1\} = \{b^2_1\} \prec A$.

Step 8. We prove that $\{b^3_{m-1}\} \succ A$. We have $\{b^3_1\} \sim \{b^1_1, b^2_2\}$, $\{b^3_2\} \sim \{b^3_1, b^2_2\}$, $\{b^2_2\} \sim \{b^1_2, b^2_3\}$ and $\{b^2_2\} \succ \{b^3_1\}$. Hence, by clause (i) of condition C, $\{b^3_2\} \succ \{b^1_1, b^1_2, b^2_3\}$. We also have $\{b^3_3\} \sim \{b^3_2, b^2_3\}$, $\{b^2_3\} \sim \{b^1_3, b^2_4\}$ and $\{b^2_3\} \succ \{b^3_2\}$. Hence, by clause (i) of condition C, $\{b^3_3\} \succ \{b^1_1, b^1_2, b^1_3, b^2_4\}$. Continuing this process, we obtain $\{b^3_{m-2}\} \succ \{b^1_1, b^1_2, \ldots, b^1_{m-2}, b^2_{m-1}\}$. Repeatedly applying Lemma 7, we find c^1_i such that $\{c^1_i\} \succ \{b^1_i\}$, for $i = 1 \ldots m-2$ such that $\{b^3_{m-2}\} \succ \{c^1_1, c^1_2, \ldots, c^1_{m-2}, b^2_{m-1}\}$. This, combined with $\{b^3_{m-1}\} \sim \{b^2_{m-1}, b^3_{m-2}\}$, $\{b^2_{m-1}\} \sim \{b^0_1, b^1_{m-2}\}$, $\{b^2_{m-1}\} \succ \{b^3_{m-2}\}$and clause (i) of condition C, implies that:

$$\{b^3_{m-1}\} \succ \{c^1_1, c^1_2, \ldots, c^1_{m-2}, b^1_{m-2}, b^0_{m-1}\}.$$

By averaging and restricted independence:

$$\{b^3_{m-1}\} \succ \{c^1_1, \ldots, c^1_{m-3}, b^1_{m-3}, b^0_{m-2}, b^0_{m-1}\}.$$

By repeatedly combining averaging and restricted independence in this way, one is led to the conclusion that:

$$\{b^3_{m-1}\} \succ \{c^1_1, b^1_1, b^0_2, b^0_3, \ldots, b^0_{m-1}\}.$$

Repeatedly applying lemma 7, one finds d_i^1 such that $\{d_i^1\} \prec \{b_i^0\}$, for $i = 2 \ldots m-1$ for which:

$$\{b_{m-1}^3\} \succ \{c_1^1, b_1^1, d_2^0, d_3^0, \ldots, d_{m-1}^0\}.$$

Repeatedly applying averaging and restricted independence, we obtain $\{b_{m-1}^3\} \succ \{a_1, a_2, \ldots, a_{m-2}, \ldots, a_{m-1}, a_m\} = A$.

Step 9. Trivially, $\{b_{m-1}^4\} = \{b_{m-1}^3\} \succ A$.

Steps 6 to 9 can clearly be repeated for ever using the same argument and this remark completes the proof of (*iii*). Now, using certainty equivalence, let x be a consequence such that $A \sim \{x\}$. Since the function u found in Theorem 1 represents $\succsim$ as per (1), one has $u(b_1^i) \leq u(x) \leq u(b_{m-1}^i)$ for every i. Now it is easy to check that the sequence $\{u(b_h^i)\}$ for every h are just like the sequences studied in lemma 1. Because of this lemma, one has:

$$\lim_{t\to\infty} u(b_1^t) = \lim_{t\to\infty} u(b_{m-1}^t) = \sum_{a\in A} \frac{u(a)}{\#A}.$$

Hence, $u(x) = \sum\limits_{a\in A} \frac{u(a)}{\#A}$. By transitivity, $A \succsim \{g\} \Longleftrightarrow \{x\} \succsim \{g\} \Longleftrightarrow \sum_{a\in A} \frac{u(a)}{\#A} \geq u(g)$.

Proof of theorem 3.

