Generalized Linear Models

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Motivation

- The standard linear model assumes

$$y_i \sim N(\mathbf{x}_i^T \beta, \sigma^2)$$

- In other words, the conditional distribution of $Y | \mathbf{X} = \mathbf{x}$
  - is a normal distrbution
  - with the mean parameter linear in terms involving x, and
  - the variance parameter independent of the mean
- *Generalized Linear Models* (GLMs) allow the response distribution to be non-Normal
- Still retains "linearity" in the sense that the conditional distribution depends on x only through $\mathbf{x}_i^T \beta$

Important special case: binary response

- We will first focus on a special case: binary response
- This problem can be viewed from various perspectives
- Example: `Cowles` dataset from `carData` package (1421 rows):
  - `volunteer` (response): whether willing to volunteer for psychological research
  - `neuroticism` as measured by a test
  - `extraversion` as measured by a test
  - `sex`: whether male or female
- Interested in 'predicting' whether subject is willing to volunteer

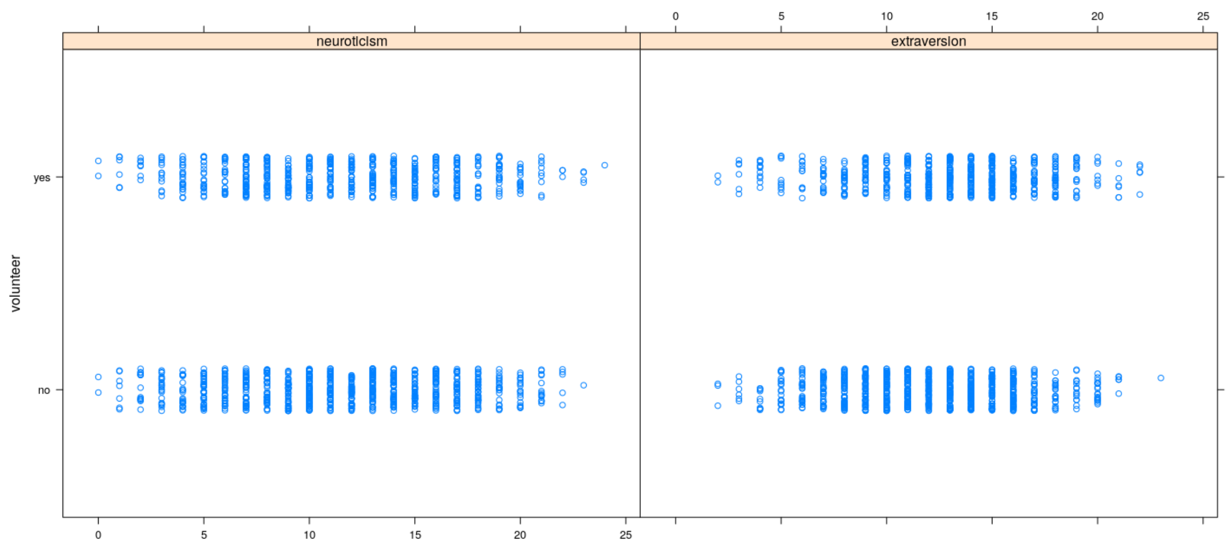
Example: Data on volunteering

```
head(Cowles, 20)
```

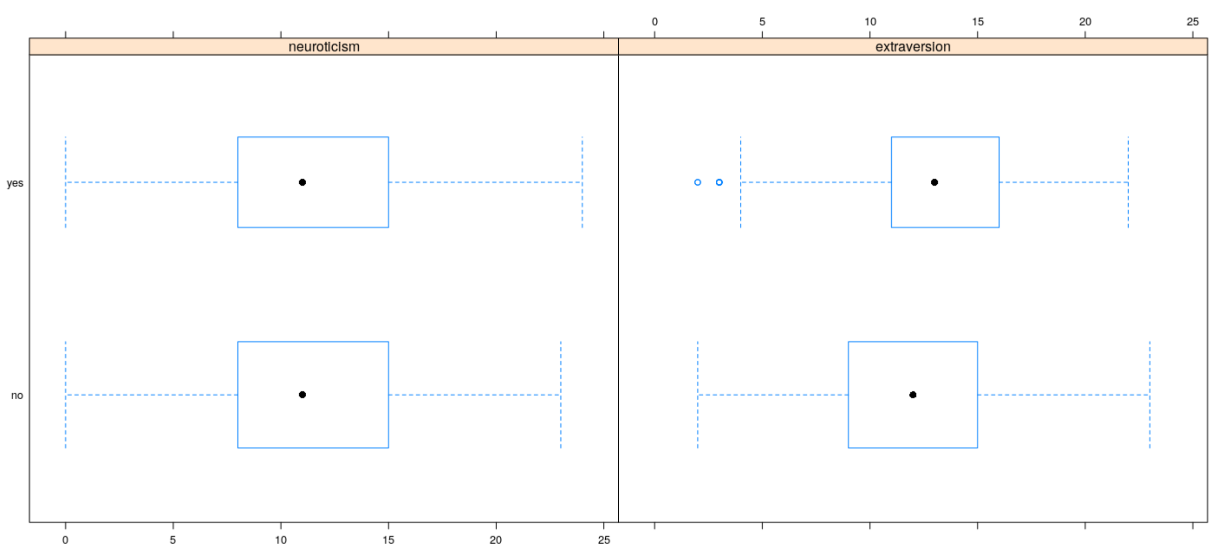
```
   neuroticism extraversion    sex volunteer
1           16           13 female        no
2            8           14   male        no
3            5           16   male        no
4            8           20 female        no
5            9           19   male        no
6            6           15   male        no
7            8           10 female        no
8           12           11   male        no
9           15           16   male        no
10          18            7   male        no
```

```
11          12          16 female       no
12           9          15   male       no
13          13          11   male       no
14           9          13   male       no
15          12          16 female       no
16          11          12   male       no
17           5           5   male       no
18          12           8   male       no
19           9           7   male       no
20           4          11 female       no
```

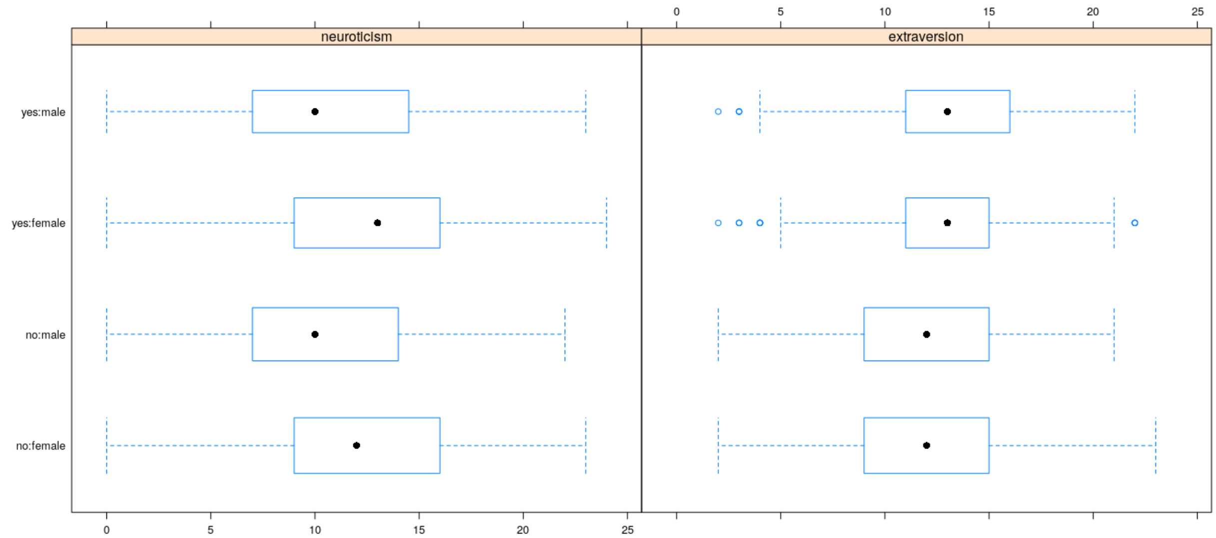
```
xyplot(volunteer ~ neuroticism + extraversion, Cowles, outer = TRUE, jitter.y = TRUE, xlab = NULL)
```



