Collinearity: Impact and Possible Remedies

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What is collinearity?

- Exact dependence between columns of $\mathbf{X}$ make coefficients non-estimable
- Collinearity refers to the situation where some columns are *almost* dependent
- Why is this a problem?
- Individual coefficient estimates $\hat{\beta}_j$ become unstable (high variance)
- Standard errors are large, tests have low power
- On the other hand, $\hat{\mathbf{y}} = \mathbf{H}\mathbf{y}$ is not particularly affected

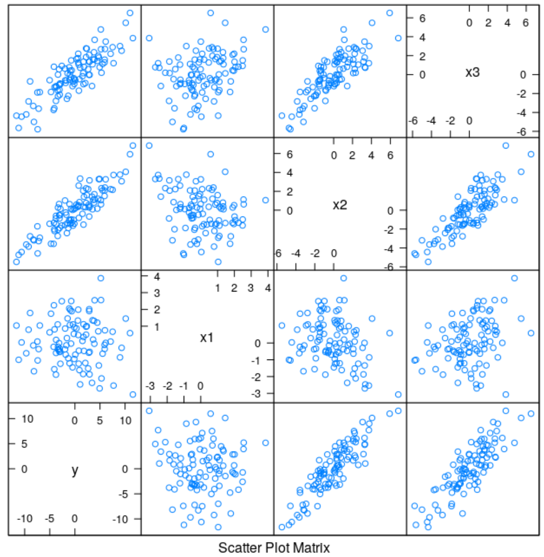
Detecting collinearity

- Collinearity in pairs of variables are easily seen in scatter plots
- However, higher dimensional collinearity may not be readily apparent
- Example:

```
n <- 100
z1 <- rnorm(n)
z2 <- rnorm(n)
x1 <- z1 + z2 + 0.1 * rnorm(n)
x2 <- z1 - 2 * z2 + 0.1 * rnorm(n)
x3 <- 2 * z1 - z2 + 0.1 * rnorm(n)
y <- x1 + 2 * x2 + 2 * rnorm(n) # x3 has coefficient 0
d3 <- data.frame(y, x1, x2, x3)
cor(d3)
```

```
             y          x1         x2        x3
y   1.00000000 -0.05350867  0.8930301 0.8498399
x1 -0.05350867  1.00000000 -0.2750082 0.3047524
x2  0.89303013 -0.27500823  1.0000000 0.8287638
x3  0.84983991  0.30475236  0.8287638 1.0000000
```

```
splom(d3)
```



- In this case, a 3-D plot is sufficient (but not enough for higher-dimensional collinearity)

```
library(rgl); with(d3, plot3d(x1, x2, x3, type = "s", col = "red", size = 1))
```

- Pairwise scatter plots do not indicate unusual dependence
- However, each X_{*j} is highly dependent on others

```
summary(lm(x1 ~ x2 + x3, d3))$r.squared
```

```
[1] 0.9816998
```

```
summary(lm(x2 ~ x1 + x3, d3))$r.squared
```

```
[1] 0.9936826
```

```
summary(lm(x3 ~ x1 + x2, d3))$r.squared
```

```
[1] 0.9938004
```

Impact of collinearity

- This results in increased uncertainty in coefficient estimates

```
summary(fm3 <- lm(y ~ x1 + x2 + x3, d3))
```

```

Call:
lm(formula = y ~ x1 + x2 + x3, data = d3)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.5620 -1.3326 -0.0007  1.6717  4.5924 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)  
(Intercept)  0.04719    0.20629   0.229   0.8195  
x1           1.09743    1.12399   0.976   0.3313  
x2           2.38916    1.12119   2.131   0.0357 *
```

```
x3          -0.33411    1.12900  -0.296   0.7679
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.042 on 96 degrees of freedom
Multiple R-squared:  0.8376,	Adjusted R-squared:  0.8325 
F-statistic:   165 on 3 and 96 DF,  p-value: < 2.2e-16
```

- Even though overall regression is highly significant, individual predictors are (marginally) not
- The situation changes dramatically if any one of the predictors is dropped

```
summary(lm(y ~ x2 + x3, d3)) # incorrect model, but still high R^2

Call:
lm(formula = y ~ x2 + x3, data = d3)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.8389 -1.3677  0.0208  1.7567  4.4134 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.07408    0.20439   0.362    0.718    
x2           1.30556    0.15921   8.200 1.01e-12 ***
x3           0.75725    0.15882   4.768 6.54e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.041 on 97 degrees of freedom
Multiple R-squared:  0.836,	Adjusted R-squared:  0.8326 
F-statistic: 247.1 on 2 and 97 DF,  p-value: < 2.2e-16
```

