LINEAR MODELS AND GLM: ASSIGNMENT 1

Exercise 1. *(2 points) Linear models are called "linear" because they are linear in the parameters* $\boldsymbol{\beta}$*, or more technically, the expected value* $E(\boldsymbol{y} \mid \boldsymbol{\beta})$ *as a function of* $\boldsymbol{\beta}$ *is linear in* $\boldsymbol{\beta}$*. Make this statement more precise and prove it.*

A model $\{P_{\boldsymbol{\theta}},\ \boldsymbol{\theta} \in \Theta\}$ is said to be *identifiable* if for any $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \Theta$, $\boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2 \implies P_{\boldsymbol{\theta}_1} \neq P_{\boldsymbol{\theta}_2}$. We have already seen that the one-way model is not identifiable.

Exercise 2. *(2 points) Consider the model given by* $\mathcal{N}(\theta, 1), \theta \in \mathbb{R}$*. Show that the model is identifiable; that is, if* $\theta_1 \neq \theta_2$*, then the distributions defined by them are different.*

Some basic linear algebra concepts reviewed

A set of vectors $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$ is *linearly dependent* if there exist coefficients $c_1, \ldots, c_n$, not all zero, such that $\sum_i c_i \boldsymbol{x_i} = \boldsymbol{0}$. This can be rephrased in matrix notation as follows: the columns of a matrix $\boldsymbol{X}$ are said to be *linearly dependent* if there exists $\boldsymbol{c} \neq \boldsymbol{0}$ such that $\boldsymbol{Xc} = \boldsymbol{0}$.
A set of vectors $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$ (the columns of matrix $\boldsymbol{X}$) is *linearly independent* if

$$\sum_i c_i \boldsymbol{x_i} = \boldsymbol{Xc} = \boldsymbol{0} \implies \boldsymbol{c} = \boldsymbol{0}$$

In $\mathbb{R}^d$, no more than d vectors can be linearly independent.
Two vectors are *orthogonal* if their inner product is zero, i.e.,

$$\boldsymbol{x}'\boldsymbol{y} = \boldsymbol{y}'\boldsymbol{x} = \sum_i x_i y_i = 0$$

A set of vectors is *mutually orthogonal* if they are pairwise orthogonal. If the columns of a matrix $\boldsymbol{Q}$ are orthogonal, then $\boldsymbol{Q}'\boldsymbol{Q}$ is a diagonal matrix. In addition, if the columns of $\boldsymbol{Q}$ are of unit length ($\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}'\boldsymbol{y}} = 1$), then $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}$, and $\boldsymbol{Q}$ is said to be *orthonormal.*

Exercise 3. *(2 points) Show that a set of mutually orthogonal vectors is linearly independent.*

A *vector space* is a set of vectors closed under addition and scalar multiplication.
The *span* of a set of vectors $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$ is the set of all linear combinations

$$span\{\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\} = \left\{ \boldsymbol{x} \mid \boldsymbol{x} = \sum_i c_i \boldsymbol{x_i} \text{ for some } c_i, i = 1, \ldots, n \right\}$$

$span\{\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\}$ is a vector space, and is said to be generated by $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$.
A *basis* for a vector space $\mathcal{V}$ is a set of linearly independent vectors $\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}$ that generate $\mathcal{V}$ (i.e., $\mathcal{V} = span\{\boldsymbol{x_1}, \ldots, \boldsymbol{x_n}\}$).
The *dimension* of a vector space is the number of vectors in a basis for the vector space.
Dimension is unique, although basis is not. The set of elementary vectors $\{\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}\}$ is an orthonormal basis for $\mathbb{R}^n$, where $\boldsymbol{e_i} \in \mathbb{R}^n$ has i^{th} element 1 and all other elements 0.
The *rank* of a matrix $\boldsymbol{X}$, denoted by $rank(\boldsymbol{X})$, is the number of linearly independent columns (or rows). $\boldsymbol{X}$ has full column rank if rank equals the number of columns (similarly full row rank).

Date: 8 January, 2010.

The *column space* of a matrix $\boldsymbol{X}$, denoted by $\mathcal{C}(\boldsymbol{X})$, is the span of the columns of $\boldsymbol{X}$; that is, $\mathcal{C}(\boldsymbol{X}) = \{\boldsymbol{z} \mid \boldsymbol{z} = \boldsymbol{X}\boldsymbol{y} \text{ for some } \boldsymbol{y}\}$.
$dim(\mathcal{C}(\boldsymbol{X})) = rank(\boldsymbol{X})$, the number of linearly independent columns.

