Continuity and Incentive Compatibility in Cardinal Voting Mechanisms *

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Abstract

We show that every cardinal incentive compatible voting mechanism satisfying a continuity condition, must be *ordinal.* Our results apply to many standard models in mechanism design without transfers, including the standard voting models with *any* domain restrictions.

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1 Introduction

Many important models in mechanism design preclude the use of monetary transfers. This may be due to ethical (school choice problems, kidney exchange problems) or institutional (voting) reasons. A typical model in such an environment consists of a set of alternatives and ordinal preferences (strict linear orders) of agents over these alternatives. A mechanism or a social choice function (scf) selects a lottery over the set of alternatives. A notable feature of these models is that they only use ordinal information about preferences over alternatives. In this paper, we ask if there is any loss of generality in restricting attention to ordinal mechanisms in such environments.

Casual reasoning suggests that a mechanism that uses "a lot of" information regarding agents' preferences is more susceptible to manipulation by agents. Two well-known results in this literature are illustrations of this informal principle. If all preferences are admissible, the Gibbard-Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975) states that the only information that an incentive compatible mechanism can use, is the preference information of a *single* (dictator) agent. If preferences are single-peaked, only *peaks* of agent preferences can be used in computing the social outcome at a profile (Moulin, 1983). Our results accord with this principle. Although cardinal information is available, we show that incentive compatibility in (private values) voting models implies that most of it cannot be used by the designer.

If the solution concept is dominant strategy and the mechanism is deterministic (i.e., lotteries produced by the scf are degenerate), then incentive compatibility immediately implies that the scf cannot be sensitive to cardinal information.[1] However, the answer to this question is not obvious if either (a) we allow for randomization and/or (b) we use the weaker solution concept of Bayesian incentive compatibility. [2] The central issue in such cases is that an agent has to evaluate lotteries based on cardinal utilities over alternatives. We follow the standard practice of evaluating lotteries using the expected utility criterion. Consequently, the evaluation of lotteries uses cardinal preference information of agents. Moreover, it implicitly imposes a domain restriction on agent preferences over the outcomes of the mechanism

[1] To see this, suppose the scf picks alternative a when agent i reports some utility representation of his true preference ordering P_i (a strict ordering), but picks an alternative b for some other utility representation. If $a \neq b$, then agent i will clearly manipulate to the utility representation that picks the higher ranked alternative according to P_i.

[2] The demanding requirement of dominant strategies with deterministic mechanisms often leads to impossibility results in standard voting models (Gibbard, 1973; Satterthwaite, 1975). Using randomization or the weaker solution concept of Bayesian incentive compatibility expands the set of incentive compatible mechanisms. Randomization is a also a natural device for achieving fairness.

(i.e. lotteries). It seems natural therefore to believe that this cardinal preference information can be used by the mechanism designer to determine the value of the scf at various profiles.

Our main result shows this to be false. We establish that every dominant strategy incentive compatible, cardinal and random scf that satisfies some version of continuity, is ordinal. We discuss two notions of ordinality and, correspondingly, two notions of continuity. To understand them, note that the *type* of an agent in our cardinal model is a utility vector over alternatives and agents use expected utility to evaluate lotteries. Now, consider an agent i and fix the utility vectors of other agents at u_{-i}.

Our first notion of ordinality requires that we treat two utility vectors representing the same ordinal ranking over *lotteries* in the same manner. In other words, we say that an scf is *vNM-ordinal* if u_i and u'_i are two utility vectors which are affine transformations of each other, then the outcome of the scf is the same at (u_i, u_{-i}) and (u'_i, u_{-i}).

To understand vNM-ordinality better, note that the outcome of an scf in our model is a lottery over the set of alternatives. Agents evaluate lotteries over alternatives using the expected utility criterion. By the expected utility theorem, the set of all utility vectors over alternatives can be partitioned into equivalence classes where utility vectors within each equivalence class rank lotteries in the same way. vNM-ordinality requires that scfs assign the same lottery to all utility vectors within an equivalence class.

Note however that even though these utility vectors rank the lotteries the same manner, they *do not* assign them the same expected utility values. vNM-ordinality amounts to ignoring the expected values of the lotteries and only considering ordinal ranking of lotteries. Though this is without loss of generality in a model where the designer only cares about ordinal ranking of lotteries (i.e., utility is ordinal), as we show, vNM-ordinality is *not* implied by incentive compatibility. Hence, it is not an assumption that can be imposed without loss of generality when utility is cardinal.

Our second notion of ordinality, which we call *strong ordinality*, requires that the scf be sensitive to only ordinal ranking of *alternatives*. In other words, strong ordinality says that if u_i and u'_i are two utility vectors which represent the same ordinal ranking over alternatives, then the outcome of the scf is the same at (u_i, u_{-i}) and (u'_i, u_{-i}). Strong ordinality implies vNM-ordinality because if u_i and u'_i are affine transformations of each other, then they represent the same ordinal ranking over alternatives.

We impose two versions of continuity. Fix the types of all agents except an arbitrary agent i. If i varies the cardinal intensities of her preferences over alternatives without changing her ordinal ranking over the alternatives, then the outcome of the scf must vary continuously. Our notion of continuity is clearly weaker than the standard notion in that it applies only to subdomains where the ordinal rankings over alternatives are unchanged. Thus, our formulation allows scfs to be sensitive to cardinal intensities but restricts them to be continuous in a limited sense. We call this requirement *cone continuity (c-continuity)*. We also consider a stronger version of continuity where we require the outcome of the scf to vary *uniformly* continuously. We call this requirement *uc-continuity*. [3]

Our main result shows that every dominant strategy incentive compatible scf is vNM-ordinal *almost everywhere*. In other words, there is a dense subset of the domain where the scf is vNM-ordinal and the complement of this subset has measure zero. We give an example to illustrate that dominant strategy incentive compatibility does not imply vNM-ordinality (everywhere) - this is in contrast to the trivial conclusion in the deterministic scf case, where incentive compatibility implies (strong) ordinality. We use our main result to establish two striking results: (a) every c-continuous and dominant strategy incentive compatible scf is vNM-ordinal and (b) every uc-continuous and dominant strategy incentive compatible scf is strongly ordinal. We emphasise here that incentive compatibility itself implies c-contunuity of the scf almost everywhere. Our assumption implies that if we strengthen it to everywhere, we get vNM-ordinal scfs. Further, if we strengthen it to uc-continuity everywhere, we get strongly ordinal scfs.

It is well known that there are incentive compatible scfs that are not strongly ordinal - we give some examples later. The uc-continuity condition requires that as the cardinal value of all the alternatives approach zero for the agent, the value of the scf must converge to any single value. This drives the strong ordinality result.

If the solution concept is Bayesian incentive compatibility, our conclusions are in terms of interim allocation probabilities. [4] In particular, we show that every cardinal Bayesian incentive compatible mechanism satisfying c-continuity property must be such that an agent's interim allocation probability of each alternative does not change whenever she changes her type by using an affine transformation. Similarly, every cardinal Bayesian incentive compatible mechanism satisfying uc-continuity property must be such that an agent's interim allocation probability of each alternative does not change whenever she changes her type such that the ordinal ranking over alternatives do not change. We note that these results for the Bayesian incentive compatibility case requires no assumptions on the priors of the agents.