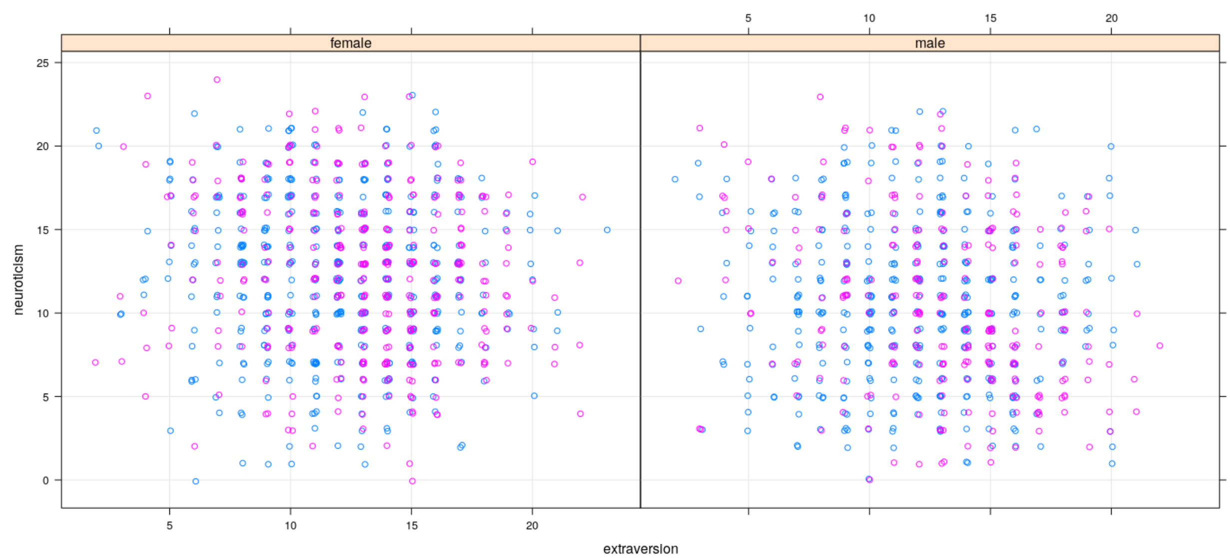

```
bwplot(volunteer ~ neuroticism + extraversion, Cowles, outer = TRUE, xlab = NULL, varwidth = TRUE)
```



```r
bwplot(volunteer:sex ~ neuroticism + extraversion, Cowles, outer = TRUE, xlab = NULL, varwidth = TRUE)
```




```r
xyplot(neuroticism ~ extraversion | sex, Cowles, groups = volunteer, grid = TRUE, jitter.x = TRUE, jitt
```



Summary

- Dependence of response on predictors does not seem to be very strong
- However, there is some information, and there appears to be some interaction
- To proceed, we need to decide
  - How can we predict willingness to volunteer?
  - What is a suitable loss function?

Possible loss functions

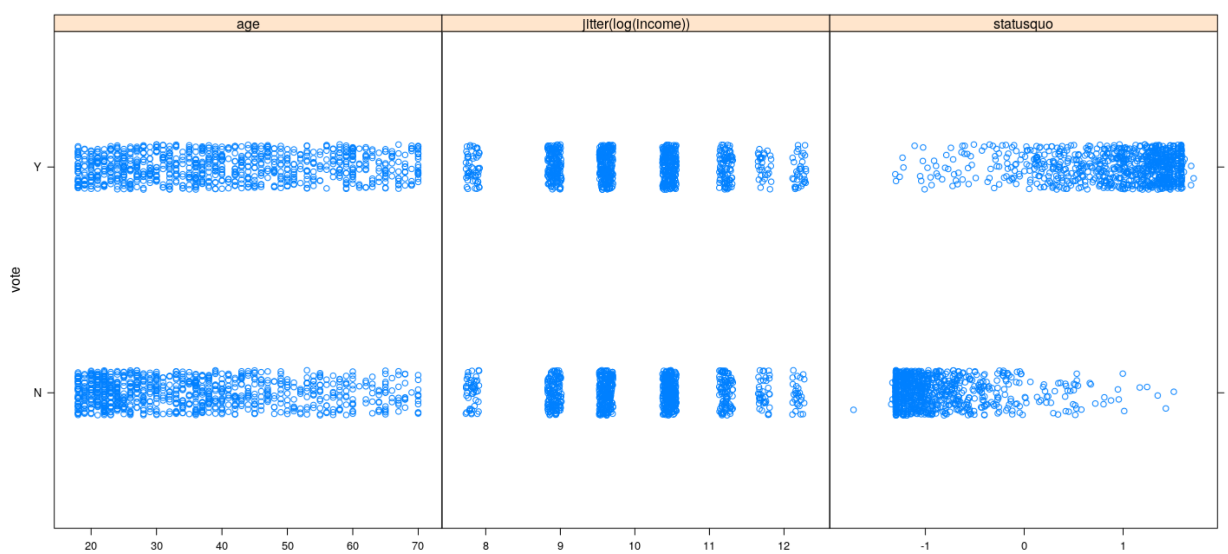
- Loss based on conditional negative log-likelihood
  - Needs a model for the conditional distribution of response
  - Leads to GLM (logistic regression in this case)
- Misclassfication loss (0 is correctly classified, 1 if misclassfied)
- Even if we use GLM, this is often the loss function we are actually interested in
- We will try some "simple" alternatives before we try logistic regression

Another example: Voting intentions in the 1988 Chilean plebiscite

- Before proceeding, we look at another example where dependence is more clear-cut
- Context: Chile was a military dictatorship under Augusto Pinochet from 1973–1990
- A referendum was held in October 1988 to decide if Chile should
  - Continue with Pinochet (Yes; result: 45%)
  - Return to democracy (No; result: 55%)
- The `Chile` data (package `carData`): National survey conducted 5 months before the referendum
- Response: intended vote (Yes / No / Abstain / Undecided)
- Other variables are sex, age, income, etc., and `statusquo` which measures support for the status quo.

Example: Voting intentions data

```
xyplot(vote ~ age + jitter(log(income)) + statusquo, Chile, subset = vote %in% c("Y", "N"),
       outer = TRUE, jitter.y = TRUE, scales = list(x = "free"), xlab = NULL)
```



Mis-classification loss: goal is to minimize false classifications

Volunteering data

```
(x <- xtabs(~ volunteer, data = Cowles))
```

```
volunteer
 no yes
824 597
```

```
min(x) / sum(x) # loss when classifying everything as modal class
```

```
[1] 0.4201267
```

Voting intentions data

```
Chile <- droplevels(subset(Chile, vote %in% c("Y", "N"))) # remove Abstain / Undecided
(x <- xtabs(~ vote, Chile))
```

```
vote
  N   Y
889 868
```

```
min(x) / sum(x) # loss when classifying everything as modal class
```

```
[1] 0.4940239
```

A simple non-parametric classification method: k-NN

- Given x, find k nearest neighbours
- Classify as modal (most common) class among these k observations
- Similar in spirit to LOWESS

```
library(class)
p <- knn.cv(Cowles[, "extraversion", drop = FALSE], cl = Cowles$volunteer, k = 11)
(x <- xtabs(~ p + Cowles$volunteer))
```

```
     Cowles$volunteer
p      no yes
  no  673 429
  yes 151 168
```

```
1 - sum(diag(x)) / sum(x)
```

```
[1] 0.4081633
```

- Slight improvement over baseline
- More variables not necessarily better (worse than baseline)

```
p <- knn.cv(Cowles[, c("extraversion", "neuroticism")], cl = Cowles$volunteer, k = 11)
(x <- xtabs(~ p + Cowles$volunteer))
```

```
     Cowles$volunteer
p      no yes
  no  628 436
  yes 196 161
```

```
1 - sum(diag(x)) / sum(x)
```

```
[1] 0.4447572
```

- Substantial improvement in voting intentions data

```
Chile2 <- subset(Chile, !(is.na(statusquo) | is.na(vote)))
p <- knn.cv(Chile2[, "statusquo", drop = FALSE], cl = Chile2$vote, k = 11)
(x <- xtabs(~ p + Chile2$vote))
```

```
   Chile2$vote
p     N   Y
  N 824  71
  Y  64 795
```

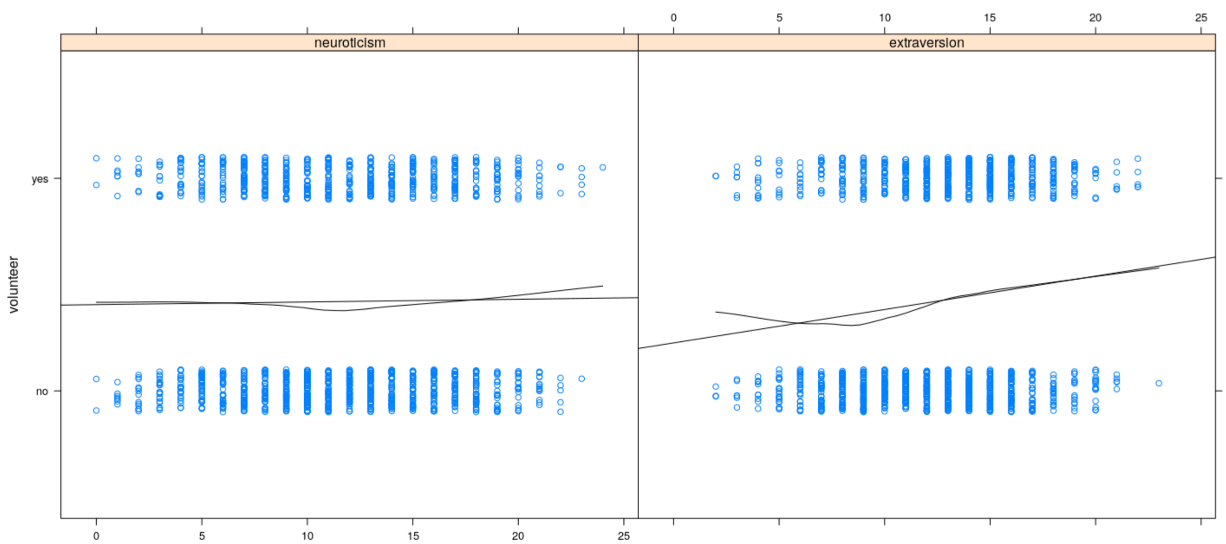
```
1 - sum(diag(x)) / sum(x)
```

```
[1] 0.07696693
```