- The correct model (dropping `x3`, whose true coefficient is 0) performs equally well (not better)

```
summary(fm2 <- lm(y ~ x1 + x2, d3)) # correct model, will use later

Call:
lm(formula = y ~ x1 + x2, data = d3)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.6542 -1.3492  0.0212  1.7051  4.5374 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.05486    0.20369   0.269    0.788    
x1           0.76811    0.15740   4.880 4.16e-06 ***
x2           2.05850    0.09225  22.314  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.032 on 97 degrees of freedom
Multiple R-squared:  0.8374,	Adjusted R-squared:  0.8341 
F-statistic: 249.8 on 2 and 97 DF,  p-value: < 2.2e-16
```

- The difference is reflected in the estimated variance-covariance matrix of $\hat{\beta}$

```
cov2cor(vcov(fm3))[-1, -1] # exclude intercept
```

```
           x1         x2         x3
x1  1.0000000  0.9898617 -0.9900518
x2  0.9898617  1.0000000 -0.9965770
x3 -0.9900518 -0.9965770  1.0000000
```

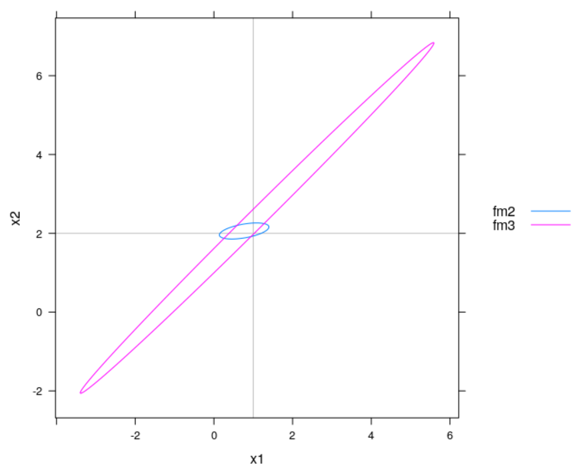
```
cov2cor(vcov(fm2))[-1, -1] # exclude intercept
```

```
          x1        x2
x1 1.0000000 0.2750082
x2 0.2750082 1.0000000
```

- The situation is more clearly seen in the confidence ellipsoids for $\hat{\beta}$

```
C3 <- chol(vcov(fm3)[2:3, 2:3]) # only x1 and x2
C2 <- chol(vcov(fm2)[2:3, 2:3])
tt <- seq(0, 1, length.out = 101)
circle <- rbind(2 * cos(2 * pi * tt), sin(2 * pi * tt))
E3 <- coef(fm3)[2:3] + 2 * t(C3) %*% circle
E2 <- coef(fm2)[2:3] + 2 * t(C2) %*% circle
E <- as.data.frame(rbind(t(E2), t(E3))); E$model <- rep(c("fm2", "fm3"), each = 101)
```

```
xyplot(x2 ~ x1, data = E, groups = model, abline = list(v = 1, h = 2, col = "grey"), type = "l",
       aspect = "iso", auto.key = list(lines = TRUE, points = FALSE, space = "right"))
```



- Many different $(\beta_1, \beta_2, \beta_3)$ combinations give essentially equivalent fit

Variance inflation factor

- It can be shown that the sampling variance of $\hat{\beta}_j$ is

$$V(\hat{\beta}_j) = \sigma^2 (\mathbf{X}^T\mathbf{X})^{-1}_{jj} = \frac{1}{1 - R_j^2} \times \frac{\sigma^2}{(n-1)s_j^2}$$

- where
  - $s_j^2 = \frac{1}{n-1} \sum_i (X_{ij} - \bar{X}_j)^2$ (sample variance of X_{*j})

- R_j^2 is the multiple correlation coefficient of X_{*j} on the remaining columns of $\mathbf{X}$

- The Variance Inflation Factor (VIF) is defined as

$$VIF_j = \frac{1}{1 - R_j^2}$$

- VIF_j directly reflects the effect of collinearity on the precision of $\hat{\beta}_j$

- Length of the confidence interval for $\hat{\beta}_j$ is proportional to $\sqrt{V(\hat{\beta}_j)}$, so more useful to compare $\sqrt{VIF_j}$

```
library(car)
sqrt(vif(fm3))
```

```
       x1        x2        x3 
 7.392177 12.581416 12.700439 
```

```
sqrt(vif(fm2))
```

```
      x1       x2 
1.040104 1.040104 
```

- For a more intuitive justification, recall partial regression of
  - residuals from regression of $\mathbf{y}$ on $\mathbf{X}_{(-j)}$, and
  - residuals from regression of $\mathbf{X}_{*j}$ on $\mathbf{X}_{(-j)}$