[3] In general, uniform continuity is stronger than continuity, but they become equivalent if the domain is compact. Since we apply our continuity requirements on subdomains which are open, this equivalence is not true for us.

[4] The interim allocation probability of an alternative being chosen at a type of agent i is the expected probability of that alternative being chosen, where the expectation is the conditional expectation over all possible types of other agents conditional on the type of agent i.

As an application of our result, we show that the utilitarian scf is not Bayesian incentive compatible under mild conditions on the priors of the agents and the type space. Further, we show that among all weighted utilitarian scfs, except the dictatorship (which assigns positive weight on one agent and zero weight on all other agents), no other scf is Bayesian incentive compatible.

Our results apply to standard voting models but can also be applied to private good allocation problems (for instance, matching models) with the additional assumption of *non-bossiness*. Non-bossiness requires the following: a type change by an agent that does not change her allocation also leaves the allocations of all other agents unchanged. Introduced by Satterthwaite and Sonnenschein (1981), non-bossiness is a commonly used condition in the literature (Papai, 2000; Ehlers, 2002).

We believe that the paper contributes to the literature on mechanism design without transfers in several ways. It provides a foundation for the use of ordinal mechanisms. Moreover, we believe that our paper makes a methodolgical contribution. We use techniques from multidimensional mechanism design with transfers, particularly subgradient techniques used in that literature, to prove our results. Related methods have been used in some restricted one dimensional problems in the voting literature recently (Borgers and Postl, 2009; Goswami et al., 2014; Gershkov et al., 2014; Hafalir and Miralles, 2015). To the best of our knowledge, ours is the first paper to use the multidimensional versions of these results in such a setting.

We define the type space of an agent by specifying an arbitrary set of permissible linear orders over alternatives and considering all positive utility functions representing these orders. Thus our specification requires maximal cardinal richness consistent with an arbitrary set of ordinal restrictions. Our results therefore hold for standard unrestricted ordinal preferences, single-peaked ordinal preferences, and all standard domain restrictions studied in strategic social choice theory - here, domain restriction means restrictions on the ordinal ranking of alternatives (degenerate lotteries).

1.1 Cardinal versus Ordinal Utility

A critical feature of our model is the cardinality assumption, i.e. the assumption that every agent can associate a real number (or value) with every alternative. We believe this is a reasonable assumption in many situations of interest. For instance, the alternatives could be possible locations of a public facility. The utility associated with each location by an agent is the (privately observed) transportation or time cost to each location. Or the alternatives could be production plans relevant to several managers of a firm. The value of each plan to a manager is the profit associated with it. Moreover, each manager's profit is private information because of non-publicly observed costs. Notice that these utilities are measured in specific units in these models - time/money in the first example and money in the second example. Hence, the utility numbers are easily measurable and carry significance. Without these utility numbers, a planner will have to disregard intensity of preferences of agents in these models.

We note that the use of cardinal utilities in models without transfers, is widespread - recent instances include Schmitz and Troger (2012); Nandeibam (2013); Azrieli and Kim (2014); Ben-Porath et al. (2014); Ashlagi and Shi (2015); Gershkov et al. (2014). [5]

There is a long standing debate in utility theory on the issue whether utility is ordinal or cardinal (Alchian, 1953; Strotz, 1953; Baumol, 1958). Our results provide a mechanism design perspective to this debate by showing that *even if* utility is cardinal, incentive compatibility and continuity imply that this cardinal information must be ignored. Hence, our results can be seen as providing a foundation for using ordinal utility models.

2 The Model

We first present a simple one-agent model. We show later how the single-agent results can be extended to the multiple agent framework. There is a finite set of alternatives denoted by $A = \{a, b, c, \ldots\}$ with $|A| \geq 2$. Let $\mathcal{P}$ be the set of all strict (linear) orderings over the set of alternatives A. For every $P \in \mathcal{P}$, we say a utility function $v : A \to \mathbb{R}_+$ represents P if for all $a, b \in A$, aPb if and only if $v(a) > v(b)$. For every $P \in \mathcal{P}$, let V^P be the set of all utility functions that represent P. Note that V^P is an open cone in $\mathbb{R}_+^{|A|}$. [6]

Throughout the paper, we will fix a domain $\mathcal{D} \subseteq \mathcal{P}$ and let $V = \cup_{P \in \mathcal{D}} V^P$. The type of the agent is a vector $v \in V$. A **social choice function (scf)** is a map $f : V \to \mathcal{L}(A)$, where $\mathcal{L}(A)$ is the set of lotteries over A. Notice that our scfs allow for randomization. For every $v \in V$, we denote the probability that the scf f chooses $a \in A$ at v by $f^a(v)$.

We investigate two notions of incentive compatibility: dominant strategy and Bayesian. When we show how we extend our results to a multi-agent model, we also clarify how our results adapt in these two notions of incentive compatibility. In the one-agent model, incentive compatibility is defined as follows.

[5] Gershkov et al. (2014) consider deterministic dominant strategy incentive compatible mechanisms, where restricting attention to ordinal mechanisms is without loss of generality. However, they consider cardinal utilities to compute an *optimal* mechanism.

[6] Our results can be extended if V^P consists of all utility vectors, including the negative ones, representing P.

Definition 1 *An scf f is* **incentive compatible** *if for all $v, v' \in V$, we have*

$$v \cdot f(v) \geq v \cdot f(v').$$

Depending on the solution concept, incentive compatibility here may refer to dominant strategy incentive compatibility or Bayesian incentive compatibility. [7]

2.1 Ordinal Mechanisms

We introduce two notions of ordinality here. The first one is a strong notion of ordinality in this context.

Definition 2 *An scf f is* **strongly ordinal** *if for all $P \in \mathcal{D}$, for all $v, v' \in V^P$, we have $f(v) = f(v')$.*

Strong ordinality requires that we completely ignore cardinal intensities of agents on the alternatives and only consider ordinal rankings. Note that strong ordinality only requires the scf to produce the same outcome in each *subdomain*, i.e., fixing the ordinal ranking of alternatives in A. Hence, it does not require the scf to be a constant scf - in particular, it can be sensitive to ordinal ranking over alternatives.

However, since the outcome of the scf is a lottery over alternatives, the appropriate notion of ordinality in this context must ignore cardinal utilities of lotteries and only consider ordinal ranking of lotteries. Since we are using expected utility to compute cardinal utilities of lotteries, the expected utility theorem defines a specific ordinal ranking of lotteries.

For any $u, v \in V$, u is an affine transformation of v if there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u(a) = \alpha v(a) + \beta$ for all $a \in A$. Any utility function v induces a ranking over lotteries in $\mathcal{L}(A)$ via the expected utility theorem. If u is an affine transformation of v, then u and v induce the same ranking over lotteries. Consequently, the set of utility functions V can be partitioned into equivalence classes, each representing a unique ranking of lotteries in $\mathcal{L}(A)$. This gives rise to the following notion of ordinality.

