Problem Set 2

Due: March 9, 2017

Please submit in groups of three/four.

1. Let $A = [0, 1] \times [0, 1]$. Each voter i has the following preference ordering over A: she has a maximal element or peak $x_i \in A$ and for all $y, z \in A$, she prefers y to z if $d(x_i, y) \leq d(x_i, z)$. Here $d(a, b)$ is the Euclidean distance between a and b. Observe that a voter's preferences are described fully by her peak. A preference profile is therefore a probability distribution over the unit square. Assume that this distribution is uniform.

   (a) Let $B = A \cap \{(x, y) \in \Re^2 : x + y = 1.5\}$. Consider a voter whose peak does not lie on B. Are her preferences over alternatives in B, single-peaked?

   (b) Let $y = (0.7, 0.2)$ and $z = (0.8, 0.9)$. Suppose there is a majority vote between y and z. Which alternative would win?

   (c) Is there an alternative that would beat every other alternative in a pairwise majority contest?

2. The goal of this problem is to characterize $\succeq$ on $\Re^2$ when it satisfies Ordinal Measurability, Full Comparability (OMFC) and some additional axioms. Recall some definitions:

   OMFC: $\succeq$ satisfies OMFC if it satisfies invariance with respect to the the set $\Phi = \{(\phi_0, \phi_0, \ldots, \phi_0)$ for all strictly increasing maps $\phi_0 : \Re \to \Re\}$. Note that the *same* transformation applies to all agents.

   Anonymity (defined here only for two agents): $\succeq$ satisfies Anonymity (AN) if, for all $\alpha \equiv (\alpha_1, \alpha_2) \in \Re^2$, we have $(\alpha_1, \alpha_2) \sim (\alpha_2, \alpha_1)$. In other words, every vector is indifferent to its reflection with respect to the 45^0 line - names of individuals do not matter.

   Max and Min Dictatorships: $\succeq$ is a max dictatorship if, for all $\alpha, \beta \in \Re^2$, $[\max_i \alpha_i > \max_i \beta_i] \to [\alpha \succ \beta]$. It is a min dictatorship if, for all $\alpha, \beta \in \Re^2$, $[\min_i \alpha_i > \min_i \beta_i] \to [\alpha \succ \beta]$. Thus $(6, -10) \succ (2, 3)$ in a max dictatorship while the reverse is true in a min dictatorship.

   We are going to prove the following:

   Theorem: Let $n = 2$. If $\succeq$ satisfies WP, AN and OMFC, it is either a max dictatorship or a min dictatorship.

   Refer to the diagram. We shall try to draw the psuedo-social indifference curve passing through α. Note that $\bar{\alpha}$ is the reflection of α with respect to the 45^0 line.

(a) Show that WP and AN can be used to rank all Regions except IV, IX, VIII and X.

(b) Pick arbitrary β, γ in Region VIII (shown in the diagram). Prove that OMFC implies $\succeq|_{\alpha,\beta} = \succeq|_{\alpha,\gamma}$. (You will need to use the fact that $\alpha_1 > \beta_1 > \beta_2 > \alpha_2$ etc).

(c) Show that $\beta \sim \alpha$ is not possible.

(d) Prove analogous properties for Regions IV, IX and X.

(e) Show that if β is in Region VIII and γ is in Region X, they must be ranked in opposite ways with respect to α, i.e. $\beta \succ \alpha$ implies $\alpha \succ \gamma$ etc.

(f) What are the two possibilities for the pseudo-social indifference curve through α. Complete the proof of the Theorem.

3. Suppose $\succeq$ is an ordering on $\Re^n$ satisfying WP and the following property: for $\alpha, \beta \in \Re^n$, if $\sum_i \alpha_i = \sum_i \beta_i$, then $\alpha \sim \beta$. Prove that if $\sum_i \alpha_i > \sum_i \beta_i$, then $\alpha \succ \beta$.