The Shapley Value as the Maximizer of Expected Nash Welfare (Extended Version)

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Abstract

In this paper we provide an alternative interpretation of the Shapley value in TU games as the unique maximizer of expected Nash welfare.

Keywords and Phrases: Shapley value, Nash Bargaining solution, Maximizing expected Nash welfare.

JEL Classification: C71, C78

1 Introduction

The Shapley value (Shapley (1953)) is a central concept in cooperative game theory. Consider an arbitrary TU game (N, v) where $N = \{1, ..., n\}$ is the set of players and v is a function which associates a real number $v(S)$ with every non-empty subset (or coalition) of N. The interpretation of the Shapley value in Shapley's own words is as follows: "The players in N agree to play the game v in a grand coalition, formed in the following way:
1. Starting with a single member, the coalition adds one player at a time until everyone has been admitted.
2. The order in which players are to join is determined by chance, with all arrangements equally probable.
3. Each player, on his admission, demands and is promised the amount which his adherence contributes to the valuation of the coalition (as determined by the function v). The grand

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coalition then plays the game "efficiently" so as to obtain $v(N)$ - exactly enough to meet all promises."
In other words, the Shapley value for a player i is the uniform average of all *marginal contributions* made by player i to the coalition of players which precede her in the randomly drawn order.

In this paper we propose an alternative interpretation of the Shapley value as a maximizer of expected welfare where the welfare function is exactly the one proposed by Nash in his celebrated paper on bargaining (Nash (1950)). Our interpretation is as follows: The grand coalition N has to decide, *ex-ante* how to divide the value of the grand coalition, $v(N)$. Suppose it proposes the division or pre-imputation $x \equiv (x_1, .., x_n)$ where $\sum_{i \in N} x_i = v(N)$. Following the division x, one of 2^N "states" occurs. Each state consists of a pair $(S, N \setminus S)$ where $S \subset N$ is a coalition, i.e. the set of players is split into two coalitions S and $N \setminus S$. Members within the two coalitions divide their proposed allocations *equally* amongst themselves so that each member of S gets $\frac{x(S)}{|S|}$ and each member of $N \setminus S$ gets $\frac{x(N \setminus S)}{|N \setminus S|}$ where $x(S) = \sum_{i \in S} x_i$ and $x(N \setminus S) = \sum_{i \in N \setminus S} x_i$ respectively. Welfare in state $(S, N \setminus S)$ is evaluated according to the Nash welfare function or the Nash product formula $\frac{(x(S)-v(S))(x(N \setminus S)-v(N \setminus S))}{|S||N-S|}$ with disagreement payoffs of $\frac{v(S)}{|S|}$ and $\frac{v(N-S)}{|N-S|}$ for coalitions S and $N \setminus S$ respectively. The probability of state $(S, N \setminus S)$ occurring is given by $p(S, N \setminus S) = \frac{|S|!|N \setminus S|!}{(|N|+1)!}$. This can be interpreted in the following way. In the first stage, an integer between $[0, |N|]$ is also chosen, with equal probability. Suppose s is picked. In the second stage, coalition S of size s is chosen with equal probability amongst all coalitions of size s. The ex-ante welfare associated with choosing the allocation x is given by $W(x) = \sum_{S \subset N} p(S, N \setminus S) \frac{(x(S)-v(S))(x(N \setminus S)-v(N \setminus S))}{|S||N \setminus S|}$. Our result is that Shapley value is the unique maximizer of $W(x)$ over all possible pre-imputations x.

In our interpretation therefore, players are randomly split into two coalitions (with specified probabilities) after deciding on a pre-imputation. Players within a coalition are treated identically. Welfare in this event is computed according to the Nash welfare function with the per-capita worths of the two coalitions serving as the disagreement payoffs [1]. Then the Shapley value uniquely maximizes expected welfare in the set of all pre-imputations. An equivalent (and almost identical) interpretation is that players believe that they will face a bilateral bargaining situation in the future where the "success" of their already agreed upon allocation will be measured by the Nash objective function. It is obvious that this interpretation of the Shapley value is completely different from the standard one based on marginal contributions. It also provides an interesting connection between two important but seemingly unrelated concepts, the Shapley value and the Nash bargaining solution.