Definition 3 *An scf f is* **vNM-ordinal** *if for every $u, v \in V$ such that u is an affine transformation of v, we have $f(u) = f(v)$.*

[7] The incentive compatibility of an ordinal scf is straightforward to define if the solution concept is dominant strategies and the scf is deterministic. If the solution concept is Bayesian or if we consider randomized scfs, then incentive compatibility in an ordinal model is usually defined in terms of a first-order stochastic dominance relation (Gibbard, 1977; d'Aspremont and Peleg, 1988; Majumdar and Sen, 2004).

A vNM-ordinal scf does not distinguish between utility functions belonging to the same equivalence class. It should be noted that vNM ordinality is *much weaker* than strong ordinality that we have defined.

If we use a vNM-ordinal scf, then it is without loss of generality to restrict attention to the following domain.

Definition 4 *A domain $V^{vNM} \subseteq \mathbb{R}^{|A|}$ is a* **vNM domain** *if for every $v \in V^{vNM}$, $v(a) = 1$ if $v(a) > v(b)$ for all $b \neq a$ and $v(a') = 0$ if $v(a') < v(b)$ for all $b \neq a'$.*

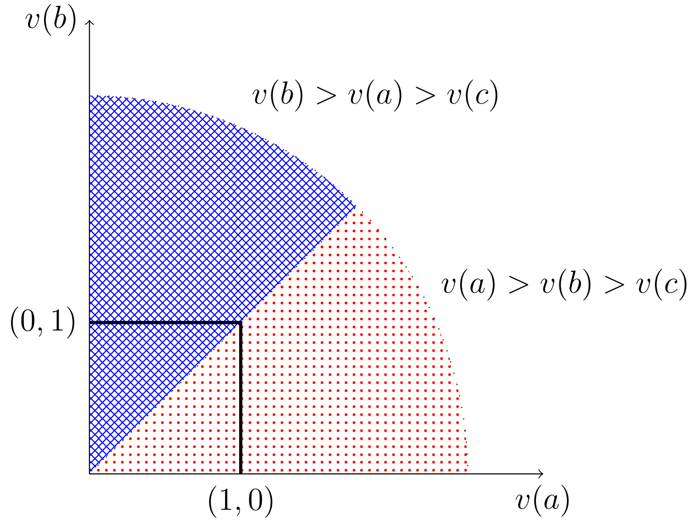

Figure 1: Our domain and the vNM domain

Consider an example with three alternatives $A = \{a, b, c\}$ and two possible strict linear orders in $\mathcal{D}$: for every $v \in V$, either $v(a) > v(b) > v(c)$ or $v(b) > v(a) > v(c)$. The projection of V to the hyperplane $v(c) = 0$ is shown in Figure 1, which has two cones (above and below the 45-degree line), each representing the strict linear orders in $\mathcal{D}$. The vNM domain in this case consists of two lines as shown in Figure 1.

If an scf f is vNM-ordinal, then its domain can be restricted to V^{vNM} in the following sense: for every $v \in V$, there exists a unique $v' \in V^{vNM}$ such that v' is an affine transformation of v and $f(v) = f(v')$. Moreover, any f defined on the vNM domain can be uniquely extended to a vNM-ordinal scf over the entire domain. Further, such an extension is incentive compatible if and only if f is incentive compatible on the vNM domain.

3 Incentive Compatibility and vNM-ordinality

We are not assuming vNM-ordinality to begin with - we do not see any compelling reason to do so, particularly in a cardinal setting. By not assuming vNM-ordinality we are allowing the scf to be sensitive to the *expected value* of lotteries. Moreover, incentive compatibility does not imply vNM-ordinality. The following example illustrates that we can have incentive compatible scfs that are not vNM-ordinal.

Example 1 Let $A = \{a, b, c\}$. Fix a strict linear order P, and let the top, middle, and bottom ranked alternatives be a, b, and c respectively. We now define an scf f as follows. For any $v \in V^P$, we define $f(v) \equiv (f^a(v), f^b(v), f^c(v))$ as:

$$f(v) = \begin{cases} (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) & \text{if } [v(a) - v(b)] > [v(b) - v(c)] \\ (0, \frac{3}{4}, \frac{1}{4}) & \text{if } [v(a) - v(b)] < [v(b) - v(c)] \\ (0, \frac{3}{4}, \frac{1}{4}) & \text{if } [v(a) - v(b)] = [v(b) - v(c)] \text{ and } v(a) - v(c) > 1 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) & \text{if } [v(a) - v(b)] = [v(b) - v(c)] \text{ and } v(a) - v(c) \leq 1 \end{cases}$$

Figure 2 shows the example in $\mathbb{R}^2$ by projecting the type space onto the hyperplane $v(c) = 0$. The regions that assign the lottery $(0, \frac{3}{4}, \frac{1}{4})$ are shown above the line $v(a) = 2v(b)$ (in red color with dashes), while the region that assign the lottery $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ is shown below the line $v(a) = 2v(b)$ (in blue color with dots). On the line $v(a) = 2v(b)$, f assigns the lottery $(0, \frac{3}{4}, \frac{1}{4})$ for $v(a) > 1$ and the lottery $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ for $v(a) \leq 1$.

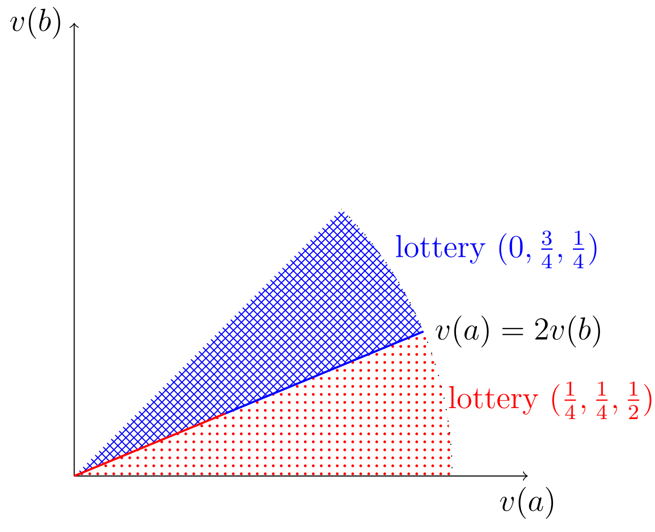

Figure 2: An example of an scf that violates vNM-ordinality

We first claim that f is incentive compatible. Pick $v, v' \in V^P$. We consider two cases.