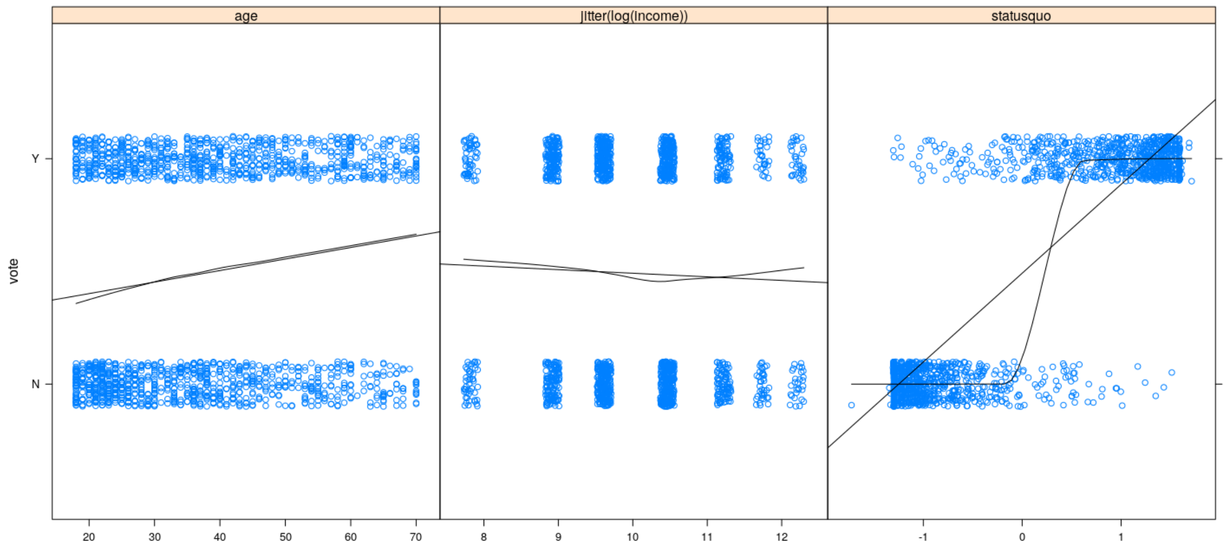
- Many other classification approaches available (but not in the scope of this course)
- We want to view this as a regression problem with a binary (0/1) response

A simple option: pretend that linear regression is valid

```
xyplot(volunteer ~ neuroticism + extraversion, Cowles, outer = TRUE, jitter.y = TRUE, xlab = NULL,
       type = c("p", "r", "smooth"), degree = 1, col.line = "black")
```




```
xyplot(vote ~ age + jitter(log(income)) + statusquo, Chile,
       outer = TRUE, jitter.y = TRUE, scales = list(x = "free"), xlab = NULL,
       type = c("p", "r", "smooth"), degree = 1, col.line = "black")
```



Can use linear regression to predict: volunteering

```
Cowles <- transform(Cowles, dvol = ifelse(volunteer == "no", 0, 1))
fm1 <- lm(dvol ~ extraversion, Cowles)
anova(fm1)
```

```
Analysis of Variance Table

Response: dvol
               Df Sum Sq Mean Sq F value    Pr(>F)
extraversion    1   5.32  5.3171  22.135 2.789e-06 ***
Residuals    1419 340.87  0.2402
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Cowles$predvol1 <- as.numeric(predict(fm1) > 0.5)
(x <- xtabs(~ predvol1 + volunteer, Cowles))
```

```
        volunteer
predvol1  no yes
       0 765 526
       1  59  71
```

```
1 - sum(diag(x)) / sum(x) # misclassification rate
```

```
[1] 0.4116819
```

```
fm2 <- lm(dvol ~ (neuroticism + extraversion) * sex, Cowles)
anova(fm2)
```

```
Analysis of Variance Table

Response: dvol
                   Df Sum Sq Mean Sq F value    Pr(>F)
neuroticism         1   0.06  0.0552  0.2298   0.63172
extraversion        1   5.49  5.4863 22.8623 1.922e-06 ***
```

```
sex                  1   1.07  1.0696  4.4571   0.03493 *
neuroticism:sex      1   0.02  0.0153  0.0640   0.80038
extraversion:sex     1   0.00  0.0006  0.0024   0.96109
Residuals         1415 339.56  0.2400
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Cowles$predvol2 <- as.numeric(predict(fm2) > 0.5)
(x <- xtabs(~ predvol2 + volunteer, Cowles))
```

```
        volunteer
predvol2  no yes
       0 745 512
       1  79  85
```

```
1 - sum(diag(x)) / sum(x) # misclassification rate
```

```
[1] 0.4159043
```

Can use linear regression to predict: voting intentions

```
Chile <- transform(Chile, dvote = ifelse(vote == "N", 0, 1))
fm3 <- lm(dvote ~ statusquo, Chile, na.action = na.exclude)
anova(fm3)
```

```
Analysis of Variance Table

Response: dvote
            Df Sum Sq Mean Sq F value    Pr(>F)
statusquo    1 320.21  320.21  4745.2 < 2.2e-16 ***
Residuals 1752 118.23    0.07
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Chile$predvote <- as.numeric(predict(fm3) > 0.5)
(x <- xtabs(~ predvote + vote, Chile))
```

```
        vote
predvote   N   Y
       0 838  82
       1  50 784
```

```
1 - sum(diag(x)) / sum(x) # misclassification rate
```

```
[1] 0.07525656
```

```
fm4 <- lm(dvote ~ statusquo + age, Chile, na.action = na.exclude)
anova(fm4)
```

```
Analysis of Variance Table

Response: dvote
            Df Sum Sq Mean Sq   F value Pr(>F)
statusquo    1 320.21  320.21 4747.5401 <2e-16 ***
age          1   0.13    0.13    1.8752 0.1711
Residuals 1751 118.10    0.07
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Chile$predvote <- as.numeric(predict(fm4) > 0.5)
(x <- xtabs(~ predvote + vote, Chile))
```

```
        vote
predvote   N   Y
       0 839  85
       1  49 781
```

```
1 - sum(diag(x)) / sum(x) # misclassification rate
```

```
[1] 0.07639681
```

Drawbacks of linear regression

- Model is clearly wrong: expected value should be in $[0, 1]$
- Expected value of response should be a non-linear function (of parameters)
- Squared error is not a meaningful loss function
- However, maximum likelihood approach is still reasonable
- Natural response distribution is Bernoulli
- Probability of "success" depends on covariates
- Logistic regression assumes that this dependence is through a linear combination $\mathbf{x}^T\beta$

Model and terminology

- Model:

$$Y|X = x \sim Ber(\mu(x)) \text{ where } \mu : \mathbb{R}^p \to [0, 1]$$

- Linear predictor

$$\eta = x^T\beta$$

- Link function $g(\cdot)$:

$$\eta = g(\mu) \text{ where } g : [0, 1] \to \mathbb{R}$$

- Inverse link function $g^{-1}(\cdot)$ (also called the mean function):

$$\mu = g^{-1}(\eta) \text{ where } g^{-1} : \mathbb{R} \to [0, 1]$$

Likelihood

- Observations $(x_1, y_1), \ldots, (x_n, y_n)$; linear predictors $\eta_i = x_i^T\beta$; mean responses $\mu_i = g^{-1}(\eta_i)$
- Likelihood

$$\prod_{i=1}^{n} [g^{-1}(x_i^T\beta)]^{y_i} [1 - g^{-1}(x_i^T\beta)]^{1-y_i} = \prod_{i=1}^{n} \mu_i^{y_i}(1 - \mu_i)^{1-y_i}$$

- log-likelihood

$$\sum_{i=1}^{n}[y_i \log \mu_i + (1-y_i)\log(1-\mu_i)]$$

- log-likelihood for the simplest case of one predictor

$$\sum_{i=1}^{n}[y_i \log g^{-1}(\alpha+\beta x_i) + (1-y_i)\log(1-g^{-1}(\alpha+\beta x_i))]$$

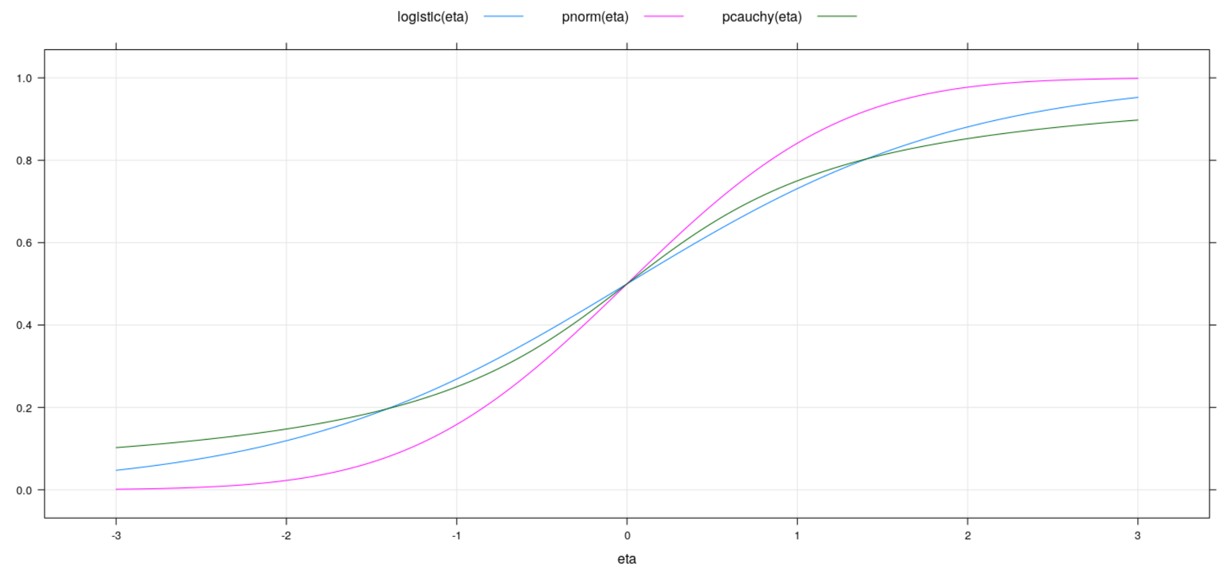
- Will be completely specified once we specify the link function $g(\cdot)$
- There are often multiple choices, with no reason to specifically prefer one over others