A possible criticism of our approach is that the model underlying our interpretation is

[1] The Nash objective function has been used as a social welfare function. See for instance d'Aspremont and Gevers (2002).

ad-hoc - in particular, the bargaining problem is bilateral and the assumptions regarding the welfare function and probabilities are specific. We note that there is some literature connecting the Shapley value to bilateral bargaining problems (see below). However, more generally, the standard interpretation of the Shapley value as the average marginal contribution of a player is also arbitrary because the average is computed in a specific way and supported by a specific "story". We believe that our interpretation illustrates a surprising connection between a plausible bargaining model and the Shapley value - clearly more work is required to explore the relationship between various bargaining models and solution concepts.

Two papers related to ours are Evans (1996) and Ruiz et al. (1998) (henceforth, RVZ). Evans proposes a bargaining protocol similar to ours.[2] The players are split randomly into two groups and a representative player is chosen, again at random, from each group with equal probability. These two players then bargain with each other over how to split between them, the total resources available to the grand coalition. Out of the proceeds of this bargain, each representative has to pay each member of his own coalition according to a 'proposed' feasible allocation and can keep the rest for herself. Evans imposes a consistency condition according to which the proposed feasible allocation must equal the ex-ante expected payoff from this procedure. He shows that the Shapley value is the only consistent feasible allocation when the excess surplus is split equally (at the stage where the two representatives bargain).

RVZ propose a family of values obtained by minimizing the weighted variances of coalitional excesses on the set of allocations. This family yields the Shapley value for a particular choice of coalitional weights. The motivation and substance of the RVZ results (relating to the Shapley value) are quite different from ours. However, they are related in the sense that in both cases, the Shapley value is obtained as a solution to an optimization problem. Although the two optimization problems appear different, it can be shown that the RVZ objective function is a negative monotone transformation of our own.

A natural question is the following: what is the allocation obtained by maximizing the expected Nash welfare function $W(x)$ but using a general form of the probability function p? If $p(S, S \setminus N)$ depends only on the cardinalities of S and N, we obtain exactly the least square family, introduced by RVZ. This yields an alternative and attractive interpretation of the least squares family as the family of allocations arising from maximizing expected Nash welfare from various probability assessments regarding the coalitions which can form.

In Section 3, we formally state our result. We provide three alternative proofs of the result which we believe highlight different aspects of the result. The first proof demonstrates that the maximizer of our welfare function satisfies the Shapley axioms of Additivity, Anonymity and Dummy. The second proof is a direct computation while the third proof uses the RVZ result. In Section 4, we discuss the connection between expected Nash welfare maximization and the least squares family.

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[2]We are grateful to a Referee for bringing this paper to our attention.

2 Preliminaries

The set of players is denoted by $N = \{1, ..., n\}$. A transferable utility or TU game is a pair (N, v) where $v : 2^n \to \mathcal{R}$.The function v is the *value function*; for every coalition $S \subset N$, $v(S)$ is interpreted as the worth of S. We assume without loss of generality that $v(\emptyset) = 0$. Throughout the paper we shall hold the set of players constant, so that we shall denote a TU game simply by v. Note that v is a point in $\mathcal{R}^{2^n-1}$.

An *pre-imputation* x in the game v is a vector $(x_1, ...x_n) \in \mathcal{R}^n$ such that $\sum_{i\in N} x_i = v(N)$. The set of all pre-imputations in the game v is denoted by $X(v)$. An *allocation* ϕ is a mapping $\phi : \mathcal{R}^{2^n-1} \to \mathcal{R}^n$ such that $\phi(v) \in X(v)$ for all $v \in \mathcal{R}^{2^n-1}$.

Definition 1 *The Shapley value of the game v is defined as follows. For all $k \in N$,*

$$\phi_k^{Sh}(N, v) = \sum_{S\subset N:\, k\notin S} \frac{|S|!\,(|N \setminus S| - 1)!}{|N|!} \left[v(S \cup \{k\}) - v(S) \right]$$

The formula for the Shapley value makes it clear that, for each player k, it is the average of the marginal contributions of k to various coalitions S not containing k. Shapley (1953) characterizes ϕ^{Sh} as the unique allocation satisfying the axioms of *Additivity*, *Dummy* and *Anonymity*. We define these below.