Case 1. Suppose $f(v) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Notice that, by definition of f, $[v(a) - v(b)] \geq [v(b) - v(c)]$. If $f(v') = f(v)$, then the agent cannot manipulate from v to v'. So, suppose that $f(v') = (0, \frac{3}{4}, \frac{1}{4})$. Then, truthtelling at v gives the agent an utility equal to

$$\frac{1}{4}[v(a) + v(b)] + \frac{1}{2}v(c).$$

Manipulating to v' gives the agent an utility equal to

$$\frac{3}{4}v(b) + \frac{1}{4}v(c).$$

The difference in utility between truthtelling and manipulation is

$$\frac{1}{4}[v(a) - 2v(b) + v(c)] = \frac{1}{4}\big[[v(a) - v(b)] - [v(b) - v(c)]\big] \geq 0,$$

where the inequality followed from our earlier conclusion that $[v(a) - v(b)] \geq [v(b) - v(c)]$. Hence, f is incentive compatible.

Case 2. Suppose $f(v) = (0, \frac{3}{4}, \frac{1}{4})$. Notice that, by definition of f, $[v(a) - v(b)] \leq [v(b) - v(c)]$. If $f(v') = f(v)$, then the agent cannot manipulate from v to v'. So, suppose that $f(v') = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Then, truthtelling at v gives the agent an utility equal to

$$\frac{3}{4}v(b) + \frac{1}{4}v(c).$$

Manipulating to v' gives the agent an utility equal to

$$\frac{1}{4}[v(a) + v(b)] + \frac{1}{2}v(c).$$

The difference in utility between truthtelling and manipulation is

$$\frac{1}{4}[-v(a) + 2v(b) - v(c)] = \frac{1}{4}\big[[v(b) - v(c)] - [v(a) - v(b)]\big] \geq 0,$$

where the inequality followed from our earlier conclusion that $[v(a) - v(b)] \leq [v(b) - v(c)]$. Hence, f is incentive compatible.

To verify that f is not vNM-ordinal, pick $v \equiv (2, 1, 0)$ and $v' \equiv (1, 0.5, 0)$. Note that $f(v') = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and $f(v) = (0, \frac{3}{4}, \frac{1}{4})$, but v is an affine scaling of v'. This implies that f is **not** vNM-ordinal.

3.1 Almost vNM-ordinality

Though Example 1 showed that incentive compatibility does not imply vNM-ordinality, one can see that vNM-ordinality is violated in that example in a small subset of the domain. We formalize this intuition and show that incentive compatibility implies vNM-ordinality *almost* everywhere.

Definition 5 *A social choice function f is* **almost vNM-ordinal** *if for every $P \in \mathcal{D}$, there exists $\bar{V}^P \subseteq V^P$ such that*

(i) $\bar{V}^P$ is dense in V^P and $V^P \setminus \bar{V}^P$ has measure zero.

(ii) if $u \in \bar{V}^P$ and v is an affine transformation of u, then $v \in \bar{V}^P$.

(iii) for every $u, v \in \bar{V}^P$, if u and v are affine transformations of each other, then $f(v) = f(u)$.

A scf is almost vNM-ordinal if it is vNM-ordinal over a set $\bar{V}^P$ that is "generic" with respect to V^P. Part (ii) of Definition 5 ensures that vNM-ordinality applies non-vacuously - whenever u belongs to $\bar{V}^P$, so do all its affine transformations. In Figure 1, there are sets that are generic in each of the blue and red cones. Pick u in one of these sets, say in the red cone. Then every v on the infinite ray containing the origin and u (excluding the origin) belongs to the generic set for the red cone and vNM ordinality applies, i.e. $f(u) = f(v)$.

The main result of the paper is the following.

Theorem 1 *Every incentive compatible scf is almost vNM-ordinal.*

Proof: Define the indirect utility function $\mathcal{U}^f$ for any incentive compatible scf f as follows: for all $v \in V$, $\mathcal{U}^f(v) = v \cdot f(v)$. Fix a $P \in \mathcal{D}$ and let $v, v' \in V^P$. An immediate consequence of incentive compatibility is:

$$\mathcal{U}^f(v) \geq \mathcal{U}^f(v') + (v - v') \cdot f(v')$$

Consequently, $\mathcal{U}^f$ is convex and $f(v')$ is a subgradient of $\mathcal{U}^f$ at v' (see Chapter 23 in Rockafellar (1970)). Moreover, $f(v')$ is the gradient of $\mathcal{U}^f$ at all points v' where $\mathcal{U}^f$ is differentiable (Theorem 25.1 in Rockafellar (1970)). Let $\bar{V}^P$ be the set of all such points. Since a convex function is differentiable almost everywhere, $\bar{V}^P$ is dense in V^P and its complement $V^P \setminus \bar{V}^P$ has measure zero. We show below that $\bar{V}^P$ satisfies parts (ii) and (iii) of Definition 5.

Consider any $v \in \bar{V}^P$ and consider $u \in V^P$ such that $u(a) = \alpha v(a) + \beta$ for all $a \in A$, where $\alpha > 0$ and $\beta \in \mathbb{R}$. Using vector notation, we will write $u = \alpha v + \mathbf{1}_\beta$, where $\mathbf{1}_\beta$ is the vector in $\mathbb{R}^{|A|}$ all of whose components are β. We will show that (a) $f(u) = f(v)$ and (b) $u \in \bar{V}^P$. We do this in three steps.

Step 1 - A property of $\mathcal{U}^f$. Incentive constraints imply that

$$v \cdot f(v) \geq v \cdot f(u),$$

$$\alpha v \cdot f(u) + \beta = u \cdot f(u) \geq u \cdot f(v) = \alpha v \cdot f(v) + \beta.$$

The second inequality implies that $v \cdot f(u) \geq v \cdot f(v)$, and this along with the first inequality implies that

$$v \cdot f(v) = v \cdot f(u). \tag{1}$$

Now,

$$\begin{aligned}\mathcal{U}^f(u) &= u \cdot f(u) = \alpha v \cdot f(u) + \beta \\ &= \alpha v \cdot f(v) + \beta \\ &= \alpha \mathcal{U}^f(v) + \beta,\end{aligned}$$

where the third equality follows from Equation 1.

Step 2. For every $\delta > 0$ and $a \in A$, let $\mathbf{1}^a_\delta$ denote the vector in $\mathbb{R}^{|A|}$ whose component for alternative a is δ with all other components being zero. Let $\delta' = \frac{\delta}{\alpha}$. We have

$$\begin{aligned}\mathcal{U}^f(u + \mathbf{1}^a_\delta) &= \mathcal{U}^f(\alpha v + \mathbf{1}_\beta + \mathbf{1}^a_\delta) \\ &= \mathcal{U}^f(\alpha(v + \mathbf{1}^a_{\delta'}) + \mathbf{1}_\beta) \\ &= \alpha \mathcal{U}^f(v + \mathbf{1}^a_{\delta'}) + \beta,\end{aligned}$$

where the last equality follows from Step 1.