- $\hat{\beta}_j$ from this partial regression is the same as $\hat{\beta}_j$ from the full model

- In presence of collinearity, residuals from regression of $\mathbf{X}_{*j}$ on $\mathbf{X}_{(-j)}$ will have very low variability

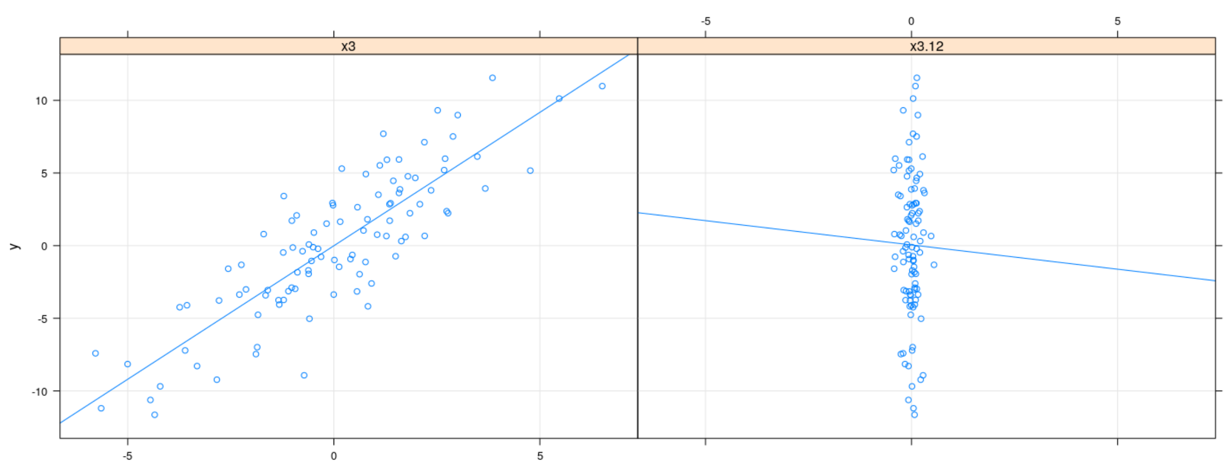
```
sd(d3$x3)
```

```
[1] 2.308524
```

```
sd(x3.12 <- residuals(lm(x3 ~ x1 + x2, d3)))
```

```
[1] 0.1817672
```

```
xyplot(y ~ x3 + x3.12, data = d3, outer = TRUE, grid = TRUE, xlab = NULL, type = c("p", "r"))
```



- Resulting $\hat{\beta}_j$ is highly unstable

Coping with collinearity

- There is no real solution if we want to estimate individual coefficients
- Recall interpretation of β_j: increase in $E(y)$ for unit increase in x_j keeping *other covariates fixed*
- For collinear data, this cannot be reliably estimated
- However, there are several approaches to "stabilize" the model

Approaches to deal with collinearity

- Variable selection:
  - Use criteria such as AIC and BIC in conjunction with stepwise / all-subset search
  - As discussed earlier, this is usually a misguided approach
  - In presence of collinearity, choice of model very sensitive to random error
- Respecify model: perhaps combine some predictors
- Principal component analysis (PCA):
  - An automated version of model respecification
  - Linearly transform covariates to make them orthogonal
  - Reduce dimension of covariate space by dropping "unimportant" variables
- Penalized regression:
  - Add some sort of penalty for "unlikely" estimates of β (e.g., many large components)
  - This is essentially a Bayesian approach
  - Results in biased estimates, but usually much more stable
  - For certain kinds of penalties, also works well as a variable selection mechanism

Standardization

- We will briefly discuss principal components and penalized regression
- Both these approaches have a practical drawback: they are not invariant to variable rescaling
- Recall that for linear regression, location-scale changes of covariates does not change fitted model
- This is no longer true if we used PCA or penalized regression
- There is no real solution to this problem: usual practice is to standardize all covariates
- Specifically, subtract mean, divide by standard deviation (so covariates have mean 0, variance 1)
- For prediction, the same scaling must be applied to new observations
- Can use R function `scale()` which also returns mean and SD for subsequent use
- More details later as necessary

Principal components

- Will be studies in more details in Multivariate Analysis course
- In what follows, the intercept is not considered as a covariate
- Let $\mathbf{z}_j$ denote the j-th covariate (column on $\mathbf{X}$) *after* standardization