Choice of link function for binary response

- The inverse link function $g^{-1}(\eta)$ should have the following properties
  - Should map $\mathbb{R}$ to $[0,1]$
  - Should be monotone (increasing, without loss of generality)
  - Should decrease to 0 as $\eta \to -\infty$, increase to 1 as $\eta \to -\infty$
- These are properties satisfied by cumulative distribution functions
- We are usually interested in smooth functions
- Three particular choices are most commonly used:
  - The logistic function $\mu = \frac{e^\eta}{1+e^\eta}$
  - The Normal CDF $\mu = \Phi(\eta)$
  - The Cauchy CDF
- The logistic function is also a CDF, although the corresponding distribution is not very common
- It is a more "natural" choice in some sense, as we will see later

Common inverse link functions

```
logistic <- function(x) exp(x) / (1 + exp(x))
eta <- seq(-3, 3, 0.01)
xyplot(logistic(eta) + pnorm(eta) + pcauchy(eta) ~ eta, type = "l", ylab = NULL,
       auto.key = list(columns = 3, lines = TRUE, points = FALSE), grid = TRUE)
```



Common link functions

- The corresponding link functions $\eta = g(\mu)$ have standard names
  - Logit: $\eta = \log \frac{\mu}{1-\mu}$
  - Probit: $\eta = \Phi^{-1}(\mu)$
  - Cauchit: $\eta = F^{-1}(\mu)$ where F is the Cauchy CDF
- Link functions connect linear predictor η to mean response μ
- Choice of coefficients (e.g., α and β) control location and slope

How can we estimate parameters?

- We can think of this as a general optimization problem
- Can be solved using general numerical optimizer (will see examples)
- However, we study GLMs in detail for a different reason
- For a specific but quite general class of distributions (exponential family)
  - There is a simple and elegant way to estimate parameters
  - Like M-estimation, this approach is an example of IRLS
  - This allows tools developed for linear models to be easily adapted for GLMs

Examples revisited: volunteering data

- Before we study the general approach, let us try numerical optimization

```
negLogLik.logit <- function(beta)
{
    with(Cowles,
         {
```

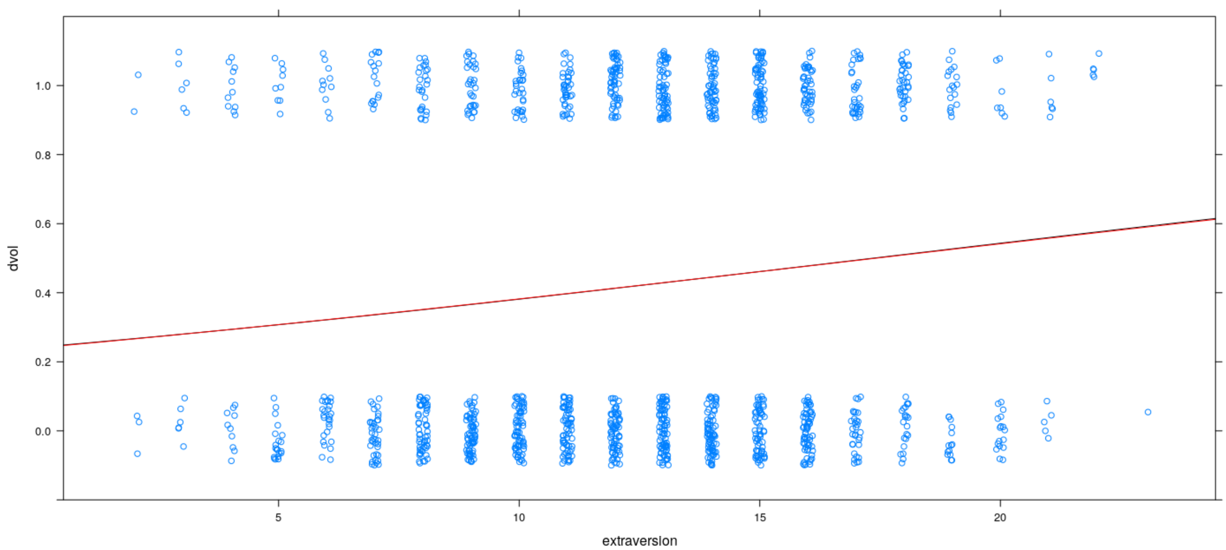
```
            mu <- logistic(beta[1] + beta[2] * extraversion)
            -sum(dvol * log(mu) + (1-dvol) * log(1-mu))
        })
}
negLogLik.probit <- function(beta)
{
    with(Cowles,
        {
            mu <- pnorm(beta[1] + beta[2] * extraversion)
            -sum(dvol * log(mu) + (1-dvol) * log(1-mu))
        })
}
opt.logit <- optim(par = c(0, 1), fn = negLogLik.logit)
opt.probit <- optim(par = opt.logit$par, fn = negLogLik.probit)
opt.logit$par
```

```
[1] -1.14145947  0.06577276
```

```
opt.probit$par
```

```
[1] -0.70500897  0.04048727
```

```
xyplot(dvol ~ extraversion, Cowles, jitter.x = TRUE, jitter.y = TRUE, ylim = c(-0.2, 1.2)) +
    layer(panel.curve(logistic(opt.logit$par[1] + opt.logit$par[2] * x), col = "black")) +
    layer(panel.curve(pnorm(opt.probit$par[1] + opt.probit$par[2] * x), col = "red"))
```




```
pred.logit <- with(Cowles, logistic(opt.logit$par[1] + opt.logit$par[2] * extraversion) > 0.5)
(x <- xtabs(~ pred.logit + volunteer, Cowles))
```

```
          volunteer
pred.logit  no yes
     FALSE 765 526
     TRUE   59  71
```

```
1 - sum(diag(x)) / sum(x) # misclassification rate
```

```
[1] 0.4116819
```

```
pred.probit <- with(Cowles, logistic(opt.probit$par[1] + opt.probit$par[2] * extraversion) > 0.5)
(x <- xtabs(~ pred.probit + volunteer, Cowles))
```

```
           volunteer
pred.probit  no yes
      FALSE 765 526
      TRUE   59  71
```

```
1 - sum(diag(x)) / sum(x) # misclassification rate
```

```
[1] 0.4116819
```

Examples revisited: voting intentions data

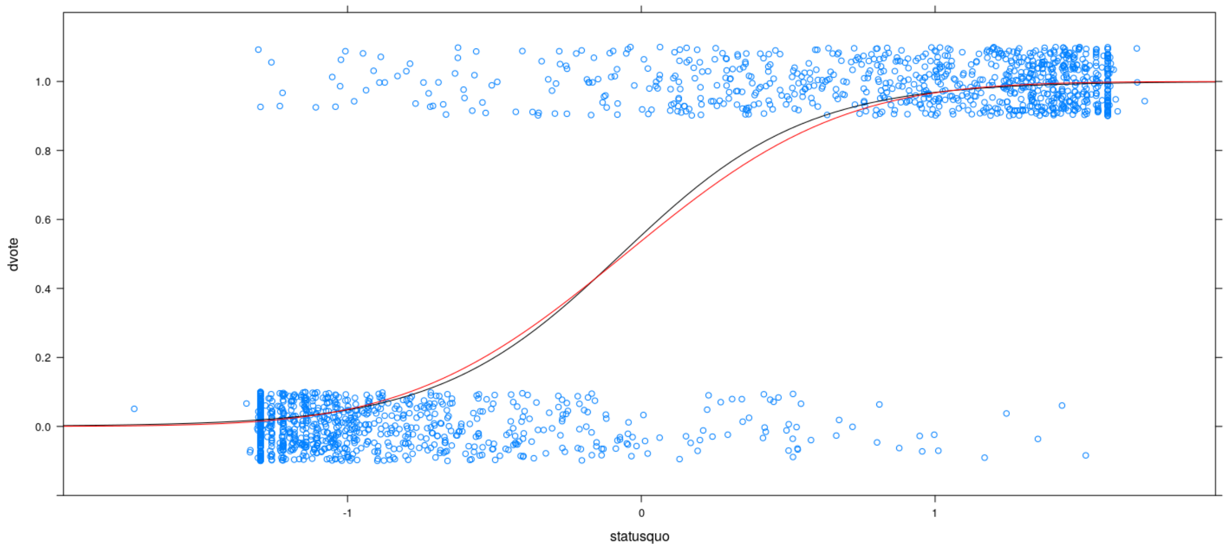
```
negLogLik.logit <- function(beta)
{
    with(Chile,
         {
             mu <- logistic(beta[1] + beta[2] * statusquo)
             -sum(dvote * log(mu) + (1-dvote) * log(1-mu), na.rm = TRUE)
         })
}
negLogLik.probit <- function(beta)
{
    with(Chile,
         {
             mu <- pnorm(beta[1] + beta[2] * statusquo)
             -sum(dvote * log(mu) + (1-dvote) * log(1-mu), na.rm = TRUE)
         })
}
opt.logit <- optim(par = c(0, 1), fn = negLogLik.logit)
opt.probit <- optim(par = opt.logit$par, fn = negLogLik.probit)
opt.logit$par
```

```
[1] 0.2153074 3.2054346
```

```
opt.probit$par
```

```
[1] 0.09379718 1.74529391
```

```
xyplot(dvote ~ statusquo, Chile, jitter.y = TRUE, ylim = c(-0.2, 1.2)) +
    layer(panel.curve(logistic(opt.logit$par[1] + opt.logit$par[2] * x), col = "black")) +
    layer(panel.curve(pnorm(opt.probit$par[1] + opt.probit$par[2] * x), col = "red"))
```




```
pred.logit <- with(Chile, logistic(opt.logit$par[1] + opt.logit$par[2] * statusquo) > 0.5)
(x <- xtabs(~ pred.logit + vote, Chile))
```

```
          vote
pred.logit   N   Y
     FALSE 829  76
     TRUE   59 790
```

```
1 - sum(diag(x)) / sum(x) # misclassification rate
```

```
[1] 0.07696693
```

```
pred.probit <- with(Chile, logistic(opt.probit$par[1] + opt.probit$par[2] * statusquo) > 0.5)
(x <- xtabs(~ pred.probit + vote, Chile))
```

```
           vote
pred.probit   N   Y
      FALSE 829  76
      TRUE   59 790
```

```
1 - sum(diag(x)) / sum(x) # misclassification rate
```

```
[1] 0.07696693
```