Step 3. We can now conclude the proof. Consider any $\delta > 0$ and $a \in A$. Let $\delta' = \frac{\delta}{\alpha}$. Now, since $v \in \bar{V}^P$, $f(v)$ is the gradient of $\mathcal{U}^f$ at v. Using this, we can write

$$\begin{aligned}
\lim_{\delta \to 0} \frac{\mathcal{U}^f(u + \mathbf{1}^a_\delta) - \mathcal{U}^f(u)}{\delta} &= \lim_{\delta \to 0} \frac{\big[\alpha \mathcal{U}^f(v + \mathbf{1}^a_{\delta'}) + \beta\big] - \big[\alpha \mathcal{U}^f(v) + \beta\big]}{\delta} \\
&= \lim_{\delta \to 0} \frac{\alpha \mathcal{U}^f(v + \mathbf{1}^a_{\delta'}) - \alpha \mathcal{U}^f(v)}{\delta} \\
&= \lim_{\delta' \to 0} \frac{\mathcal{U}^f(v + \mathbf{1}^a_{\delta'}) - \mathcal{U}^f(v)}{\delta'} \\
&= f^a(v),
\end{aligned}$$

where the property derived in Step 2 was used in the first equality and the assumption that $v \in \bar{V}^P$, in the last equality. This shows that the partial derivative of $\mathcal{U}^f$ along a exists at u. This implies that $\mathcal{U}^f$ is differentiable at u [8]. Hence, $u \in \bar{V}^P$. But the above equations also show that the gradient at u equals $f(v)$. Hence, $f(u) = f(v)$. This completes the proof. ■

The proof of Theorem 1 reveals that an incentive compatible scf is continuous almost everywhere in the following sense. We record this fact as a corollary below. To state the corollary, we denote by f_P the restriction of an scf f to any arbitrary $P \in \mathcal{D}$. Note that f_P is a map $f_P : V^P \to \mathcal{L}(A)$.

Corollary 1 *Suppose f is an incentive compatible scf. For every P, f_P is continuous almost everywhere in V^P.*

3.2 Continuous Mechanisms

Corollary 1 shows that some form of continuity is already implied almost everywhere by incentive compatibility. In this section, we begin by modifying this implication in a minor way. While the modifications we talk about are technical, we believe continuity is a natural assumption to impose on mechanisms - small changes in player announcements should not lead to dramatic changes in outcomes. All our continuity properties are imposed on f_P for arbitrary P. We begin with the standard notion of continuity.

Definition 6 *An scf f is* **cone continuous (c-continuous)** *if for every $P \in \mathcal{D}$, f_P is continuous in V^P, i.e., for every $v \in V^P$, for every $\epsilon > 0$, there exists δ such that for all $v' \in V^P$ with $\|v - v'\| < \delta$, we have $\|f(v) - f(v')\| < \epsilon$.*

[8] A convex function defined on an open convex set is differentiable at a point if and only if its partial derivatives exist at that point, see Chapter 23 in Rockafellar (1970).

Fix an scf f and $P \in \mathcal{P}$. By allowing for cardinal scfs, we are allowing f_P to be a non-constant map, i.e., for any $v, v' \in V^P$, $f(v)$ can be different from $f(v')$. Note that an ordinal scf trivially satisfies c-continuity since for every P, $f_P(v) = f_P(v')$ for all $v, v' \in V^P$. c-continuity does not impose any restriction between f_P and $f_{P'}$ if $P \neq P'$.

Also, note that Theorem 1 already implies that an incentive compatible scf is c-continuous almost everywhere. So, c-continuity assumption strengthens it in a minor way. As a result, Theorem 1 is extended in the following manner.

THEOREM **2** *Every incentive compatible and c-continuous scf is vNM-ordinal.*

Proof: Let f be an incentive compatible and c-continuous scf. By Theorem 1, f is almost vNM-ordinal. Fix a $P \in \mathcal{D}$. Almost vNM-ordinality implies that there exists $\bar{V}^P \subseteq V^P$ such that $\bar{V}^P$ is dense in V^P and $V^P \setminus \bar{V}^P$ has measure zero. Further, for every $v \in \bar{V}^P$ and $u \in V^P$ such that $u = \alpha v + \mathbf{1}_\beta$ with $\alpha > 0$ and $\beta \in \mathbb{R}$, we have $u \in \bar{V}^P$ and $f(v) = f(u)$.

Pick $v \in V^P \setminus \bar{V}^P$. Since $\bar{V}^P$ is dense in V^P, there exists a sequence $\{v^k\}$ such that $v^k \in \bar{V}^P$ and the sequence converges to v. By continuity of f, the sequence $\{f(v^k)\}$ converges to $f(v)$.

Consider $u = \alpha v + \mathbf{1}_\beta$, where $\alpha > 0$ and $\beta \in \mathbb{R}$. For every v^k in the sequence $\{v^k\}$, define $u^k = \alpha v^k + \mathbf{1}_\beta$. Clearly the sequence $\{u^k\}_k$ converges to u. Since $v^k \in \bar{V}^P$, it follows from the properties of $\bar{V}^P$ that $u^k \in \bar{V}^P$. The continuity of f implies that the sequence $\{f(u^k)\}$ converges to $f(u)$. By Theorem 1, $f(v^k) = f(u^k)$. Hence, the sequence $\{f(u^k)\}$ also converges to $f(v)$, i.e. $f(v) = f(u)$. ■

The following example illustrates Theorem 2.

EXAMPLE **2** Let $A = \{a, b, c\}$. Let P be such that the top, middle, and bottom ranked alternatives are a, b, and c respectively. The scf is specified over the vNM domain and extended to the the entire domain. In the vNM domain, a utility function v representing P will have $v(a) = 1, v(c) = 0$, and $v(b) \in (0, 1)$. Let $\theta = v(b)$. A utility function or type in this domain is therefore determined entirely by θ. Define an scf f_P over this domain as follows:

$$\begin{aligned}
f_P^a(\theta) &= \frac{1}{2} - \frac{\theta^2}{4} \\
f_P^b(\theta) &= \frac{\theta}{2} \\
f_P^c(\theta) &= \frac{1}{2} - \frac{\theta}{2} + \frac{\theta^2}{4}.
\end{aligned}$$

Here $f_P^a(\theta), f_P^b(\theta)$ and $f_P^c(\theta)$ denote the probabilities assigned to alternatives a, b and c respectively for type θ in the vNM domain.

The scf is now extended to the domain V^P in the following way. For all $v \in V^P$, let $\theta(v) = \frac{v(b)-v(c)}{v(a)-v(c)}$. Then $f_P^x(v) = f_P^x(\theta(v))$ for all alternatives $x \in \{a, b, c\}$. Observe that f_P is cardinal in the sense that the agent's intensity of preference for the middle-ranked alternative b is used to generate the outcome of the scf.

We first claim that f_P is incentive compatible over V^P. It follows from the construction of f_P that it suffices to prove incentive- compatibility over the vNM domain. Consider this domain and observe that the agent with type θ who reports a type θ', has net utility is

$$\frac{1}{2} - \frac{\theta'^2}{4} + \frac{\theta\theta'}{2}.$$

On the other hand, truth telling gives a net utility of

$$\frac{1}{2} - \frac{\theta^2}{4} + \frac{\theta^2}{2} = \frac{1}{2} + \frac{\theta^2}{4}.$$

The difference between truth telling and deviating to θ' is thus given by

$$\frac{\theta^2}{4} + \frac{\theta'^2}{4} - \frac{\theta\theta'}{2} = \Big(\frac{\theta}{2} - \frac{\theta'}{2}\Big)^2 \geq 0.$$

Hence, f is incentive compatible.