- This means that the length of each $\mathbf{z}_j$ is $\|\mathbf{z}_j\| = \sqrt{\sum_i z_{ij}^2} = n - 1$
- This is not necessary for PCA, but is usually not meaningful
- Consider the matrix $\mathbf{Z} = [\mathbf{z}_1\, \mathbf{z}_2\, \cdots\, \mathbf{z}_k]$
- Suppose the rank of $\mathbf{Z}$ is p; for our purposes, $p = k$
- Our goal is to find $\mathbf{W} = [\mathbf{w}_1\, \mathbf{w}_2\, \cdots\, \mathbf{w}_p] = \mathbf{ZA}$ such that
  - $\mathcal{C}(\mathbf{W}) = \mathcal{C}(\mathbf{Z})$
  - Columns of $\mathbf{W}$ are mutually orthogonal
  - The first principal component $\mathbf{w}_1$ has the largest variance among linear combinations of columns of $\mathbf{Z}$
  - The second principal component $\mathbf{w}_2$ has the largest variance among linear combinations of columns of $\mathbf{Z}$ that are orthogonal to $\mathbf{w}_1$
  - ... and so on
- More precisely, we only consider *normalized* linear combinations $\mathbf{Za}$ such that $\|\mathbf{a}\|^2 = \mathbf{a}^T\mathbf{a} = 1$
- Otherwise, the variance of $\mathbf{Za}$ can be made arbitrarily large
- Note that by construction any linear combination of $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ has mean 0
- The variance of any such $\mathbf{w} = \mathbf{Za}$ is given by

$$s^2(\mathbf{a}) = \frac{1}{n-1}\mathbf{w}^T\mathbf{w} = \frac{1}{n-1}\mathbf{a}^T\mathbf{Z}^T\mathbf{Za} = \mathbf{a}^T\mathbf{Ra}$$

- where $\mathbf{R} = \frac{1}{n-1}\mathbf{Z}^T\mathbf{Z}$ is the correlation matrix of the *original* predictors $\mathbf{X}$
- We can maximize $s^2(\mathbf{a})$ subject to the constraint $\mathbf{a}^T\mathbf{a} = 1$ using a Lagrange multiplier:

$$F = \mathbf{a}^T\mathbf{Ra} - \lambda(\mathbf{a}^T\mathbf{a} - 1)$$

- Differentiating w.r.t. $\mathbf{a}$ and λ and equating to 0, we get

$$\begin{aligned} \frac{\partial F}{\partial \mathbf{a}} &= 2\mathbf{Ra} - 2\lambda\mathbf{a} = 0 \quad \implies \quad \mathbf{Ra} = \lambda\mathbf{a} \\ \frac{\partial F}{\partial \lambda} &= -(\mathbf{a}^T\mathbf{a} - 1) = 0 \quad \implies \quad \mathbf{a}^T\mathbf{a} = 1 \end{aligned}$$

- In other words, potential solutions are the normalized eigenvectors of $\mathbf{R}$
- Which of these k solutions maximizes $s^2(\mathbf{a})$?
- For any solution $\mathbf{a}$, the variance $s^2(\mathbf{a}) = \mathbf{a}^T\mathbf{Ra} = \lambda\,\mathbf{a}^T\mathbf{a} = \lambda$
- So the first principal component is given by the eigenvector $\mathbf{a}_1$ corresponding to the largest eigenvalue λ_1
- Let the eigenvalues in decreasing order be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$
- Not surprisingly, the principal components are given by the corresponding eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$
- The desired transformation matrix $\mathbf{A}$ is given by $\mathbf{A} = [\mathbf{a}_1\, \mathbf{a}_2\, \cdots\, \mathbf{a}_k]$
- As the eigenvectors are normalized, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$
- The variance-covariance matrix of the principal components $\mathbf{W}$ is

$$\frac{1}{n-1}\mathbf{W}^T\mathbf{W} = \frac{1}{n-1}\mathbf{A}^T\mathbf{Z}^T\mathbf{Z}\mathbf{A} = \mathbf{A}^T\mathbf{R}\mathbf{A} = \mathbf{A}^T\mathbf{A}\mathbf{\Lambda} = \mathbf{\Lambda}$$

- Here $\mathbf{\Lambda}$ is the diagonal matrix with entries $\lambda_1, \lambda_2, \ldots, \lambda_k$
- A general indicator of the degree of collinearity present in the covariates is the *condition number*

$$K \equiv \sqrt{\frac{\lambda_1}{\lambda_k}}$$

- Large condition number indicates that small changes in data can cause large changes in $\hat{\beta}$
- In theory, using (all) principal components as covariates leads to the same fit (i.e., same $\mathbf{H}$, $\hat{\mathbf{y}}$, etc.)
- To "stabilize" collinearity, we can instead regress on the first few principal components