Inference: sampling distribution and testing

- Inference approaches usually based on asymptotic properties of MLEs
- In particular, estimates are asymptotically normal, and Wald tests are possible
- Likelihood ratio tests can also be performed to compare models (asymptotically χ^2)

The general formulation: Exponential family

- A p.d.f. or p.m.f. of Y that can be written as

$$p(y;\theta,\varphi) = \exp\left[\frac{y\theta - b(\theta)}{a(\varphi)} + c(y,\varphi)\right]$$

- where
  - $a(\cdot), b(\cdot), c(\cdot)$ are known functions; in most common cases, $a(\varphi) = \varphi/a$ for some known a
  - θ is known as the canonical parameter, and is essentially a location parameter
  - φ is a dispersion parameter (absent in some cases)
- This representation can be made more general, but is sufficient (and more suitable) for our needs
- Advantage: We can use general results for exponential families
- Expectation and variance: it can be shown that
  - $E(Y) = \mu = b'(\theta)$
  - $V(Y) = \sigma^2 = b''(\theta)a(\varphi) = a(\varphi)v(\mu)$
  - In the simplified case, $V(Y) = \varphi v(\mu)/a$
- In general, variance is function of mean (and possibly a dispersion parameter)
- The function $g_c(\cdot)$ such that $\theta = g_c(\mu) = {b'}^{-1}(\mu)$ is known as the canonical link function

Digression: expectation and variance of exponential family

- Using shorthand notation $a \equiv a(\varphi)$ and $c(y) = c(y,\varphi)$, we note that

$$p(y) = \exp\left[\frac{y\theta - b(\theta)}{a} + c(y)\right] = e^{-\frac{b(\theta)}{a}} \exp\left[\frac{y\theta}{a} + c(y)\right]$$

- Assuming that $p(y)$ is a density (analogous calculations are valid if $p(y)$ is a mass function),

$$\begin{aligned}
\int p(y)dy = 1 \quad &\Longrightarrow \quad e^{\frac{b(\theta)}{a}} = \int \exp\left[\frac{y\theta}{a} + c(y)\right] dy \\
&\Longrightarrow \quad \frac{b(\theta)}{a} = \log \int \exp\left[\frac{y\theta}{a} + c(y)\right] dy = \log Q(\theta)
\end{aligned}$$

- Thus, we have

$$\frac{b'(\theta)}{a} = \frac{Q'(\theta)}{Q(\theta)} \quad \text{and} \quad \frac{b''(\theta)}{a} = \frac{Q''(\theta)}{Q(\theta)} - \left(\frac{Q'(\theta)}{Q(\theta)}\right)^2$$

- Now, interchanging $\int$ and $\frac{d}{d\theta}$ as necessary (Leibniz's rule), we have

$$\begin{aligned}
\frac{Q'(\theta)}{Q(\theta)} &= \frac{\int \exp\left[\frac{y\theta}{a} + c(y)\right] \frac{y}{a} dy}{\int \exp\left[\frac{y\theta}{a} + c(y)\right] dy} = \frac{\int \exp\left[\frac{y\theta - b(\theta)}{a} + c(y)\right] \frac{y}{a} dy}{\int \exp\left[\frac{y\theta - b(\theta)}{a} + c(y)\right] dy} = \frac{E(Y)}{a} \\
\frac{Q''(\theta)}{Q(\theta)} &= \frac{\int \frac{y}{a} \frac{d}{d\theta} \exp\left[\frac{y\theta}{a} + c(y)\right] dy}{\int \exp\left[\frac{y\theta}{a} + c(y)\right] dy} \\
&= \frac{\int \exp\left[\frac{y\theta}{a} + c(y)\right] \frac{y^2}{a^2} dy}{\int \exp\left[\frac{y\theta}{a} + c(y)\right] dy} = \frac{\int \exp\left[\frac{y\theta - b(\theta)}{a} + c(y)\right] \frac{y^2}{a^2} dy}{\int \exp\left[\frac{y\theta - b(\theta)}{a} + c(y)\right] dy} = \frac{E(Y^2)}{a^2}
\end{aligned}$$

- It immediately follows that $E(Y) = b'(\theta)$ and $V(Y) = a(\varphi)b''(\theta)$

Examples of exponential families

Family	$a(\varphi)$	$b(\theta)$	$c(y, \varphi)$
Gaussian	φ	$\theta^2/2$	$-\frac{1}{2}[\frac{y^2}{\varphi} + \log(2\pi\varphi)]$
Binomial proportion	$1/n$	$\log(1 + e^\theta)$	$\log \binom{n}{ny}$
Poisson	1	e^θ	$-\log y!$
Gamma	φ	$-\log(-\theta)$	$\log(y/\varphi)/\varphi^2 - \log y - \log\Gamma(1/\varphi)$
Inverse-Gaussian	φ	$-\sqrt{-2\theta}$	$-\frac{1}{2}[\log(\pi\varphi y^3) + 1/(\varphi y)]$

- Exercise: Verify
- What are the corresponding canonical link functions?

GLM with response distribution given by exponential family

- Observations $(x_i, y_i); i = 1, \ldots, n$
- Basic premise of model: Location parameter $\mu_i = E(Y|X = x_i)$ depends on predictors x_i
- Variance depends on μ_i, but apart from that no dependence on predictors
- In other words, dispersion parameter φ is a constant nuisance parameter
- Dependence of μ_i on x_i given by a *link function* $g(\cdot)$ through the relationship

$$g(\mu_i) = g(b'(\theta_i)) = x_i^T\beta = \eta_i$$

- In other words, a GLM can be thought of as a *linear model* for the *transformation* $g(\mu)$ of the mean μ

- If $g(\cdot)$ is chosen to be the canonical link $g_c(\cdot)$, then $g(\mu_i) = \theta_i = x_i^T\beta = \eta_i$
- This choice leads to some simplications
- However, no reason for effects of covariates to be additive on this particular (transformed) scale

Common link functions

Link	$\eta = g(\mu)$	$\mu = g^{-1}(\eta)$
Identity	μ	η
Log	$\log \mu$	e^η
Inverse	$1/\mu$	$1/\eta$
Inverse square	$1/\mu^2$	$1/\sqrt{\eta}$
Square root	$\sqrt{\mu}$	η^2
Logit	$\log \frac{\mu}{1-\mu}$	$\frac{e^\eta}{1+e^\eta}$
Probit	$\Phi^{-1}(\mu)$	$\Phi(\eta)$
Log-log	$-\log(-\log \mu)$	$e^{-e^{-\eta}}$
Complementary log-log	$\log(-\log \mu)$	$1 - e^{-e^{\eta}}$

- Last four are for Binomial proportion; last two are asymmetric (Exercise: plot and compare)

Comparison with variable transformation

- GLM assumes that a transformation of the *mean* is linear in parameters
- This is somewhat similar to transforming the *response* to achieve linearity in a linear model
- However, in a linear model, transforming the response also changes its distribution / variance
- In contrast, distribution of response and linearizing transformation are kept separate in GLM
- A practical problem: transformation may not be defined for all observations
  - Bernoulli response 0 / 1 is mapped to $\pm\infty$ by all link functions
  - Poisson count of 0 is mapped to $-\infty$ by log link

Maximum likelihood estimation

- Log-likelihood (assuming for the moment that $a(\varphi)$ may depend on i)

$$\ell(\theta(\beta), \varphi|y) = \log L(\theta(\beta), \varphi|y) = \sum_{i=1}^{n} \left[\frac{y_i\theta_i - b(\theta_i)}{a_i(\varphi)} + c(y_i, \varphi) \right] = \sum_{i=1}^{n} \ell_i$$

- Suppose link function is $g(\mu_i) = \eta_i = x_i^T\beta$ where $\mu_i = b'(\theta_i)$
- To obtain score equations / estimating equations, we need to calculate (for $i = 1, \dots, n; j = 1, \dots, p$)

$$\frac{\partial \ell_i}{\partial \beta_j} = \frac{\partial \ell_i}{\partial \theta_i} \times \frac{\partial \theta_i}{\partial \mu_i} \times \frac{\partial \mu_i}{\partial \eta_i} \times \frac{\partial \eta_i}{\partial \beta_j}$$