The scf f_P is continuous. Observe that the scf restricted to the vNM domain is continuous in θ by construction, i.e. if $\{\theta^k\}$ is a sequence converging to θ, then the sequence $\{f_P(\theta^k)\}$ converges to $f_P(\theta)$. Let $\{v^k\} \in V^P$ be a sequence converging to $v \in V^P$. Clearly, the sequence $\{\theta(v^k)\}$ converges to $\theta(v)$. By our earlier remark, $\{f(v^k)\}$ converges to $f(v)$.

Pick $v, v' \in V^P$ such that $v' = \alpha v + \beta$ with $\alpha > 0$. Observe that $\theta(v) = \theta(v')$ so that $f_P(v) = f_P(v')$. Thus f_P is vNM-ordinal as required by Theorem 2.

Although, f_P is continuous in v, it does have an unattractive feature. To see this, pick $\theta, \theta' \in (0, 1)$ such that $\theta \neq \theta'$ and let $v, v' \in V^P$ be such that $\theta(v) = \theta$ and $\theta(v') = \theta'$. According to the construction of f_P, $f_P(v) \neq f_P(v')$. Also $f_P(\epsilon v) = f_P(v)$ and $f_P(\epsilon v') = f_P(v')$ for all $\epsilon > 0$. By choosing ϵ small enough, the distance between ϵv and $\epsilon v'$ can be made arbitrarily small. Thus *every* neighbourhood of types around the zero type (the utility function that assigns zero to all alternatives), contains the *entire range* of f_P. The zero type does not belong to V^P but the domain does contain types arbitrarily close to it. Very small variations or mistakes in announcements when utilities of all alternatives are close to zero will lead to very substantial variations in the lotteries generated by f_P. More formally, f_P fails to be *uniformly continuous* although it is continuous.

We show below that the failure of uniform continuity in this example is not accidental - every scf satisfying incentive-compatibility and uniform continuity must be strongly ordinal.

DEFINITION **7** *An scf f is* **uninformly cone continuous (uc-continuous)** *if for every $P \in \mathcal{D}$, f_P is uniformly continuous in V^P, i.e., for every $\epsilon > 0$, there exists δ such that for all $v, v' \in V^P$ with $\|v - v'\| < \delta$, we have $\|f(v) - f(v')\| < \epsilon$.*

A uniformly continuous function is continuous - the converse is true if the domain is compact Rudin (1976). The uc-continuity assumption strengthens Theorem 1 even further.

THEOREM **3** *Every incentive compatible and uc-continuous scf is strongly ordinal.*

Proof: We do the proof in two steps. In Step 1, we prove a stronger result and show that the theorem follows from this step in Step 2.

STEP 1. Consider the following regularity condition.

DEFINITION **8** *An scf f is* **0***-regular, if for all $P \in \mathcal{D}$ and all sequences $\{v^k\}, \{v'^k\}$ in V^P converging to* **0***, we have*

$$\lim_{v^k \to \mathbf{0}} f(v^k) = \lim_{v'^k \to \mathbf{0}} f(v'^k).$$

The **0**-regularity condition applies to type sequences converging to the origin in a given subdomain. It requires the scf to converge to the same value along all such sequences. We first show that every incentive compatible, c-continuous, and **0**-regular scf is strongly ordinal.

LEMMA **1** *Every incentive compatible, c-continuous, and* **0***-regular scf is strongly ordinal.*

Proof: Let f be a incentive compatible, c-continuous, and **0**-regular scf. By Theorem 2, f is a vNM-ordinal scf. Pick any $v, v' \in V^P$, where $P \in \mathcal{D}$ and v, v' represent P. By vNM-ordinality, for every $\epsilon > 0$, we have $f(v) = f(\epsilon v)$ and $f(v') = f(\epsilon v')$. Hence, $f(v) = \lim_{\epsilon \to 0} f(\epsilon v)$ and $f(v') = \lim_{\epsilon \to 0} f(\epsilon v')$. By **0**-regularity, $\lim_{\epsilon \to 0} f(\epsilon v) = \lim_{\epsilon \to 0} f(\epsilon v')$. Therefore, $f(v) = f(v')$ as required. ■

STEP 2. In this step, we show that every uc-continuous scf is c-continuous and **0**-regular. Clearly, a uc-continuous scf is c-continuous (since uniform continuity implies continuity). Let f be an incentive compatible and uc-continuous scf.

CLAIM **1** *Suppose f_P is uniformly continuous. Choose a sequence of types $\{v_k\}$ in V^P converging to* **0**. *The sequence $\{f_P(v_k)\}$ must converge to a point that is independent of the chosen sequence $\{v_k\}$.*

The proof of this claim is provided in the Appendix. The proof follows from the fact that every uniformly continuous function can be extended to its closure in a uniformly continuous manner, and moreover, such an extension is unique - see Exercise 13 in Chapter 4 of Rudin (1976). It is then easy to verify that Claim 1 is true. We provide a self contained proof in the Appendix.

Claim 1 shows that if f is uc-continuous, then it is $\mathbf{0}$-regular, and this concludes our proof. ■

We conclude this section with two remarks.

Remark 1. If $|A| = 2$, Theorem 3 holds without any further assumptions regarding uc-continuity or domains. [9] To see this, suppose $A = \{a, b\}$ and f is a incentive compatible scf. Note that $f(v, a) + f(v, b) = 1$ for all $v \in V$. Using this, for any $v, v' \in V$, incentive compatibility implies that

$$[v(a) - v(b)]f^a(v) + v(b) \geq [v(a) - v(b)]f^a(v') + v(b)$$
$$[v'(a) - v'(b)]f^a(v') + v'(b) \geq [v'(a) - v'(b)]f^a(v) + v'(b).$$

Combining these two inequalities we get, $f^a(v) = f^a(v')$. This also highlights the fact that we can always fix the lowest ranked alternative value at zero and work with the rest of the alternatives, reducing the dimensionality of the problem.

Remark 2. None of the results in this section, Theorems 1, 2 and 3 depend on any restriction of ordinal preferences over alternatives. They remain valid for arbitrary domains of ordinal preferences over alternatives. However the domain must include all cardinalisations of admissible ordinal preferences.

4 Extension to Many Agents

In this section, we discuss extensions of our result to a multi-agent model. Let $N = \{1, \ldots, n\}$ be the set of agents. The type space of agent i will be denoted as V_i, and, as before, it is the set of all non-negative utility functions consistent with some set of strict orderings $\mathcal{D}_i \subseteq \mathcal{P}$. Let $\mathcal{V} = V_1 \times \ldots \times V_n$ be the set of all profiles of types. As before, A is the set of alternatives.