Exercise 4. *(2 points) Show that* $rank(\boldsymbol{AB}) \leq \min(rank(\boldsymbol{A}), rank(\boldsymbol{B}))$

Exercise 5. *(3 points) Show that* $\mathcal{C}(\boldsymbol{A}) \subseteq \mathcal{C}(\boldsymbol{B}) \iff \boldsymbol{A} = \boldsymbol{BC}$ *for some* $\boldsymbol{C}$.

The *null space* or *kernel* of a matrix $\boldsymbol{A}$ is $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{x} \mid \boldsymbol{Ax} = \boldsymbol{0}\}$

Exercise 6. *(0 points) Show that if* $\boldsymbol{A}$ *has full column rank, then* $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{0}\}$.

Vector spaces $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}^n$ are said to form *orthogonal complements* if (1) $\mathcal{U} \cap \mathcal{V} = \{\boldsymbol{0}\}$, (2) $\dim(\mathcal{U}) + \dim(\mathcal{V}) = n$, and (3) $\boldsymbol{u} \perp \boldsymbol{v}$ for all $\boldsymbol{u} \in \mathcal{U}, \boldsymbol{v} \in \mathcal{V}$. Then, we write $\mathcal{U} \oplus^\perp \mathcal{V} = \mathbb{R}^n$.

Exercise 7. *(4 points) Show that if* $\mathcal{U} \oplus^\perp \mathcal{V} = \mathbb{R}^n$, *then any* $\boldsymbol{x} \in \mathbb{R}^n$ *can be written as* $\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{v}$ *where* $\boldsymbol{u} \in \mathcal{U}, \boldsymbol{v} \in \mathcal{V}$, *and the decomposition is unique. Note that for such a decomposition,* $\|\boldsymbol{x}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$.

Exercise 8. *(6 points) Prove that for any matrix* $\boldsymbol{A}_{m\times n}$, $\dim(\mathcal{N}(\boldsymbol{A})) + \dim(\mathcal{C}(\boldsymbol{A})) = n$.

Exercise 9. *(3 points) Show that for any matrix* $\boldsymbol{A}_{m\times n}$, $\mathcal{C}(\boldsymbol{A})$ *and* $\mathcal{N}(\boldsymbol{A}')$ *are orthogonal complements in* $\mathbb{R}^m$.

Exercise 10. *(3 points) Prove the following results:*

- $\mathcal{U}, \mathcal{V}$ *vector spaces,* $\mathcal{U} \subseteq \mathcal{V}, \dim(\mathcal{U}) = \dim(\mathcal{V}) \implies \mathcal{U} = \mathcal{V}$.
- $\boldsymbol{Ax} + \boldsymbol{b} = \boldsymbol{0} \ \forall \ \boldsymbol{x} \in \mathbb{R}^n \implies \boldsymbol{A} = \boldsymbol{0}, \boldsymbol{b} = \boldsymbol{0}$.
- $\boldsymbol{Ax} = \boldsymbol{Bx} \ \forall \ \boldsymbol{x} \implies \boldsymbol{A} = \boldsymbol{B}$.
- *If* $\boldsymbol{A}$ *has full column rank, then* $\boldsymbol{AB} = \boldsymbol{AC} \implies \boldsymbol{B} = \boldsymbol{C}$.

Exercise 11. *(3 points) Show that* $\mathcal{N}(\boldsymbol{X}'\boldsymbol{X}) = \mathcal{N}(\boldsymbol{X})$, *and as a corollary, that* $\mathcal{C}(\boldsymbol{X}'\boldsymbol{X}) = \mathcal{C}(\boldsymbol{X}')$.

Exercise 12. *(3 points) The goal of this exercise is to prove the existence of a g-inverse for an arbitrary matrix* $\boldsymbol{A}$ *by constructing one. Let* $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *be nonsingular matrices such that* $\boldsymbol{BAC} = \boldsymbol{\Delta}$ *is a diagonal matrix (not necessarily square). Find a candidate* $\boldsymbol{\Delta}^-$. *Prove that* $\boldsymbol{C\Delta}^-\boldsymbol{B}$ *is a g-inverse of* $\boldsymbol{A}$.