Definition 2 *The allocation ϕ satisfies Additivity, if for all games $v, w \in \mathcal{R}^{2^n-1}$, we have $\phi(v + w) = \phi(v) + \phi(w)$.*

Definition 3 *Player k is a dummy in game v is $v(S\cup\{k\}) = v(S)$ for all coalitions $S \subset N$ such that $k \notin S$. The allocation ϕ satisfies the Dummy Property if $\phi_k(v) = 0$ whenever k is a dummy in v.*

Let $\sigma : N \to N$ be a permutation. For all games v, the game v^σ is defined as follows: for all $S \subset N$, $v^\sigma(S) = v(\{k \in N|\sigma(k) \in S\})$.

Definition 4 *The allocation ϕ satisfies Anonymity, if for all games v and permutations σ of the set N, we have $\phi_k(v) = \phi_{\sigma(k)}(v^\sigma)$.*

Let v be a game, let $x \in X(v)$ and let $\emptyset \neq S \subset N$. The *excess* corresponding to x at S, denoted by $e(x, S)$ is defined as $\sum_{i\in S} x_i - v(S)$. Note that $e(x, N) = 0$ by definition. The *average excess* associated with $x \in X(v)$, denoted by $\overline{e}(x, v)$ is defined as $\overline{e}(x, v) = \frac{1}{2^n-1} \sum_{S\subset N} e(x, S)$. A *weight function* on N is a map m which associates with every non-empty coalition S, a non-negative real number $m(S)$.

Definition 5 *(Ruiz et al. (1998)) The allocation ψ belongs to the least squares family if there exists a weight function m such that for all games v, $\psi(v)$ solves*

$$\min_{x \in X(v)} \sum_{S \subseteq N} m(S)(e(x,S) - \overline{e}(x,v))^2$$

RVZ show that there is a unique solution to the problem above. Moreover they show (Remark 14) that in the special case where the weight function is given by $m^{Sh}(|S|) = \frac{1}{|N|-1}\binom{|N|-2}{|S|-1}^{-1}$, the associated least square allocation coincides with the Shapley value, i.e. $\psi(v) = \phi^{Sh}(v)$ for all games v [3].

3 The Result

Let v be a game and let $x \in X(v)$. The *welfare* associated with x, denoted by $W(x,v)$ is given by

$$W(x,v) = \sum_{S \subset N,\, S \neq \emptyset} \left[p(S, N \setminus S) \, \frac{x(S) - v(S)}{|S|} \, \frac{x(N \setminus S) - v(N \setminus S)}{|N \setminus S|} \right] \tag{1}$$

where,

$$p(S, N \setminus S) = \frac{|S|! \, |N \setminus S|!}{(|N|+1)!} \tag{2}$$

As outlined in the Introduction, players propose a pre-imputation x after which one of 2^n states can occur. Each state is associated with a partition of the grand coalition into two coalitions where the state $(S, N \setminus S)$ occurs with probability $\frac{|S|! \, |N \setminus S|!}{(|N|+1)!}$. Players within each coalition divide their shares of the pre-imputation equally. Welfare in each state is evaluated according to the Nash welfare function; $W(x)$ is therefore the ex-ante Nash welfare associated with x.

Theorem 1 *$\phi^{Sh}(v) = argmax_{x \in X(v)} W(x,v)$ for all v.*

We give three proofs of the Theorem. However before we do, we observe that $W(x,v)$ is a strictly quasi-concave function in x. Since the constraint set is convex, it follows that the optimization problem has a unique solution. Moreover, the optimum is characterized by the first-order conditions.

First Proof: We will show that the argmax of the optimization problem satisfies Additivity, the Dummy Property and Anonymity. The result then follows immediately from Shapley's characterization result.

[3] They also attribute this result to Keane (1969).

Observe that $W(x,v)$ can be rewritten (upto a constant $\frac{1}{n+1}$) as

$$\left[\sum_{m=1}^{(n-1)} \sum_{k\in S:\,|S|=m} \frac{[(m-1)!\,(n-m-1)!]}{n!} \left(\sum_{i\in S} x_i - v(S)\right) \left(\sum_{i\in N\setminus S} x_i - v(N\setminus S)\right)\right]$$

The Lagrangean for the optimization problem is therefore,

$$\begin{aligned}\mathcal{L} &= \left[\sum_{m=1}^{(n-1)} \sum_{k\in S:\,|S|=m} \frac{[(m-1)!\,(n-m-1)!]}{n!} \left(\sum_{i\in S} x_i - v(S)\right) \left(\sum_{i\in N\setminus S} x_i - v(N\setminus S)\right)\right] \\ &+ \lambda(v(N) - \sum_{i\in N} x_i)\end{aligned}$$