- Note that $b'(\theta_i) = \mu_i$, $\frac{\partial \mu_i}{\partial \theta_i} = b''(\theta_i) = v(\mu_i)$, $\frac{\partial \eta_i}{\partial \mu_i} = g'(\mu_i)$, and $\frac{\partial \eta_i}{\partial \beta_j} = x_{ij}$
- After simplification, we obtain the score equations (one for each β_j)

$$s_j(\beta) = \frac{\partial}{\partial \beta_j} \ell(\theta(\beta), \varphi|y) = \sum_{i=1}^{n} \frac{y_i - \mu_i}{a_i(\varphi)v(\mu_i)} \times \frac{x_{ij}}{g'(\mu_i)} = 0$$

- To proceed further, we need to assume the form $a_i(\varphi) = \varphi/a_i$, which gives

$$s_j(\beta) = \sum_{i=1}^{n} \frac{a_i(y_i - \mu_i)}{v(\mu_i)} \times \frac{x_{ij}}{g'(\mu_i)} = 0$$

- In other words, score equations for β do not depend on the dispersion parameter φ
- In practice, a_i is constant for most models; for binomial proportion, $\varphi = 1$ and $a_i = n_i$

Maximum likelihood estimation with canonical link

- Further simplification when $g(\cdot)$ is the canonical link $g_c(\cdot)$, where $\eta_i = \theta_i$

$$\frac{\partial \ell_i}{\partial \beta_j} = \frac{\partial \ell_i}{\partial \theta_i} \times \frac{\partial \eta_i}{\partial \beta_j} = \frac{y_i - \mu_i}{a_i(\theta)} x_{ij}$$

- Score equations become (when $a_i(\varphi) = \varphi/a_i$)

$$\sum_{i=1}^{n} a_i y_i x_{ij} = \sum_{i=1}^{n} a_i \mu_i x_{ij}$$

Analogy with normal equations in linear model

- With $\mathbf{A} = diag(a_1, \ldots, a_n)$ diagonal matrix of prior weights, these can be written as

$$\mathbf{X}^T \mathbf{A} \mu(\beta) = \mathbf{X}^T \mathbf{A} \mathbf{y}$$

- In particular, when $\mathbf{A} = \mathbf{I}$ (for Binomial, use individual Bernoulli trials)

$$\mathbf{X}^T (\mathbf{y} - \hat{\mu}) = \mathbf{0}$$

- In other words, "residuals" are orthogonal to column space of $\mathbf{X}$
- In the linear model, $\mu(\beta) = \mathbf{X}\beta$, giving the usual normal equations
- In general, the score equations are non-linear in β because $\mu(\beta)$ is non-linear
- This is true whether or not we use the canonical link
- How can we solve them? Need some kind of iterative method

Digression: Newton-Raphson and Fisher scoring

- The Newton-Raphson method is a general numerical algorithm to solve $f(x) = 0$
- Suppose we have an approximate solution x_0
- Locally approximate $f(x)$ by a line (first order Taylor series approximation)

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

- A hopefully "closer" solution of $f(x) = 0$ is the root of this approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Treat x_1 as an updated estimate and iterate until convergence

$$x^{(t+1)} = x^{(t)} - \frac{f(x^{(t)})}{f'(x^{(t)})}$$

- This usually works as long as we get a good starting estimate $x^{(0)}$ and f is well behaved
- In our case, f is the score function $s(\beta)$
- This is actually a set of p separate equations $s_j(\beta) = 0$ (one for each β_j)
- In other words, $s(\cdot)$ is a vector function

$$s : \mathbb{R}^p \to \mathbb{R}^p$$

- Fortunately, the algorithm is still valid, giving

$$\hat{\beta}^{(t+1)} = \hat{\beta}^{(t)} - \left(H\left(\hat{\beta}^{(t)}\right)\right)^{-1} s(\hat{\beta}^{(t)})$$

- where $H\left(\hat{\beta}^{(t)}\right)$ is the Jacobian matrix of $s(\cdot)$ or the Hessian of the log-likelihood function at $\hat{\beta}^{(t)}$
- The only potential difficulty is in computing H
- In the context of maximum likelihood estimation, H is closely related to Fisher information
- Recall that for a scalar parameter θ,

$$E_\theta(s(\theta; X)) = E_\theta\left[\frac{\partial}{\partial\theta}\log f(X;\theta)\right] = \int \frac{\frac{\partial}{\partial\theta}f(x;\theta)}{f(x;\theta)} f(x;\theta) = \frac{\partial}{\partial\theta}1 = 0$$

- Also, under regularity conditions, the variance of the score function (a.k.a. Fisher information) is given by

$$I(\theta) = V_\theta(s(\theta;X)) = E_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(X;\theta)\right)^2\right] = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right] = -E_\theta H(\theta;X)$$

- In other words, Fisher information is the expected value of the Hessian
- $-H(\theta; X)$ is often referred to as the *observed* information (as it depends on the observations X)
- These results hold for vector-valued parameters as well
- The Newton-Raphson algorithm described above uses the observed information
- If we use Fisher information instead, we get the so-called "Fisher scoring" algorithm
- Before we try to see how this turns out, we look at a more "intuitive" iterative method

Iteratively Reweighted Least Squares for GLM

- Write $y = \mu + (y - \mu) = \mu + \epsilon$. Can we transform both μ and ϵ to the linear scale?
- $\eta = g(\mu)$ is the "mean" in the linear scale, and a first order Taylor approximation of g around μ gives

$$\tilde{g}(y) = g(\mu) + g'(\mu)(y - \mu)$$

- Use this to define "error" ε_i and "response" z_i on the linear scale as

$$z_i \equiv \tilde{g}(y_i) = g(\mu_i) + g'(\mu_i)(y_i - \mu_i) = \eta_i + \varepsilon_i$$

- It follows that

$$E(z_i) = \eta_i = x_i^T\beta \quad \text{and} \quad V(z_i) = [g'(\mu_i)]^2 v(\mu_i)/a_i$$

- This is a weighted linear model that can be fitted using weighted least squares problem…
- apart from the slight inconvenience that μ_i-s depend on the unknown parameter β

However, this immediately suggests the following iterative approach:

1. Start with initial estimates $\hat{\mu}_i^{(0)}$ and $\hat{\eta}_i^{(0)} = g(\hat{\mu}_i^{(0)})$
2. For each iteration, set
   - working response $z_i^{(t)} = \hat{\eta}_i^{(t)} + g'(\hat{\mu}_i^{(t)})\left(y_i - \hat{\mu}_i^{(t)}\right)$

- working weights

$$w_i^{(t)} = \frac{a_i}{\left[g'(\hat{\mu}_i^{(t)})\right]^2 v(\hat{\mu}_i^{(t)})}$$

3. Fit a weighted least squares model for $\mathbf{z}$ on X with weights $\mathbf{w}$ to obtain $\hat{\beta}^{(t+1)}$
4. Define $\hat{\eta}_i^{(t+1)} = x_i^T \hat{\beta}^{(t+1)}$ and $\hat{\mu}_i^{(t+1)} = g^{-1}\left(\hat{\eta}_i^{(t+1)}\right)$
5. Repeat steps 2–4 until convergence

- In matrix notation, the iteration can be written as

$$\begin{aligned}
\hat{\beta}^{(t+1)} &= (X^T W^{(t)} X)^{-1} X^T W^{(t)} \mathbf{z}^{(t)} \\
&= (X^T W^{(t)} X)^{-1} X^T W^{(t)} \left[X\hat{\beta}^{(t)} + \tilde{G}^{(t)}\left(\mathbf{y} - \hat{\mu}^{(t)}\right)\right] \\
&= \hat{\beta}^{(t)} + (X^T W^{(t)} X)^{-1} X^T W^{(t)} \tilde{G}^{(t)}\left(\mathbf{y} - \hat{\mu}^{(t)}\right)
\end{aligned}$$

- Where
  - $W^{(t)}$ is a diagonal matrix with elements $w_i^{(t)}$
  - $\tilde{G}^{(t)}$ is a diagonal matrix with elements $g'(\hat{\mu}_i^{(t)})$
- It turns out that this is equivalent to the Fisher scoring algorithm
- Enough to show

$$(X^T W^{(t)} X)^{-1} X^T W^{(t)} \tilde{G}^{(t)}\left(\mathbf{y} - \hat{\mu}^{(t)}\right) = -\left(E_\beta H\left(\hat{\beta}^{(t)}\right)\right)^{-1} s(\hat{\beta}^{(t)})$$

- Additionally, with the canonical link, this reduces to the Newton-Raphson algorithm

Calculation of Hessian H

- Recall that

$$\frac{\partial \ell_i}{\partial \theta_i} = \frac{y_i - b'(\theta_i)}{a_i(\varphi)} \implies \frac{\partial^2 \ell_i}{\partial \theta_i^2} = \frac{\partial}{\partial \theta_i}\left(\frac{y_i - \mu_i}{a_i(\varphi)}\right) = \frac{-b''(\theta_i)}{a_i(\varphi)} = -\frac{v(\mu_i)}{a_i(\varphi)}$$