Principal components in R

```
pr <- prcomp(~ x1 + x2 + x3, data = d3, scale. = TRUE)
pr
```

```
Standard deviations (1, .., p=3):
[1] 1.35254256 1.08071572 0.05178952

Rotation (n x k) = (3 x 3):
           PC1        PC2        PC3
x1 -0.02887707 -0.9244271  0.3802640
x2 -0.70174549  0.2896624  0.6508832
x3 -0.71184225 -0.2480529 -0.6570771
```

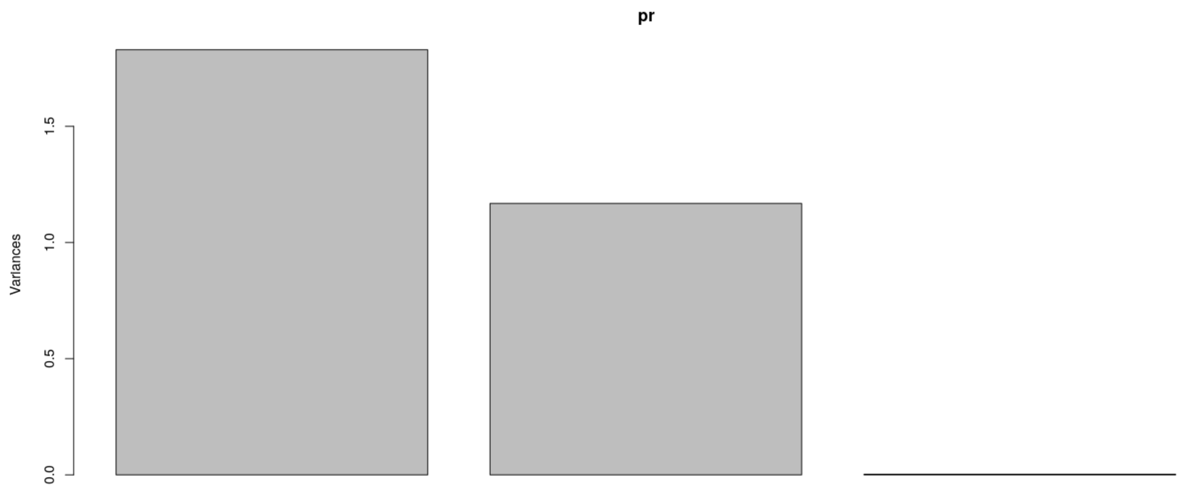
```
head(pr$x)
```

```
         PC1           PC2          PC3
1  1.1023871 -0.0005434015  0.072144023
2  0.1583731 -1.9368798962 -0.023836194
3 -0.8266643 -0.1014248304 -0.088178195
4  0.3620938  0.6099882730 -0.020273616
5  1.2346433  0.2956850245 -0.023914456
6  1.7477325  1.2684577099 -0.002962102
```

```
d3 <- cbind(d3, pr$x)
```

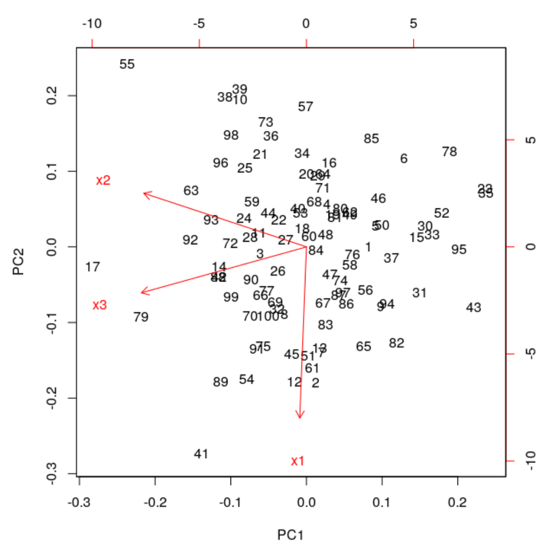
Principal components in R: scree plot

```
plot(pr)
```