[9] The two alternatives case occupies an important place in the strategic voting literature because the Gibbard-Satterthwaite theorem does not hold in this case. Schmitz and Troger (2012) considers an optimal mechanism design problem in a two-alternative model with cardinal intensities. Similarly, Azrieli and Kim (2014) consider Pareto optimal mechanism design in a two alternatives model with cardinal intensities.

A social choice function f is a map

$$f : \mathcal{V} \to \mathcal{L}(A)^n.$$

Observe that f picks n lotteries at each type profile - one for each agent. This formulation allows us to capture both public good and private good problems. We assume **no externalities**, i.e., the utility of an agent only depends on the lottery chosen for her. For any scf f, $f_i(v)$ and $f_i^a(v)$ will denote the lottery chosen for agent i at type profile v and the corresponding probability of choosing alternative a respectively.

We distinguish between two kinds of models. A **voting** model is one where every scf f satisfies $f_i(v) = f_j(v)$ for every pair of agents $i, j \in N$ and every type profile v. In these models, an scf must choose the *same* lottery for all the agents. This covers the standard strategic voting models. Any model that is not a voting model will be called a **private good** model. In these models, there is no requirement of choosing the same lottery for all agents although there may be restrictions on the choices. For instance, in the one-sided matching model (here A is the set of objects), an scf need not choose the same lottery over objects for each agent, but the lotteries must satisfy feasibility conditions.

As before, we consider two notions of incentive compatibility.

Dominant Strategy Incentive Compatibility. Dominant strategy incentive compatibility requires that for every $i \in N$, $v_i, v_i' \in V_i$ and $v_{-i} \in V_{-i}$, we have $v_i \cdot f_i(v_i, v_{-i}) \geq v_i \cdot f_i(v_i', v_{-i})$. The definitions of c-continuity and uc-continuity can now be straightforwardly adapted with additional qualifiers for each $i \in N$ and each v_{-i}.

To define the notions of ordinality here, we say that two type profiles $v, v' \in \mathcal{V}$ are ordinally equivalent if for every $i \in N$, v_i and v_i' are affine transformations of each other. Similarly, we say two profiles $v, v' \in \mathcal{V}$ are strongly ordinally equivalent if for every $i \in N$, v_i and v_i' represent the same strict ordering in $\mathcal{D}_i$. An scf f is **vNM-ordinal** if for every pair of type profiles $v, v' \in \mathcal{V}$ such that v and v' are ordinally equivalent, we have $f_i(v) = f_i(v')$ for all $i \in N$. An scf f is **strongly ordinal** if for every pair of type profiles $v, v' \in \mathcal{V}$ such that v and v' are strongly ordinally equivalent, we have $f_i(v) = f_i(v')$ for all $i \in N$.

We can now extend Theorems 2 and 3. Note here that we do not require any assumption on $\{\mathcal{D}_i\}_{i \in N}$.

THEOREM 4 *In the voting model, (a) every dominant strategy incentive compatible and c-continuous scf is vNM-ordinal and (b) every dominant strategy incentive compatible and uc-continuous scf is strongly ordinal.*

Proof: Let f be a dominant strategy incentive compatible and c-continuous scf. By virtue of the voting model assumption, $f_i(v'') = f_j(v'')$ for all $i, j \in N$ and v''. Fix any pair of ordinally equivalent type profiles $v, v' \in \mathcal{V}$. We move from v to v' by changing the types of agents one at a time and apply Theorem 2 at each step. This immediately implies that f is vNM-ordinal. A similar proof using Theorem 3 establishes (b). ■

This argument does not work in the private good models for well-known reasons. In the move from (v_i, v_{-i}) to (v'_i, v_{-i}) (where (v_i, v_{-i}) and (v'_i, v_{-i}) are ordinally equivalent), Theorem 2 guarantees $f_i(v_i, v_{-i}) = f_i(v'_i, v_{-i})$ but not $f_j(v_i, v_{-i}) = f_j(v'_i, v_{-i})$ for any $j \neq i$. This can be restored by the familiar non-bossiness condition adapted to our model. An scf f is **non-bossy** if for all $i \in N$, for all $P \in \mathcal{D}$, and for all $v_i, v'_i \in V^P$ and for all $v_{-i} \in V_{-i}$, $f_i(v_i, v_{-i}) = f_i(v'_i, v_{-i})$ implies $f_j(v_i, v_{-i}) = f_j(v'_i, v_{-i})$ for all $j \neq i$. Non-bossiness requires that if an agent is not able to change his allocation, then the allocation of other agents should remain unchanged. It is now easy to see that Theorem 4 holds in private good problem with the additional non-bossiness condition.

Bayesian Incentive Compatibility. Each agent has a conditional distribution over the types of other agents. Thus agent i of type v_i has a probability measure $G_i(\cdot|v_i)$ over V_{-i}. We do not impose any restriction on the distributions and allow for arbitrary correlation. For any scf f, $\Pi_i^f(v_i)$ reflects the interim allocation probability vector at v_i for agent i. Formally, for every $a \in A$, $i \in N$ and $v_i \in V_i$,

$$\Pi_i^f(v_i, a) := \int_{V_{-i}} f_i^a(v_i, v_{-i}) dG_i(v_{-i}|v_i).$$

An scf f is **Bayesian incentive compatible** if for every $i \in N$ and for every $v_i, v'_i \in V_i$, we have $v_i \cdot \Pi_i^f(v_i) \geq v_i \cdot \Pi_i^f(v'_i)$. It is straightforward to extend the definitions of c-continuity and uc-continuity requiring continuity and uniform continuity of Π^f instead of f.

Definition **9** *An scf f is* **ec-continuous** *if for every $i \in N$, for every $P \in \mathcal{D}$, Π_i^f is continuous in V^P.*
An scf f is **euc-continuous** *if for every $i \in N$, for every $P \in \mathcal{D}$, Π_i^f is uniformly continuous in V^P.*

The scf f is **vNM-ordinal in expectation (ev-ordinal)** if for all $i \in N$ and pairs of ordinally equivalent types $v_i, v'_i \in V_i$, we have $\Pi_i^f(v_i) = \Pi_i^f(v'_i)$. Similarly, the scf f is **strongly ordinal in expectation (es-ordinal)** if for all $i \in N$ and pairs of strongly ordinally equivalent types $v_i, v'_i \in V_i$, we have $\Pi_i^f(v_i) = \Pi_i^f(v'_i)$. It is important to observe that the notion of e-ordinality is different from that of ordinality in the dominant strategy case because interim probabilities for an agent depends only on her type rather than on the entire type profile.

With these modifications, Theorems 2 and 3 can be directly extended as follows: every BIC and ec-continuous scf is ev-ordinal and every BIC and euc-continuous scf is es-ordinal. In view of the notion of ordinality in the Bayesian model (see our earlier remark), the distinction between the voting and private good models disappears and the non-bossiness assumption is redundant. Hence, an identical proof as in Theorem 4 gives us the following theorem.