- $H_{jk} = \sum_i h_{ijk}$, where

$$\begin{aligned}
h_{ijk} = \frac{\partial^2 \ell_i}{\partial \beta_j \beta k} &= \frac{\partial}{\partial \beta_j}\left[\frac{\partial \ell_i}{\partial \beta_k}\right] = \frac{\partial}{\partial \beta_j}\left[\frac{\partial \ell_i}{\partial \theta_i} \times \frac{\partial \theta_i}{\partial \mu_i} \times \frac{\partial \mu_i}{\partial \eta_i} \times \frac{\partial \eta_i}{\partial \beta_k}\right] \\
&= \frac{\partial}{\partial \beta_j}\left[\frac{y_i - \mu_i}{a_i(\varphi)} \times \frac{\partial \theta_i}{\partial \mu_i} \times \frac{\partial \mu_i}{\partial \eta_i} \times x_{ik}\right] \\
&= x_{ik}\left[\frac{y_i - \mu_i}{a_i(\varphi)} \frac{\partial}{\partial \beta_j}\left(\frac{\partial \theta_i}{\partial \mu_i} \times \frac{\partial \mu_i}{\partial \eta_i}\right) + \frac{\partial}{\partial \beta_j}\left(\frac{y_i - \mu_i}{a_i(\varphi)}\right) \cdot \frac{\partial \theta_i}{\partial \mu_i} \times \frac{\partial \mu_i}{\partial \eta_i}\right] \\
&= x_{ik}\left[\frac{y_i - \mu_i}{a_i(\varphi)} \frac{\partial}{\partial \beta_j}\left(\frac{\partial \theta_i}{\partial \mu_i} \times \frac{\partial \mu_i}{\partial \eta_i}\right) - \frac{v(\mu_i)}{a_i(\varphi)} \cdot \left(\frac{\partial \theta_i}{\partial \mu_i} \times \frac{\partial \mu_i}{\partial \eta_i}\right)^2 \times x_{ij}\right]
\end{aligned}$$

Calculation of Hessian H when $g(\cdot)$ is canonical link

- $\frac{\partial \theta_i}{\partial \mu_i} \times \frac{\partial \mu_i}{\partial \eta_i} \equiv 1$
- First term vanishes
- Second term does not involve observations y_i
- Observed information equals expected information
- Results in Newton-Raphson iterations
- Hessian is given by

$$H_{jk} = -\sum_{i=1}^{n} \frac{v(\mu_i)}{a_i(\varphi)} x_{ij} x_{ik}$$

Calculation of expected Hessian H for general link

- First term vanishes after taking expectation as $E_\beta(y_i - \mu_i) = 0$ (second term does not involve y)
- Results in Fisher scoring iterations (Newton-Raphson possible, but not equivalent to IRLS)
- Second term simplifies as before
- Expected Hessian is given by (after assuming $a_i(\varphi) = \varphi / a_i$)

$$E_\beta H_{jk} = -\sum_{i=1}^{n} \frac{v(\mu_i)}{a_i(\varphi)} \left(\frac{1}{v(\mu_i) g'(\mu_i)} \right)^2 x_{ij} x_{ik} = -\sum_{i=1}^{n} \frac{a_i}{[g'(\mu_i)]^2 v(\mu_i)} x_{ij} x_{ik}$$

- In other words, Fisher information $I = -E_\beta H = X^T W X$, where W is diagonal with entries

$$w_i = \frac{a_i}{[g'(\mu_i)]^2 v(\mu_i)}$$

Equivalence of Fisher scoring and IRLS

- Recall: We need to show that

$$(X^T W^{(t)} X)^{-1} X^T W^{(t)} \tilde{G}^{(t)} (\mathbf{y} - \hat{\mu}^{(t)}) = -\left(E_\beta H \left(\hat{\beta}^{(t)} \right) \right)^{-1} s(\hat{\beta}^{(t)})$$

- We have just shown that $X^T W^{(t)} X = -E_\beta H \left(\hat{\beta}^{(t)} \right)$
- Remains to show that $X^T W^{(t)} \tilde{G}^{(t)} (\mathbf{y} - \hat{\mu}^{(t)}) = s(\hat{\beta}^{(t)})$
- Dropping the suffix $^{(t)}$ indicating iteration, the j-th element of the RHS is

$$s(\beta) = \sum_{i=1}^{n} \frac{a_i (y_i - \mu_i)}{v(\mu_i)} \cdot \frac{x_{ij}}{g'(\mu_i)} = \sum_{i=1}^{n} x_{ij} \left[\frac{a_i}{[g'(\mu_i)]^2 v(\mu_i)} \cdot g'(\mu_i) \right] (y_i - \mu_i)$$

- It is easy to see that the j-th element of the LHS is the same

Initial estimates

- Simple choice: $\hat{\mu}_i^{(0)} = y_i$
- This may cause a problem computing $\hat{\eta}_i^{(0)}$ in some cases
  - For Bernoulli response, if $\mu = y \in \{0, 1\}$, $\text{logit}(\mu) = \pm\infty$
  - For Poisson response, if $\mu = y = 0, \log(\mu) = -\infty$
- The initial values are not that critical, and can be adjusted to avoid this
- E.g., choose initial $\mu = 0.5$ for Bernoulli, or $\mu = 1$ when $y = 0$ for Poisson

Estimating the dispersion parameter

- Recall that $V(y_i) = \varphi v(\mu_i)/a_i$
- This suggests the *method of moments* estimator

$$\hat{\varphi} = \frac{1}{n-p} \sum_{i=1}^{n} \frac{a_i (y_i - \hat{\mu}_i)^2}{v(\hat{\mu}_i)}$$

- This is usually preferred over the MLE of φ

Asymptotic sampling distribution of $\hat{\beta}$

- Under mild regularity conditions, the MLE $\hat{\beta}$ is asymptotically normal
- Variance-covariance matrix is given by inverse of Fisher information

$$\hat{\beta} \sim AN(\beta, \varphi\, I_\mu^{-1}) \equiv AN(\beta, \varphi \left(X^T W_\mu X\right)^{-1})$$

- So Wald tests for linear functions of β can be performed using standard errors based on

$$\hat{V}(\hat{\beta}) = \hat{\varphi} \left(X^T W_{\hat{\mu}} X\right)^{-1}$$

- For models without a dispersion parameter, these are approximate χ^2 or z-tests
- For models with a dispersion parameter, these are approximate F or t-tests

Analysis of deviance

- F-tests to test nested models in linear regression are no longer valid
- Analogous tests can be performed using asymptotic results for likelihood ratio tests
- Recall that the log-likelihood for the model can be written as

$$\ell(\mu, \varphi | y) = \log L(\mu, \varphi | y) = \sum_{i=1}^{n} \left[\frac{a_i (y_i \theta_i - b(\theta_i))}{\varphi} + c(y_i, \varphi) \right]$$

- For any fitted model, this can be compared with the "saturated model" $\hat{\mu}_i = y_i$
- Define deviance (ignoring the dispersion parameter) as twice the difference in log-likelihoods

$$\begin{aligned} D(y;\hat{\mu}) &= 2\varphi\left[\ell(y,\varphi|y) - \ell(\hat{\mu},\varphi|y)\right] \\ &= 2\sum_{i=1}^{n} a_i\left[y_i(\theta(y_i) - \theta(\hat{\mu}_i)) - (b(\theta(y_i)) - b(\theta(\hat{\mu}_i)))\right] \end{aligned}$$

- Boundary problems can be resolved on the observation scale
- This is analogous to sum of squared errors in a linear model (exercise: check for Gaussian)
- Forms basis for (asymptotic) χ^2 tests for models without a dispersion parameter (Binomial, Poisson)
- Exercise: Compute deviance explicitly for Binomial proportion and Poisson
- The scaled deviance divides by the estimated dispersion parameter

$$D^*(y;\hat{\mu}) = D(y;\hat{\mu})/\hat{\varphi}$$

- Forms basis for (approximate) F tests for models with a dispersion parameter
- The deviance for a constant mean model (intercept only) is called the *null deviance* (say D_0)
- A GLM analogue of the coefficient of determination R^2 for a model with deviance D_1 is

$$R^2 = 1 - \frac{D_1}{D_0}$$

Fitting Generalized Linear Models in R

- GLMs are fit using the function `glm()`, which has an interface similar to `lm()`
- In addition to a formula and the `data` argument, `glm()` requires a `family` argument to be specified
- Examples (continuous):

```
gaussian(link = "identity")
gaussian(link = "log")
gaussian(link = "inverse")

Gamma(link = "inverse")
Gamma(link = "identity")
Gamma(link = "log")

inverse.gaussian(link = "1/mu^2")
inverse.gaussian(link = "inverse")
inverse.gaussian(link = "identity")
inverse.gaussian(link = "log")
```

- GLMs are fit using the function `glm()`, which has an interface similar to `lm()`
- In addition to a formula and the `data` argument, `glm()` requires a `family` argument to be specified
- Examples (discrete):

```
binomial(link = "logit")
binomial(link = "probit")
binomial(link = "cauchit")
binomial(link = "cloglog")
binomial(link = "log")
```

```r
poisson(link = "log")
poisson(link = "identity")
poisson(link = "sqrt")
```

- The link function can also be constructed by specifying the functions
  - link: g
  - linkinv: g^{-1}
  - mu.eta: $\frac{d\mu}{d\eta}$

```r
str(make.link("probit"))
```

```
List of 5
 $ linkfun :function (mu)
 $ linkinv :function (eta)
 $ mu.eta  :function (eta)
 $ valideta:function (eta)
 $ name    : chr "probit"
 - attr(*, "class")= chr "link-glm"
```