From theorem 2, we know that if $\succsim$ is an ordering on $P(X)$ satisfying averaging, restricted independence, richness and the Archimedean axiom, then there exists a function $u : X' \to \mathbb{R}$ such that, for all sets A and $B \in P(X')$, one has $A \succsim B \Longleftrightarrow \sum\limits_{a\in A} \frac{u(a)}{\#A} \geq \sum\limits_{b\in B} \frac{u(b)}{\#B}$. If $M(X) = m(X) = \varnothing$ so that $X = X'$, then the proof is done. Assume first that $M(X) \neq \varnothing$ and let t be a consequence in $M(X)$. We have $\{t\} \succ \{x\}$ for every $x \in X'$. We first show that the image of X'under u, denoted $u(X')$, is a set of real numbers that is bounded from above. That is, there is a real number $\overline{b}$such that $u(x) \leq \overline{b}$ for all $x \in X'$. Suppose indeed by contradiction that $u(X')$ is not bounded and consider, thanks to proposition 6, consequences a, $c_0$and $b \in X'$such that $\{a\} \prec \{c_0\} \prec \{b\}$. By averaging and transitivity, one has $\{a\} \prec \{a, c_0\} \prec c_0 \prec \{c_0, b\}$ and, by theorem 2, one has $u(a) < \frac{u(b_0)+u(b)}{2} \Longleftrightarrow u(a) + u(a) < u(c_0) + u(b)$. Since $u(X')$ is unbounded, there is a real number $u' \in u(X')$ such that $u' + u(a) \geq u(c_0) + u(b)$. Since $u' \in X'$, there is a consequence $c \in X'$ such that $u(c) = u'$. By theorem 2, one has $\{c, a\} \succsim \{c_0, b\} \succ \{c_0, a\}$. If follows from richness that there is a consequence c_1 such that $\{c_1, a\} \sim \{c_0, b\}$. Since $\{a\} \prec \{b\}$, it follows by restricted independence that $\{c_1\} \succ \{c_0\}$. This procedure, initiated by finding c_0 and c_1, can clearly be iterated at infinitum and generate a sequence c_k, for $k = 0, \ldots$, of consequences in X'such that $\{c_k, a\} \sim \{c_{k-1}, b\}$ for $k = 1, \ldots$By assumption, $\{t\} \succ \{c_k\}$ for every k so that the sequence is bounded by t. Hence the fact that the sequence $\{c_k\}$ $k = 0, 1...$is infinite contradicts the Archimedean axiom. Analogously, starting from the assumption that $m(X) \neq \Phi$, one can deduct that $u(X')$is bounded from below. Since the set of real number $u(X')$ is either bounded from above and/or from below, it has a least upper bound and/or a greatest lower bound. We therefore extend u to X by defining, for every $t \in M(X)$(if any) $u(t) = \sup_{x \in X'} u(t)$ and, for every $s \in m(X)$ (if any), $u(s) = \inf_{x \in X'} u(x)$. We now show that u so extended represents $\succsim$ as per (1) on the whole set X (and not only on X'). By definition $u(t) > u(x) > u(s)$ for all $t \in M(X)$, $x \in X'$and $s \in m(X)$, and u represents $\succsim$ as per (1) on X' by theorem 2. Take any $x \in X'$. By certainty equivalence, there are consequences b and $c \in X$ such that $\{b\} \sim \{x, t\}$ and $\{c\} \sim \{x, s\}$. By averaging and transitivity, we have $\{s\} \prec \{c\} \sim \{x, s\} \prec \{x\} \prec \{x, t\} \sim \{b\} \prec \{t\}$ so that both b and c belong to X'. We therefore only need to show that $\frac{u(x)+u(t)}{2} = u(b)$ and $\frac{u(x)+u(s)}{2} = u(c)$. The argument being symmetric, we only prove that $\frac{u(x)+u(t)}{2} = u(b)$. By contradiction, suppose first that $\frac{u(x)+u(t)}{2} < u(b)$. By certainty equivalence, there exists a consequence $b_1 \in X$ such that $\{b_1\} \sim \{x, b\}$. By averaging $\{x\} \prec \{b_1\} \prec \{b\}$ and, therefore, $b_1 \in X'$. By theorem 2, the statement $\{b_1\} \sim \{x, b\}$ can be written as $u(b_1) = \frac{u(x)+u(b)}{2}$. Define recursively b_n by $\{b_n\} \sim \{b_{n-1}, b\}$ for $n = 2, ...$Since $\{t\} \succ \{b\} \succ \{b_n\} \succ \{b_{n-1}\}$ by averaging and transitivity, we have that b and $b_n \in X'$ so that, by theorem 2, $u(b_n) = \frac{u(b_{n-1})+u(b)}{2} = \frac{\frac{1}{2}[u(b_{n-2})+u(b)]+u(b)}{2} = ... = \frac{u(x)}{2^{n-1}} + \frac{2^{n-1}-1}{2^{n-1}} u(b)$. Hence, for n large enough, $u(b_n) \in]\frac{u(x)+u(t)}{2}, u(b)[$. Now, we know that $\{b\} \sim \{x, t\} \succ \{b_n\} \succ \{b_{n-1}, x\}$. By richness, there exists t'such that $\{x, t'\} \sim \{b_n\}$. Since $\{x, t\} \succ \{b_n\} \sim \{x, t'\}$, it follows from restricted independence that $\{t'\} \prec \{t\}$. Hence x, b_n and $t' \in X'$so that, by theorem 2, $\frac{u(t')+u(x)}{2} = u(b_n) > \frac{u(t)+u(x)}{2}$. Yet this inequality is incompatible with the definition of $u(t)$ as $u(t) = \sup_{x \in X'} u(t)$.

