Problem Set 1

Individual and Collective Choice

Due: February 6, 2017

Please submit your answers in groups of three/four.

In the Questions 1 and 2 below, assume that voter orderings over the set A are *strict* or *antisymmetric*, i.e there are no "ties". Formally, $[aR_ib$ and $bR_ia] \Rightarrow [a = b]$. Also recall the following definitions.

- Copeland Rule: The Copeland score of an alternative a at profile R, $C(a, R)$ is the number of alternatives $b \neq a$ that a beats in a majority contest. The social ordering at R ranks alternatives according to their Copeland scores at R.

- Kramer Rule: The Kramer score of an alternative a at profile R, $K(a, R)$ is the size of the maximal majority against a at R, i.e. $K(a, R) = \max_{x \neq a} |\{i \in N | xR_ia\}|$. The Kramer social ordering ranks a above b if $K(a, R) \leq K(b, R)$.

- The French Presidential Rule (Plurality with Runoffs): Pick the two alternatives with the highest plurality score. In case there are more than two alternatives with the same (highest) Plurality score, break ties according to a fixed ranking of alternatives, i.e. pick the two alternatives that are ranked highest on the list. The winner is the outcome that is the majority winner among these two alternatives. In order to construct a social ordering, we can repeat the process after eliminating the winner etc.

1. Let $A = \{a, b, c\}$ and $N = \{1, 2, 3, 4, 5\}$. Show that the following rules violate Independence of Irrelevant Alternatives (a) Plurality (b) Anti-Plurality (c) Borda Count (d) Copland Rule (e) Kramer Rule.

2. We describe a property called *elementary monotonicity* of an ASWF F. Pick and arbitrary pair of alternatives and a profile R. Assume $aF(R)b$. Suppose a rises (or remains fixed) in the ranking of all voters while leaving the relative ranking of all other alternatives unchanged. Similarly, suppose b falls (or remains fixed) in the ranking of all voters while leaving the relative ranking of all other alternatives unchanged. Call the new profile R'. Elementary monotonicity is satisfied if $aF(R')b$.

   (a) Do scoring methods satisfy elementary monotonicity?

   (b) The French Presidential rule?

3. Let $A = \{a, b, c\}$ and $N = \{1, \ldots, n\}$. Assume that agents have strict orderings, i.e. there are six orderings $abc, cab, bac, bac, cab, cba$. Here I use abc as shorthand for the ordering a is strictly better than b is strictly better than c etc. Label these orderings 1 through 6. We are going to consider *anonymous* ASWFs. Therefore we can represent a profile as a six-tuple of real numbers $\{x_j\}$, $j = 1, \ldots 6$ where x_j is the proportion of voters who have ordering j. Hence $x_j \geq 0$ and $\sum_j x_j = 1$.

   (a) Characterize the profiles (in terms of linear inequalities) where a is the (i) Plurality winner (ii) Borda winner.

   (b) Characterize the profiles where a Condorcet cycle exists.

   (c) Characterize the profiles where a is the Condorcet loser, i.e. alternatives b and c are majority winners over a.

   (d) Is it possible for a to be a Borda winner *and* a Condorcet loser?

4. Let $|A| = m$. Show that if there exists a non-dictatorial ASWF satisfying IIA and WP when $m > 3$, there also exists a non-dictatorial ASWF satisfying IIA and WP when $m = 3$. In other words, if we want to prove Arrow's Theorem, we can start directly by assuming that there are exactly three alternatives.

5. Lt $|N| = n$. Show that if there exists a non-dictatorial ASWF satisfying IIA and WP when $n > 2$, there also exists a non-dictatorial ASWF satisfying IIA and WP when $n = 2$. In other words, if we want to prove Arrow's Theorem, we can start directly by assuming that there are exactly two voters.

6. In this question, we are going to prove a generalized version of Arrow's Theorem where we replace WP by a weaker axiom.

   The ASWF F satisfies *Non-Imposition* (NI) if, for all $a, b \in A$, there exists a profile R such that $aF(R)b$.

   The ASWF F is *null* if, for all $a, b \in A$ and profiles R, we have $a\bar{F}(R)b$, i.e. all alternatives are indifferent to each other socially, at all profiles.

   The ASWF F is *anti-dictatorial*, if there exists a voter i such that for all alternatives a, b and all profiles R, $aP_ib \Rightarrow b\hat{F}(R)a$.

   We want to prove the following result.

   (Wilson's) Theorem: Let $|A| \geq 3$. If a ASWF satisfies NI and IIA, it is either null or dictatorial or anti-dictatorial.

   (a) Show that WP implies NI.

   (b) Suppose F satisfies NI and IIA. Fix $a, b \in A$. We say $PO(a, b)$ if, for all profiles R, $[aP_ib\ \forall i] \Rightarrow [a\hat{F}(R)b]$. We say $APO(a, b)$ if for all profiles R, $[aP_ib\ \forall i] \Rightarrow [b\hat{F}(R)a]$. Show that for all $a, b, x, y \in A$ (i) $PO(a, b) \Rightarrow PO(x, y)$ and $APO(a, b) \Rightarrow APO(x, y)$. (Try and mimic the Field Expansion Lemma arguments. There are several cases as in FE. It is enough to show a few cases explicitly and indicate how to proceed in the others).

   (c) Suppose F satisfies NI and IIA. Show that one of the following must be true (i) F is null (ii) there exists a pair a, b such that $PO(a, b)$ (iii) there exists a pair a, b such that $APO(a, b)$. (This may be slightly hard. In case you have difficulties, assume this is true and proceed to (d) below.)

   (d) Prove the Theorem.