Example: volunteering

```r
fgm1 <- glm(dvol ~ (extraversion + neuroticism) * sex, Cowles, family = binomial("logit"))
summary(fgm1)
```

```

Call:
glm(formula = dvol ~ (extraversion + neuroticism) * sex, family = binomial("logit"),
    data = Cowles)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.3972  -1.0505  -0.9044   1.2603   1.6909

Coefficients:
                      Estimate Std. Error z value Pr(>|z|)
(Intercept)          -1.138048   0.329538  -3.453 0.000553 ***
extraversion          0.065547   0.019360   3.386 0.000710 ***
neuroticism           0.008910   0.015348   0.581 0.561539
sexmale              -0.191828   0.477453  -0.402 0.687851
extraversion:sexmale  0.001600   0.028627   0.056 0.955419
neuroticism:sexmale  -0.005612   0.022827  -0.246 0.805785
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1933.5  on 1420  degrees of freedom
Residual deviance: 1906.0  on 1415  degrees of freedom
AIC: 1918

Number of Fisher Scoring iterations: 4
```

```r
fgm2 <- glm(dvol ~ extraversion, Cowles, family = binomial("logit"))
anova(fgm2, fgm1, test = "LRT")
```

```
Analysis of Deviance Table

Model 1: dvol ~ extraversion
Model 2: dvol ~ (extraversion + neuroticism) * sex
  Resid. Df Resid. Dev Df Deviance Pr(>Chi)
1      1419     1911.5
2      1415     1906.0  4   5.4899   0.2406
```

Example: voting intentions

```
Chile0 <- na.omit(Chile[, c("dvote", "statusquo", "income", "age", "sex")])
fgm3 <- glm(dvote ~ ., Chile0, family = binomial("cauchit")) # ~ . means all covariates
summary(fgm3)
```

```

Call:
glm(formula = dvote ~ ., family = binomial("cauchit"), data = Chile0)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.6516  -0.3281  -0.2669   0.2883   2.5213

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)  6.723e-01  6.060e-01   1.109 0.267233
statusquo    6.164e+00  6.231e-01   9.893  < 2e-16 ***
income      -1.707e-05  4.540e-06  -3.759 0.000171 ***
age          2.529e-02  1.279e-02   1.978 0.047953 *
sexM        -6.480e-01  3.676e-01  -1.763 0.077951 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2368.68  on 1708  degrees of freedom
Residual deviance:  754.18  on 1704  degrees of freedom
AIC: 764.18

Number of Fisher Scoring iterations: 9
```

```
anova(fgm3, test = "LRT")
```

```
Analysis of Deviance Table

Model: binomial, link: cauchit

Response: dvote

Terms added sequentially (first to last)


          Df Deviance Resid. Df Resid. Dev  Pr(>Chi)
NULL                       1708    2368.68
statusquo  1  1597.76      1707     770.92 < 2.2e-16 ***
income     1     9.69      1706     761.23  0.001852 **
age        1     4.07      1705     757.17  0.043740 *
```

```
sex          1      2.99       1704      754.18   0.084023 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Example: Snow geese flock counts

- Background: Aerial surveys to estimate number of snow geese over Hudson Bay, Canada
- Approximate count visually estimated by "experienced person"
- In this experiment, two observers recorded estimates for several flocks
- Actual count was obtained from a photograph taken at the same time

```r
library(alr3)
head(snowgeese)
```

```
  photo obs1 obs2
1    56   50   40
2    38   25   30
3    25   30   40
4    48   35   45
5    38   25   30
6    22   20   20
```

Example: Poisson response for snow geese flock counts

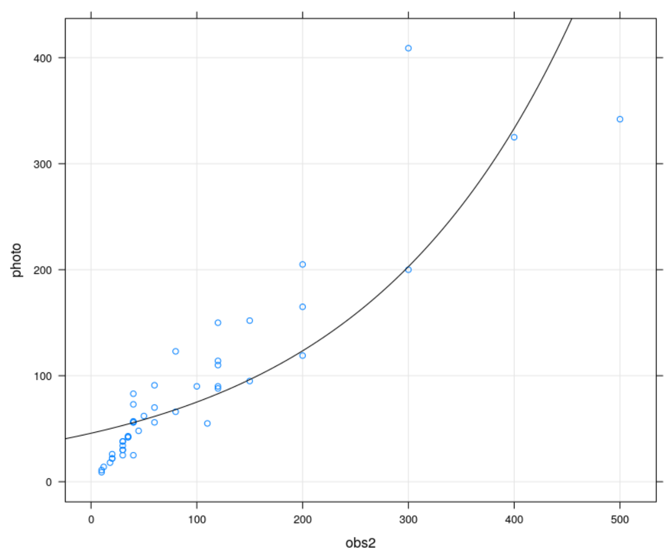
```r
fmp1 <- glm(photo ~ obs1, snowgeese, family = poisson("log"))
summary(fmp1)
```

```

Call:
glm(formula = photo ~ obs1, family = poisson("log"), data = snowgeese)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-11.516   -4.602   -1.296    2.939   14.351

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) 4.020e+00  2.098e-02  191.55   <2e-16 ***
obs1        4.759e-03  9.689e-05   49.12   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 2939.7  on 44  degrees of freedom
Residual deviance: 1274.9  on 43  degrees of freedom
AIC: 1546.8

Number of Fisher Scoring iterations: 5
```

```r
fmp2 <- glm(photo ~ obs2, snowgeese, family = poisson("log"))
summary(fmp2)
```

```
Call:
glm(formula = photo ~ obs2, family = poisson("log"), data = snowgeese)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-9.4531  -3.4545  -0.4068   1.6597  12.6966

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) 3.823e+00  2.377e-02  160.84   <2e-16 ***
obs2        4.966e-03  9.408e-05   52.78   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 2939.73  on 44  degrees of freedom
Residual deviance:  773.67  on 43  degrees of freedom
AIC: 1045.6

Number of Fisher Scoring iterations: 4
```

```
xyplot(photo ~ obs2, snowgeese, grid = TRUE, aspect = "iso") +
    layer(panel.curve(predict(fmp2, newdata = list(obs2 = x), type = "response")))
```

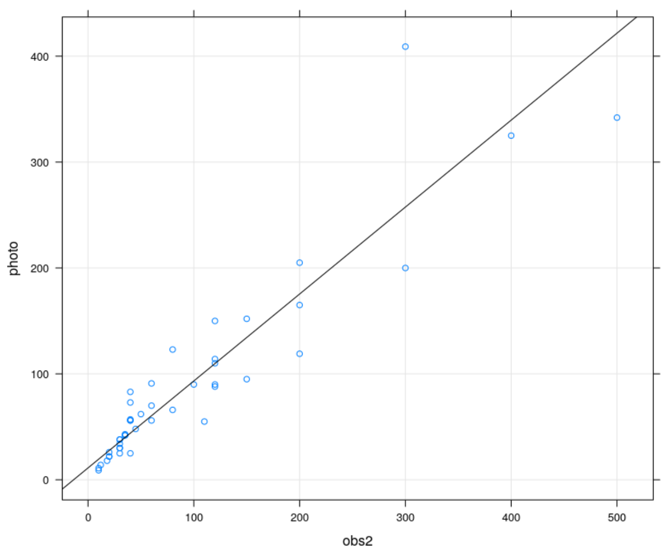



```
fmp3 <- glm(photo ~ obs2, snowgeese, family = poisson("identity"))
summary(fmp3)
```

```
Call:
glm(formula = photo ~ obs2, family = poisson("identity"), data = snowgeese)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
```

```
-5.0628  -1.6622  -0.3158   1.3064   8.6863

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) 11.22312    1.39585    8.04 8.96e-16 ***
obs2         0.82102    0.01948   42.14  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 2939.73  on 44  degrees of freedom
Residual deviance:  324.55  on 43  degrees of freedom
AIC: 596.51

Number of Fisher Scoring iterations: 6
```

```
xyplot(photo ~ obs2, snowgeese, grid = TRUE, aspect = "iso") +
    layer(panel.curve(predict(fmp3, newdata = list(obs2 = x), type = "response")))
```



Diagnostics for GLMs

- For the most part, based on (final) WLS approximation
- Hat-values: Can be taken from WLS approximation (technically depends on y as well as X)
- Residuals: can be of several types, `residuals(object, type = ...)` in R
  - `"response"` : $y_i - \hat{\mu}_i$
  - `"working"` : $z_i - \hat{\eta}_i$ (residuals from WLS approximation)
  - `"deviance"` : square root of i-th component of deviance (with appropriate sign)
  - `"pearson"` : $\frac{\sqrt{\hat{\varphi}}(y_i - \hat{\mu}_i)}{\sqrt{\hat{V}(y_i)}}$
- Other diagnostic measures and plots have similar generalizations

Quasi-likelihood families

- Binomial and Poisson families have $\varphi = 1$
- We can still pretend that there is a dispersion parameter φ during estimation
- There is no corresponding response distribution or likelihood
- The IRLS procedure still works (and gives identical estimates for β)
- However, estimated $\hat{\varphi} > 1$ indicates overdispersion
- Tests can be adjusted accordingly
- This approach is known as quasi-likelihood estimation

Example: Quasi-Poisson model for snow geese counts

```
fmp4 <- glm(photo ~ obs2, snowgeese, family = quasipoisson("identity"))
summary(fmp4)
```

```

Call:
glm(formula = photo ~ obs2, family = quasipoisson("identity"),
    data = snowgeese)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-5.0628  -1.6622  -0.3158   1.3064   8.6863

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 11.22312    3.93720   2.851  0.00668 **
obs2         0.82102    0.05496  14.939  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for quasipoisson family taken to be 7.956067)

    Null deviance: 2939.73  on 44  degrees of freedom
Residual deviance:  324.55  on 43  degrees of freedom
AIC: NA

Number of Fisher Scoring iterations: 6
```