The first-order condition with respect to x_k, $k = 1,..,n$ is given by

$$\begin{aligned}\lambda &= \sum_{m=1}^{(n-1)} \sum_{k\in S:\,|S|=m} \frac{(m-1)!\,(n-m-1)!}{n!} \left(\sum_{i\in N\setminus S} x_i - v(N\setminus S)\right) \\ &= \left[\sum_{m=1}^{(n-1)} \frac{(m-1)!\,(n-m-1)!}{n!} \sum_{k\in S:\,|S|=m} \sum_{i\in N\setminus S} x_i\right] \\ &\quad - \left[\sum_{m=1}^{(n-1)} \sum_{k\in S:\,|S|=m} \frac{(m-1)!\,(n-m-1)!}{n!} v(N\setminus S)\right]\end{aligned}$$

Note that in $\sum_{k\in S:\,|S|=m} \sum_{i\in N\setminus S} x_i$, each x_i appears $\binom{n-2}{m-1}$ times, where $i \neq k$. Thus,

$$\begin{aligned}&\sum_{m=1}^{(n-1)} \frac{(m-1)!\,(n-m-1)!}{n!} \sum_{k\in S:\,|S|=m} \sum_{i\in N\setminus S} x_i \\ &= \sum_{m=1}^{(n-1)} \frac{(m-1)!\,(n-m-1)!}{n!} \binom{n-2}{m-1} \sum_{i\neq k} x_i \\ &= \sum_{i\neq k} x_i \left[\sum_{m=1}^{(n-1)} \frac{(m-1)!\,(n-m-1)!}{n!} \binom{n-2}{m-1}\right] \\ &= \frac{1}{n} \sum_{i\neq k} x_i = \frac{1}{n}(v(N) - x_k)\end{aligned}$$

By replacing $(n-m)$ with t, the second term in the equation for λ above can be rewritten as

$$\sum_{t=1}^{(n-1)} \sum_{k\notin T:\,|T|=t} \frac{(n-t-1)!\,(t-1)!}{n!} v(T)$$

The first-order condition with respect to x_k, $k = 1,..,n$ can therefore be rewritten as

$$\lambda = \frac{1}{n}(v(N) - x_k) - \sum_{t=1}^{(n-1)} \sum_{k \notin T:\, |T|=t} \frac{(n-t-1)!\,(t-1)!}{n!}\, v(T) \tag{3}$$

The optimum is therefore characterized by the equations

$$\begin{aligned} &\frac{1}{n}(v(N) - x_j) - \sum_{t=1}^{(n-1)} \sum_{j \notin T:\, |T|=t} \frac{(n-t-1)!\,(t-1)!}{n!}\, v(T) \\ &= \frac{1}{n}(v(N) - x_k) - \sum_{t=1}^{(n-1)} \sum_{k \notin T:\, |T|=t} \frac{(n-t-1)!\,(t-1)!}{n!}\, v(T) \end{aligned} \tag{4}$$

for all $j, k = 1, ..., n$ together with the equation $\sum_{i \in N} x_i = v(N)$.

Let $\phi(v) = argmax_{x \in X(v)} W(x, v)$. We will show that ϕ satisfies Additivity.

Let $\{\hat{x}_i\}$, $i = 1, ..., n$ and $\{\bar{x}_i\}$, $i = 1..., n$ be solutions to the equation system 4 for the games v and w respectively. Adding these equations, we obtain immediately that

$$\begin{aligned} &\frac{1}{n}((v+w)(N) - \hat{x}_j + \bar{x}_j) - \sum_{t=1}^{(n-1)} \sum_{j \notin T:\, |T|=t} \frac{(n-t-1)!\,(t-1)!}{n!}\, (v+w)(T) \\ &= \frac{1}{n}((v+w)(N) - \hat{x}_k + \bar{x}_k) - \sum_{t=1}^{(n-1)} \sum_{k \notin T:\, |T|=t} \frac{(n-t-1)!\,(t-1)!}{n!}\, (v+w)(T) \end{aligned} \tag{5}$$

Since $\{\hat{x}_i + \bar{x}_i\}$, $i = 1, ..., n$ is a pre-imputation for the game $v + w$ it follows that $\{\hat{x}_i + \bar{x}_i\} = argmax_{x \in X(v+w)} W(x, v + w)$. Hence ϕ satisfies Additivity.