Principal components in R: biplot

```
biplot(pr)
```



Principal components are orthogonal

Principal component regression

```
summary(fm.pc3 <- lm(y ~ PC1 + PC2 + PC3, data = d3))
```

```
Call:
```

```
lm(formula = y ~ PC1 + PC2 + PC3, data = d3)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.5620 -1.3326 -0.0007  1.6717  4.5924 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.04883    0.20419   0.239    0.811    
PC1         -3.35460    0.15173 -22.110   <2e-16 ***
PC2          0.41578    0.18989   2.190    0.031 *  
PC3          4.65106    3.96249   1.174    0.243    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.042 on 96 degrees of freedom
Multiple R-squared:  0.8376,	Adjusted R-squared:  0.8325 
F-statistic:   165 on 3 and 96 DF,  p-value: < 2.2e-16
```

```
zapsmall(vcov(fm.pc3))
```

```
            (Intercept)      PC1      PC2      PC3
(Intercept)    0.041692 0.000000 0.000000  0.00000
PC1            0.000000 0.023021 0.000000  0.00000
PC2            0.000000 0.000000 0.036058  0.00000
PC3            0.000000 0.000000 0.000000 15.70134
```

```
summary(fm.pc2 <- lm(y ~ PC1 + PC2, data = d3))
```

```

Call:
lm(formula = y ~ PC1 + PC2, data = d3)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.9339 -1.3878  0.0018  1.7533  4.3695 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  0.04883    0.20458   0.239   0.8118    
PC1         -3.35460    0.15202 -22.067   <2e-16 ***
PC2          0.41578    0.19026   2.185   0.0313 *  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.046 on 97 degrees of freedom
Multiple R-squared:  0.8352,	Adjusted R-squared:  0.8318 
F-statistic: 245.9 on 2 and 97 DF,  p-value: < 2.2e-16
```

```
zapsmall(vcov(fm.pc2))
```

```
            (Intercept)        PC1        PC2
(Intercept)  0.04185465 0.00000000 0.00000000
PC1          0.00000000 0.02311036 0.00000000
PC2          0.00000000 0.00000000 0.03619809
```

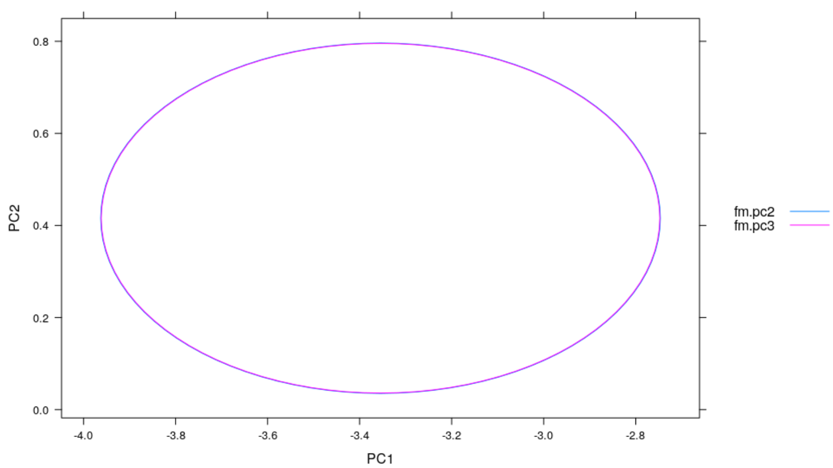
Statistical interpretation of Principal Component regression

- PCA rotates (through $\mathbf{A}$) and scales (through $\mathbf{\Lambda}$) $\mathbf{Z}$ to make columns orthonormal
- Resulting variables may be thought of as "latent variables" controlling observed covariates
- Principal components with higher variability lead to smaller sampling variance of coefficients
- Orthogonality means that estimated coefficients are uncorrelated
- Unfortunately, no longer possible to interpret effect of *individual* covariates
- Confidence ellipsoids are essentially identical (except for different residual d.f.)

```
C3 <- chol(vcov(fm.pc3)[2:3, 2:3]) # only PC1 and PC2
C2 <- chol(vcov(fm.pc2)[2:3, 2:3])
tt <- seq(0, 1, length.out = 101)
circle <- rbind(2 * cos(2 * pi * tt), sin(2 * pi * tt))
E3 <- coef(fm.pc3)[2:3] + 2 * t(C3) %*% circle
E2 <- coef(fm.pc2)[2:3] + 2 * t(C2) %*% circle
E.pc <- as.data.frame(rbind(t(E2), t(E3))); E.pc$model <- rep(c("fm.pc2", "fm.pc3"), each = 101)
```