Proof of theorem 4

We know from proposition 2 that a UEU criterion satisfies averaging and restricted independence on any environment. Conversely, let X be an arc-connected subset of $\mathbb{R}^k$and let $\succsim$ be an ordering of $P(X)$ satisfying the continuity axiom as well as averaging and restricted independence. We will prove that, under these conditions, $\succsim$ satisfies richness and the Archimedean axiom. Using theorems 1, 2 and 3, the conclusion that $\succsim$ is a UEU criterion will then follow immediately. We first notice that, under averaging, if the sets $B(A) = \{x \in X : \{x\} \succsim A\}$and $W(A) = \{x \in X : A \succeq \{x\}\}$ are closed in X for every A, then so are the sets $\widetilde{B}(A) = \{x \in X : A \cup \{x\} \succsim A\}$and $\widetilde{W}(A) = \{x \in X : A \succeq A \cup \{x\}\}$. To see that, assume by contraposition that, say, $\widetilde{B}(A)$ is not closed (the argument for $\widetilde{W}(A)$ is similar). Then, there exists a sequence $\{x^t\}t = 1, ...$converging to some limit x such that:

$$A \cup \{x^t\} \succsim A$$

for all t and

$$A \succ A \cup \{x\}$$

where the last strict ranking is obtained from the assumption that $\succsim$ is complete. Since $\succsim$ is also reflexive, this strict ranking implies therefore that $x \notin A$. By averaging one has therefore:

$$A \succ \{x\} \tag{10}$$

Now, since A is finite, and x^t is a sequence converging to x, either x^t is a finite sequence or x^t is infinite. If x^t is finite, then, by definition of a sequence converging to x, there exists some $s \leq t$ for which $x^s = x \notin X$. But given averaging, this is incompatible with the definition of the sequence x^tas satisfying $A \cup \{x^t\} \succsim A$for every t. Hence we must conclude that $x^t$is infinite. If this is the case, there must exists, since $A$is finite, an infinite subsequence $\widetilde{x}^t$of x^tconverging to $x$and such that $\widetilde{x}^t \notin A$ for every t. Since for every t, we have:

$$A \cup \{\widetilde{x}^t\} \succsim A$$

it follows from averaging that we also have:

$$\{x^t\} \succsim A$$

Given (10), this gives us the required contradiction of the closedness of the set $B(A)$. We now prove that $\succsim$ satisfies the three structural axioms.

Richness: Consider any set $B \in P(X)$ and, without loss of generality, write it as $B = \{b_1, ..., b_{\#B}\}$ with $\{b_h\} \precsim \{b_{h+1}\}$ for $h = 1, ..., \#B - 1$. By averaging (and specifically the Gardenförs principle) one has that $B \succsim \{b_1\}$ and $\{b_{\#B}\} \succsim B$ so that none of the (closed under continuity) sets $\{x \in X : \{x\} \succsim B\}$ and $\{x \in X : B \succeq \{x\}\}$ is empty. Since $\succsim$ is complete, $X = \{x \in X : \{x\} \succsim B\} \cup \{x \in X : B \succeq \{x\}\}$. Since X is arc connected, there exists a continuous function $f : [0, 1] \rightarrow X$ such that $f(0) = b_1$ and $f(1) = b_{\#B}$. By continuity, given the closedness of $\{x \in X : \{x\} \succsim B\}$ and $\{x \in X : B \succsim \{x\}\}$, there must be some $\alpha \in [0, 1]$ such that $f(\alpha) \in \{x \in X : \{x\} \succsim B\} \cap \{x \in X : B \succsim \{x\}\}$. By definition $\{f(\alpha)\} \sim B$. Hence $\succsim$ verifies the certainty equivalence condition.Consider now any sets A and B in $P(X)$ and bundles $c^*$and $c_* \in X$ such that $A \cup \{c^*\} \succsim B \succsim A \cup \{c_*\}$. If either $A \cup \{c^*\} \sim B$ or $B \sim A \cup \{c_*\}$, then richness is satisfied and there is nothing to be proved. Assume therefore that:

$$A \cup \{c^*\} \succ B \succ A \cup \{c_*\} \tag{11}$$

holds. Since, as was just shown, $\succsim$ satisfies certainty equivalence, there are consequences b and $b(c) \in X$ (for all $c \in X$) such that $\{b(c)\} \sim A \cup \{c\}$ and $B \sim \{b\}$. By continuity, the restriction of the ordering $\succsim$ to singletons is continuous. Hence, by Debreu (1954) theorem, there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x) \geq f(y)$ if and only if $\{x\} \succsim \{y\}$ for every $x, y \in X$. Define therefore the function $h : X \to \mathbb{R}$ by $h(c) = f(b(c)) - f(b)$. The function h is continuous if f is. By assumption, we have $h(c^*) > 0$ and $h(c_*) < 0$. Since X is an arc-connected subset of $\mathbb{R}^k$, there exists by the intermediate value theorem a consequence $\overline{c}$such that $h(\overline{c}) = f(b(\overline{c})) - f(b) = 0$. By definition, $\overline{c}$is such that $A \cup \{\overline{c}\} \sim \{b\} \sim B$, as required.

Archimedean axiom: If it is impossible to construct one of the standard sequence as in the antecedent clause of the Archimedean axiom, then the proof is (trivially) over. Assume therefore that such a sequence exists (we only provide the argument for the ascending sequence) and, therefore, that a and b are two points in X satisfying $\{a\} \succ \{b\}$ for which one has, for a sequence of points $\{c_t\}_{t \in \mathbb{N}_+}$:

$$\{c_t, a\} \sim \{c_{t+1}, b\} \tag{12}$$

and $\{x\} \succ \{c_t\} \succ \{y\}$ for every $t = 0, ...$and for some point $x$and $y \in X$. By restricted independence, we must have $\{c_{t+1}\} \succ \{c_t\}$ for all t. As noticed earlier, the restriction of the ordering $\succsim$ to singletons is continuous so that, by virtue of Debreu (1954) theorem, there exists a continuous function $f : X \longrightarrow \mathbb{R}$ such that $f(x) \geq f(y)$ if and only if $\{x\} \succsim \{y\}$ for every x, $y \in X$. Hence the existence of a sequence of points $\{c_t\}_{t \in \mathbb{N}_+}$ and of points x and y such that $\{x\} \succ \{c_{t+1}\} \succ \{c_t\} \succ \{y\}$ for all t implies the existence of a sequence of real numbers $\{f_t\}_{t \in \mathbb{N}_+}$as well as real numbers f_x and f_y satisfying $f_{t+1} > f_t$ where $f_x = f(x)$, $f_y = f(y)$ and, for all t, $f_t = f(c_t)$. Now, since every increasing sequence of numbers that is bounded from above is either convergent or finite, the only thing we need to check is that the sequence is not convergent. Suppose by contradiction that the sequence $\{f_t\}$is infinite and converges to some number $\overline{f}$. Since f is a continuous real-valued function from an arc-connected subset of $\mathbb{R}^k$, $\overline{f}$ belongs to the image of f so that there exists some point $\overline{c} \in X$ such that $\overline{f} = f(\overline{c})$. By restricted independence, we know that:

$$\{\overline{c}, a\} \succ \{\overline{c}, b\}$$

By continuity and restricted independence, the set $\{x : \{x, a\} \succsim \{x, b\}\}$ is closed. Because of this, there exists a number $\varepsilon > 0$ such that:

$$\{c', a\} \succ \{c'', b\}$$

for all $c'$and $c'' \in N_\varepsilon(\overline{\alpha})$. Assuming the sequence $\{f_t\}$ to be converging to $\overline{f}$ implies the existence of some positive integer s such, for all $t \geq s$, $f_t \in N_\varepsilon(\overline{f})$. By the continuity condition, we must therefore have:

$$\{c_t, a\} \succ \{c_{t+1}, b\}$$

for any such t, which contradicts the definition of f_t provided by (12). Hence the increasing sequence $\{f_t\}$ is not convergent and must therefore be finite.

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