We now show that ϕ satisfies the Dummy Property. Suppose that player i is a dummy, i.e. $v(S \cup \{i\}) = v(S)$ for all $S \subset N$ such that $i \notin S$. Pick $k \neq i$. Consider a solution $\hat{x}_k$, $k = 1, .., n$ to equation 3, i.e.

$$\begin{aligned} \lambda &= \frac{1}{n}(v(N) - \hat{x}_k) - \sum_{t=1}^{(n-1)} \sum_{k \notin T:\, |T|=t} \frac{(n-t-1)!\,(t-1)!}{n!}\, v(T) \\ &= \frac{1}{n}(v(N) - \hat{x}_k) - \sum_{t=1}^{(n-2)} \sum_{k,i \notin T:\, |T|=t} \frac{(n-t-1)!\,(t-1)!}{n!}\, v(T) \\ &\quad - \sum_{t=1}^{(n-2)} \sum_{k \notin T, i \in T:\, |T|=t} \frac{(n-t-1)!\,(t-1)!}{n!}\, v(T) \end{aligned}$$

Since player i is a dummy player, $v(N) = v(N \setminus \{i\})$ and $v(T) = v(T \setminus \{i\})$ whenever $i \notin T$. Hence the equation above can be written as

$$\begin{aligned}\lambda &= \frac{1}{n}(v(N \setminus \{i\}) - \hat{x}_k) \\ &\quad - \sum_{t=1}^{(n-2)} \sum_{k,i \notin T:\, |T|=t} \left[\frac{(n-t-1)!\,(t-1)!}{n!} + \frac{(n-t-2)!\,t!}{n!}\right] v(T) \\ &= \frac{n-1}{n}\left[\frac{1}{(n-1)}(v(N \setminus 1) - x_k) - \sum_{t=1}^{(n-2)} \sum_{k \notin T \subseteq N \setminus \{i\}: |T|=t} \frac{(n-t-2)!\,(t-1)!}{(n-1)!}\, v(T)\right]\end{aligned}$$

It follows that, for all $j, k \neq i$,

$$\begin{aligned}&\frac{1}{n-1}(v(N \setminus \{i\}) - \hat{x}_j) - \sum_{t=1}^{(n-2)} \sum_{j \notin T \subseteq N \setminus \{i\}:\, |T|=t} \frac{(n-t-2)!\,(t-1)!}{n!}\, v(T) \\ &= \frac{1}{n-1}(v(N \setminus \{i\}) - \hat{x}_k) - \sum_{t=1}^{(n-2)} \sum_{k \notin T \subseteq N \setminus \{i\}:\, |T|=t} \frac{(n-t-2)!\,(t-1)!}{n!}\, v(T) \qquad (6)\end{aligned}$$

Equation 6 implies that $\hat{x}_k$, $k \neq i$ is a solution to the optimization problem with $n-1$ players with the value function v where $v(T) = v(T \cup \{i\})$ for all $T \subseteq N \setminus \{i\}$. Therefore $\sum_{k \in N \setminus \{i\}} x_k = v(N)$which implies $x_i = 0$. Therefore ϕ satisfies the Dummy Property.

The proof of the claim that ϕ is Anonymous follows from the fact that $W(x, v)$ is an anonymous function, i.e for all permutations $\sigma : N \to N$, $W(x^\sigma, v^\sigma) = W(x, v)$ for all games v and $x \in X(v)$ where x^σ is the vector obtained by permutaing the components of x according to σ.

Since ϕ satisfies Additivity, the Dummy Property and Anonymity, it must be the case that $\phi = \phi^{Sh}$. ∎

Second Proof: This is a proof by direct computation. By adding Equation 3 over all $k \in N$, we obtain

$$n\lambda = \left(v(N) - \frac{1}{n}\sum_{k \in N} x_k\right) - \sum_{k \in N} \sum_{t=1}^{(n-1)} \sum_{k \notin T:\, |T|=t} \left(\frac{(n-t-1)!\,(t-1)!}{n!}\, v(T)\right)$$

The first term here is simply $\frac{n-1}{n}\, v(N)$, because $\sum_{k \in N} x_k = v(N)$. Let us now simplify the second term.