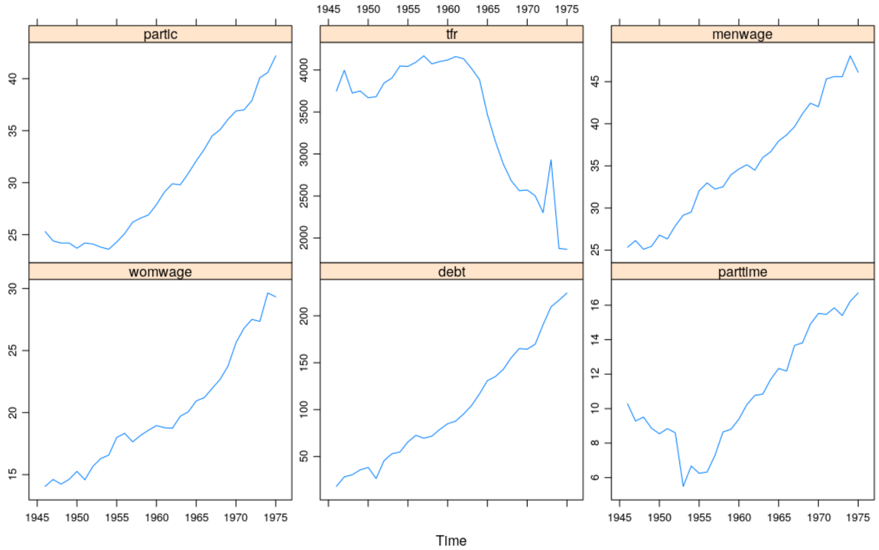
Confidence ellipsoids in principal component regression

```
xyplot(PC2 ~ PC1, data = E.pc, groups = model, abline = list(v = 1, h = 2, col = "grey"), type = "l",
       aspect = "iso", auto.key = list(lines = TRUE, points = FALSE, space = "right"))
```

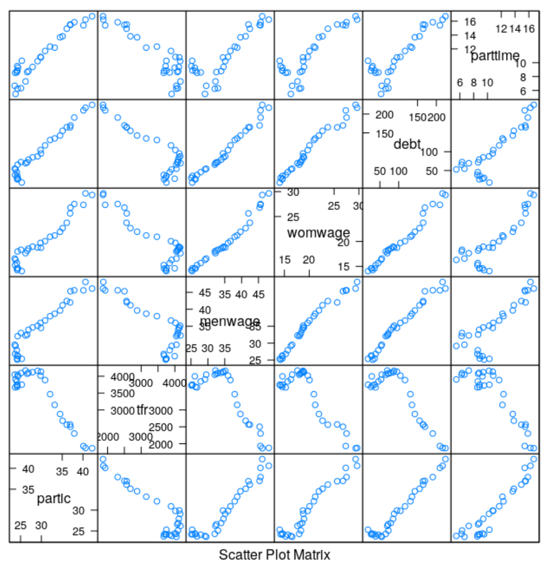


Example: Canadian Women's Labour-Force Participation

```
data(Bfox, package = "carData")
xyplot(ts(Bfox, start = 1946), aspect = "xy", layout = c(0, 6))
```




```
Bfox["1973", "tfr"] <- 1931
splom(Bfox)
```




```
Bfox$year <- as.numeric(rownames(Bfox))
summary(fm.bfox <- lm(partic ~ ., data = Bfox))
```

```

Call:
lm(formula = partic ~ ., data = Bfox)

Residuals:
     Min       1Q   Median       3Q      Max
-0.83213 -0.33438 -0.01621  0.36769  1.05048

Coefficients:
```

```
               Estimate Std. Error t value Pr(>|t|)
(Intercept)  8.142e+00  2.129e+02   0.038  0.96982
tfr         -1.949e-06  5.011e-04  -0.004  0.99693
menwage     -2.919e-02  1.502e-01  -0.194  0.84766
womwage      1.984e-02  1.744e-01   0.114  0.91041
debt         6.397e-02  1.850e-02   3.459  0.00213 **
parttime     6.566e-01  8.205e-02   8.002 4.27e-08 ***
year         4.452e-03  1.107e-01   0.040  0.96827
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.5381 on 23 degrees of freedom
Multiple R-squared:  0.9935,    Adjusted R-squared:  0.9918
F-statistic: 587.3 on 6 and 23 DF,  p-value: < 2.2e-16
```

```
sqrt(vif(fm.bfox))
```

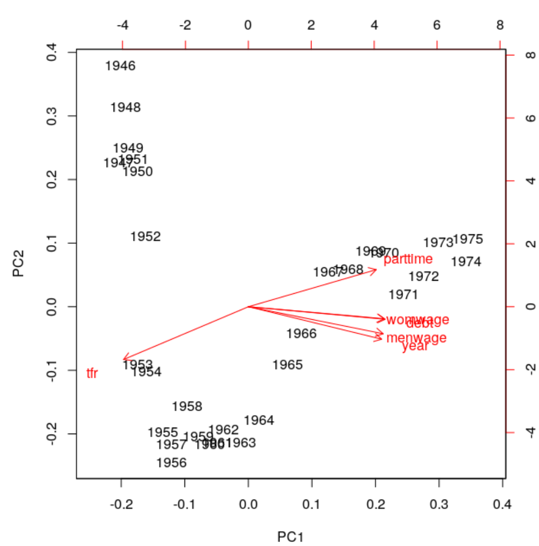
```
      tfr   menwage   womwage      debt  parttime      year
 3.891829 10.673117  8.205214 11.474235  2.748158  9.755120
```

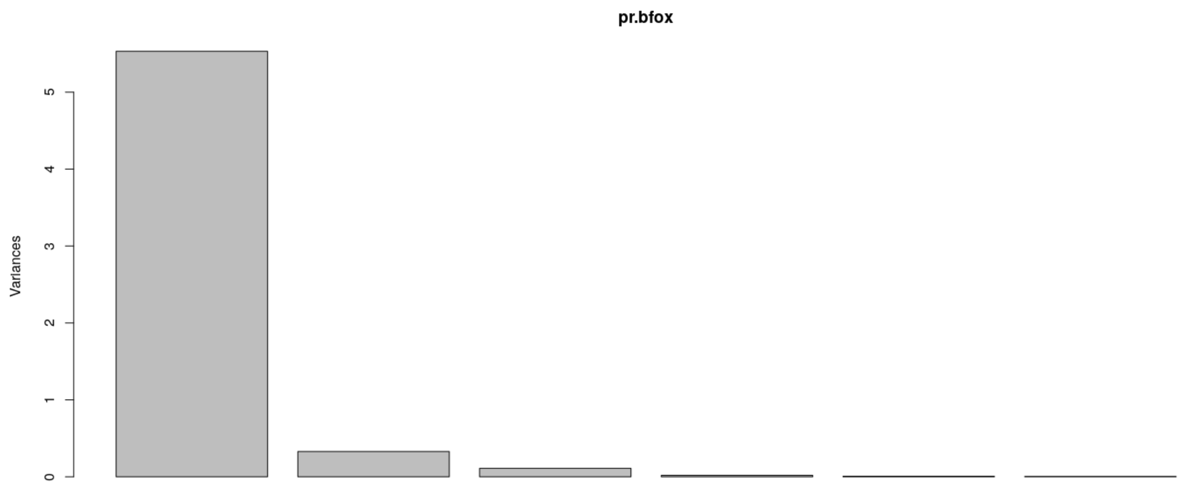
```
pr.bfox <- prcomp(~ . - partic, data = Bfox, scale. = TRUE)
pr.bfox
```

```
Standard deviations (1, .., p=6):
[1] 2.35180659 0.57341995 0.33183302 0.13614037 0.08403192 0.06698201

Rotation (n x k) = (6 x 6):
                PC1        PC2         PC3        PC4         PC5         PC6
tfr      -0.3849387 -0.6675739  0.54244962  0.2518053  0.19660094  0.09928673
menwage   0.4158879 -0.3420846 -0.02228191  0.1571042 -0.70548213  0.43258778
womwage   0.4195650 -0.1523080 -0.26579808  0.7291795  0.27909187 -0.34716538
debt      0.4220132 -0.1591200 -0.09747758 -0.2757411  0.61883636  0.57279338
parttime  0.3945669  0.4692796  0.77462008  0.1520461  0.02516761  0.01748999
year      0.4111526 -0.4105886  0.15831978 -0.5301495 -0.04646405 -0.59505293
```

- All variables contribute roughly equally to first PC
- Note non-linear pattern (PCA only accounts for linear relationships)





- First PC explains bulk of the variability (92%)

```
Bfox <- cbind(Bfox, pr.bfox$x)
summary(lm(partic ~ PC1 + PC2 + PC3 + PC4 + PC5 + PC6, data = Bfox))
```

```

Call:
lm(formula = partic ~ PC1 + PC2 + PC3 + PC4 + PC5 + PC6, data = Bfox)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.83213 -0.33438 -0.01621  0.36769  1.05048 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 29.99667    0.09823 305.356  < 2e-16 ***
PC1          2.50998    0.04248  59.080  < 2e-16 ***
PC2          0.44179    0.17424   2.535 0.018485 *  
PC3          1.30075    0.30110   4.320 0.000254 ***
PC4         -0.74497    0.73391  -1.015 0.320632    
PC5          2.67915    1.18900   2.253 0.034079 *  
PC6          2.16416    1.49166   1.451 0.160326    
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.5381 on 23 degrees of freedom
Multiple R-squared:  0.9935,    Adjusted R-squared:  0.9918 
F-statistic: 587.3 on 6 and 23 DF,  p-value: < 2.2e-16
```