$$\begin{aligned}
&\sum_{k\in N}\sum_{t=1}^{(n-1)}\sum_{k\notin T:\,|T|=t}\left(\frac{(n-t-1)!\,(t-1)!}{n!}\,v(T)\right)\\
&= \sum_{t=1}^{(n-1)}\sum_{k\in N}\sum_{k\notin T:\,|T|=t}\left(\frac{(n-t-1)!\,(t-1)!}{n!}\,v(T)\right)\\
&= \sum_{t=1}^{(n-1)}\sum_{T\subset N:|T|=t}\frac{(n-t)!\,(t-1)!}{n!}v(T)
\end{aligned}$$

The last equality follows from the fact that each $v(T)$ appears exactly $(n-t)$ times in $[\sum_{k\in N}\sum_{k\notin T:\,|T|=t} v(T)]$. Hence,

$$n\lambda = \frac{n-1}{n}v(N) - \sum_{t=1}^{(n-1)}\sum_{T\subset N:|T|=t}\frac{(n-t)!\,(t-1)!}{n!}v(T) \tag{7}$$

Now, by comparing Equation 3 with Equation 7, we get

$$\begin{aligned}
x_k &= \frac{1}{n}v(N) + \sum_{t=1}^{(n-1)}\sum_{T\subset N:|T|=t}\frac{(n-t)!\,(t-1)!}{n!}v(T)\\
&\quad -n\sum_{t=1}^{(n-1)}\sum_{k\notin T:\,|T|=t}\frac{(n-t-1)!\,(t-1)!}{n!}\,v(T)\\
&= \sum_{t=1}^{n}\sum_{T\subset N:|T|=t}\frac{(n-t)!\,(t-1)!}{n!}v(T) - n\sum_{t=1}^{(n-1)}\sum_{k\notin T:\,|T|=t}\frac{(n-t-1)!\,(t-1)!}{n!}\,v(T)\\
&= \sum_{t=1}^{n}\sum_{k\in T:|T|=t}\frac{(n-t)!\,(t-1)!}{n!}v(T)\\
&\quad -\sum_{t=1}^{(n-1)}\sum_{k\notin T:\,|T|=t}\left[n\frac{(n-t-1)!\,(t-1)!}{n!} - \frac{(n-t)!\,(t-1)!}{n!}\right]\,v(T)\\
&= \sum_{t=1}^{n}\sum_{k\in T:|T|=t}\frac{(n-t)!\,(t-1)!}{n!}v(T) - \sum_{t=1}^{(n-1)}\sum_{k\notin T:\,|T|=t}\frac{(n-t-1)!\,t!}{n!}\,v(T)\\
&= \sum_{t=1}^{n}\sum_{k\in T:|T|=t}\frac{(n-t)!\,(t-1)!}{n!}v(T) - \sum_{t=0}^{(n-1)}\sum_{k\notin T:\,|T|=t}\frac{(n-t-1)!\,t!}{n!}\,v(T)
\end{aligned}$$

The last equality follows from $v(\emptyset) = 0$. Rewriting the first term as a sum of $R \cup \{k\}$, where $k \notin R$, we get,

$$\sum_{t=1}^{n} \sum_{k \in T: |T|=t} \frac{(n-t)!\,(t-1)!}{n!} v(T) = \sum_{r=0}^{(n-1)} \sum_{k \notin R:\, |R|=r} \frac{(n-r-1)!\, r!}{n!}\, v(R)$$

Hence,

$$\begin{aligned} x_k &= \sum_{t=0}^{(n-1)} \sum_{k \notin T:\, |T|=t} \frac{(n-t-1)!\, t!}{n!} \left[v(T \cup k) - v(T) \right] \\ &= \phi_k^{Sh}(v) \end{aligned}$$

■

Third Proof: We show that maximizing $W(x, v)$ over the set $x \in X(v)$ is equivalent to solving the RVZ minimization problem with the appropriate weight function. In what follows, we write s for the cardinality of the coalition S.

$$\begin{aligned} W(x, v) &= \sum_{S \subset N,\, S \neq \emptyset} \left[p(S, N \setminus S)\, \frac{e(x, S)}{s}\, \frac{e(N \setminus S)}{n-s} \right] \\ &= \frac{1}{n^2} \sum_{S \subset N,\, S \neq \emptyset} \left[\frac{(s-1)!(n-s-1)!}{(n-1)!}\, e(S) e(N \setminus S) \right] \\ &= \frac{1}{n^2} \sum_{S \subseteq N} m^{Sh}(s)\, e(x, S) e(x, N \setminus S) \end{aligned}$$

The last equality follows from the fact that $e(x, N) = 0$ for all $x \in X(v)$ and the definition of $m^{Sh}(s)$. Note that $e(x, S) + e(x, N \setminus S) = v(N) - v(S) - v(N \setminus S)$ for all $x \in X(v)$. Let $\alpha(v) = v(N) - v(S) - v(N \setminus S)$. Thus,

$$\begin{aligned} W(x, v) &= \frac{1}{n^2} \sum_{S \subseteq N} m^{Sh}(s)\, e(x, S) e(x, N \setminus S) \\ &= \frac{1}{n^2} \sum_{S \subseteq N} m^{Sh}(s) [\alpha(v) e(x, S) - e(x, S)^2] \end{aligned}$$

However,

$$\sum_{S \subseteq N} m^{Sh}(s) \alpha(v) e(S) = \sum_{S \subseteq N} \alpha(v) m^{Sh}(s) \sum_{i \in S} x_i - \sum_{S \subseteq N} \alpha(v) m^{Sh}(s) v(S)$$

and

$$\begin{aligned}\sum_{S\subseteq N}\alpha(v)m^{Sh}(s)\sum_{i\in S}x_i &= \sum_{s=1}^{n}\alpha(v)m^{Sh}(s)\left[\sum_{\{T||T|=s\}}\sum_{i\in T}x_i\right]\\ &= \sum_{s=1}^{n}\alpha(v)m^{Sh}(s)\binom{n-1}{s-1}v(N)\end{aligned}$$

Thus $\sum_{S\subseteq N} m^{Sh}(s)\alpha(v)e(S)$ is a constant which depends only on v. Hence maximizing $W(x,v)$ over $x \in X(v)$ is equivalent to minimizing $\sum_{S\subseteq N} m^{Sh}(s)e(x,S)^2$ over $x \in X(v)$. The latter is easily seen to be equivalent to solving the RVZ problem, i.e. solving $\min_{x\in X(v)}\sum_{S\subseteq N} m(S)(e(x,S)-\overline{e}(x,v))^2$ ■

4 An Alternative Interpretation of the Least Squares Family

Our approach can also be used to provide a new and perhaps, more natural interpretation of the least squares family of allocations introduced by RVZ. Consider the welfare maximization problem $\max_{x\in X(v)} W(x,v)$ where

$$W(x,v)=\sum_{S\subset N,\,S\neq\emptyset}\left[p(S,N\setminus S)\,\frac{x(S)-v(S)}{|S|}\,\frac{x(N\setminus S)-v(N\setminus S}{|N\setminus S|}\right]$$

and $p(S,N\setminus S)$ is the probability of coalition S forming and depends only on the cardinality of S and N. Now consider the set of allocations obtained by ranging over all possible probability distributions or weight schemes p. We claim that this set of allocations is exactly the least squares family. In other words, every member of the least squares family is the solution to an expected Nash welfare maximization problem for some *beliefs* about coalition formation. Conversely every solution to an expected Nash welfare maximization problem for a probability distribution p, is a member of the least squares family.

In order to establish this claim, it can be verified that the first- order condition for the maximization here will be similar to Equation 4.

$$\begin{aligned}A(p)(v(N)-x_j)-\sum_{t=1}^{(n-1)}\sum_{j\notin T:\,|T|=t}\frac{p(T,N\setminus T)}{t\,(n-t)}\,v(T)\\ =A(p)(v(N)-x_k)-\sum_{t=1}^{(n-1)}\sum_{k\notin T:\,|T|=t}\frac{p(T,N\setminus T)}{t\,(n-t)}\,v(T)\end{aligned}\tag{8}$$

where $A(p)$ is a function of the weight scheme p. RVZ showed that the least squares family is characterized by the following axioms: Additivity, Symmetry, Inessential Game and Coalitional Monotonicity. Symmetry is a weaker version of Anonymity. The Inessential Game axiom says that if a game v is additive, that is $v(S) = \sum_{i \in S} v(\{i\})$, then an allocation ψ must allocate $v(\{i\})$ to agent i. Coalitional Monotonicity demands that if two games v and w are identical except $v(S) > w(S)$ for some $S \subseteq N$, then $\psi_i(v) \geq \psi_i(w)$ for all $i \in S$. It can be easily checked from Equation 8, that our weighted welfare maximizers also satisfy all these axioms. Hence, our family of welfare maximizing allocations is identical to the least squares family.

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