Theorem **5** *Every Bayesian incentive compatible and ec-continuous scf is ev-ordinal. Every Bayesian incentive compatible and euc-continuous scf is es-ordinal.*

5 Applications

In this section, we provide a couple of application of our results. The applications illustrate how our results can be readily used to show that certain class of social choice functions cannot be incentive compatible in our model.

As before, let $N = \{1, \dots, n\}$ be the set of agents. A type profile is denoted by $\mathbf{v} \equiv (v_1, \dots, v_n)$ and let $\mathcal{V}$ be the set of all type profiles.

Definition **10** *The scf $\hat{f} : \mathcal{V} \to \mathcal{L}(A)$ is the proportional utilitarian voting rule if, for all $\mathbf{v} \in \mathcal{V}$ and $a \in A$, we have*

$$\hat{f}^a(\mathbf{v}) = \frac{\sum_{i \in N} v_i(a)}{\sum_{b \in A} \sum_{i \in N} v_i(b)}$$

In other words, the probability of choosing alternative a in profile v is the aggregate utility of a relative to the aggregate utility of all alternatives. Note that the denominator is positive and the numerator is non-negative by our assumption.

Theorem **6** *The proportional utilitarian voting rule is not dominant strategy incentive compatible.*

Proof: We will prove the result by contradiction. Observe that $\hat{f}$ is c-continuous (since admissible orderings are strict and type 0 does not belong to $\mathcal{V}$). Suppose $\hat{f}$ is incentive compatible. Theorem 4 implies that $\hat{f}$ is vNM ordinal. We show that this is false.

Pick $i \in N$ and $\mathbf{v} \in \mathcal{V}$ such that $\frac{v_i(a)}{\sum_{b \in A} v_i(b)} \neq \frac{\sum_{j \neq i} v_j(a)}{\sum_{j \neq i} \sum_{b \in A} v_j(b)}$ for some $a \in A$. Note that such a $\mathbf{v}$ can be found in every open neighborhood of $\mathcal{V}$. Choose $\alpha_i > 0$ and $\alpha_i \neq 1$. Let

$\mathbf{v}' \in \mathcal{V}$ such that $v_i' = \alpha_i v_i$ and $v_j' = v_j$ for all $j \neq i$. Then v and v' are ordinally equivalent and Theorem 4 requires $\hat{f}(\mathbf{v}') = \hat{f}(\mathbf{v})$. However,

$$\hat{f}^a(v') = \frac{\alpha_i v_i(a) + \sum_{j\neq i} v_i(a)}{\alpha_i \sum_{b\in A} v_i(b) + \sum_{j\neq i}\sum_{b\in A} v_j(b)} \neq \frac{v_(a) + \sum_{j\neq i} v_i(a)}{\sum_{b\in A} v_i(b) + \sum_{j\neq i}\sum_{b\in A} v_j(b)} = \hat{f}^a(\mathbf{v}).$$

Therefore $\hat{f}$ is not incentive compatible. ■

Our next example is the utilitarian scf.

Definition **11** *An scf f^* is a* **utilitarian** *scf if for every $\mathbf{v} \in \mathcal{V}$, we have*

$$f^*(\mathbf{v}) \in \arg\max_{a\in A} \sum_{i\in N} v_i(a).$$

Observe that we are assuming that an utilitarian scf is deterministic - this will mean breaking ties at some subset of (measure zero) profile of types in a deterministic manner. As we have remarked earlier, dominant strategy incentive compatible deterministic scfs must be strongly ordinal. The utilitarian scf is clearly not strongly ordinal. We show below that impossibility persists if we weaken the incentive compatibility requirement to Bayesian incentive compatibility. This is done by showing that the utilitarian scf is not ev-ordinal but ec-continuous, and applying Theorem 5. We will show this for the unrestricted domain. Further, we will work with *bounded* type spaces.

Formally, let $\mathcal{P}$ be the set of all preference orderings. Fix a real number $\beta > 0$. The type space of every agent is defined as

$$V := \{v \in \mathbb{R}_+^{|A|} : \; v(a) \neq v(b) \; \forall a, b \in A, \text{ and} \max_{a\in A} v(a) \leq \beta\}. \tag{2}$$

Note that V consists of all non-negative utility functions representing preference orderings in $\mathcal{P}$ with values not more than β. All our results discussed in earlier sections continue to hold with this upper bound restriction.

We will assume that each agent $i \in N$ has a probability measure G_i over V^{n-1} which is independent of his type. The Lebesgue measure will be denoted by μ. We will make the following assumption about the probability measure G_i.

Our assumption over the priors of agents is the following.

Assumption A. Suppose for every agent $i \in N$, the prior (probability measure) G_i over V^{n-1} is independent of his type. Further, this probability measure is absolutely continuous with respect to μ and admits a positive density function g_i over V^{n-1}.

THEOREM 7 *Suppose Assumption A holds and $f^* : V^n \to A$ is a utilitarian scf, where V is as defined in (2). Then, f^* is not Bayesian incentive compatible.*

Proof: Fix an agent $i \in N$ and an alternative $a \in A$. For each $v_i \in V$, let

$$R(v_i, a) := \{v_{-i} \in V^{n-1} : f^*(v_i, v_{-i}) = a\}.$$

Note that every $v_{-i} \in R(v_i, a)$ satisfies $v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(b) + \sum_{j \neq i} v_j(b)$ for all $b \neq a$. Hence, $R(v_i, a)$ can be equivalently written as

$$R(v_i, a) := \{v_{-i} \in V^{n-1} : \sum_{j \neq i} \big[v_j(b) - v_j(a)\big] \leq v_i(a) - v_i(b) \ \forall \ b \in A\}.$$

Because of our assumptions on type spaces, $R(v_i, a)$ is a convex and bounded set. We complete the proof in three steps.

STEP 1. Pick $v_i \in V$ such that $v_i(a) = \frac{\beta}{2}$ and $v_i(b)$ is arbitrarily close to 0 for all $b \neq a$, where β is as defined in Equation (2). Now, let $v'_i = 2v_i$. Note that $v'_i(a) = \beta$ and $v'_i(b)$ is close to 0 for all $b \neq a$.

Since $v_i(a) - v_i(b) < v'_i(a) - v'_i(b)$ for all $b \neq a$, we have $R(v_i, a) \subsetneq R(v'_i, a)$. Now, the set $R(v'_i, a) \setminus R(v_i, a)$ is equal to

$$\{v_{-i} : [v_i(a) - v_i(b)] < \sum_{j \neq i} \big[v_j(b) - v_j(a)\big] \leq 2[v_i(a) - v_i(b)] \ \forall \ b \in A\}.$$

Clearly, this is a set with non-zero measure in V^{n-1}. Since our probability density function is strictly positive, we get that

$$\Pi^{f^*}(v'_i, a) - \Pi^{f^*}(v_i, a) := \int_{R(v'_i,a) \setminus R(v_i,a)} g_i(v_{-i}) dv_{-i} > 0.$$

Hence, $\Pi_i^{f^*}(v'_i, a) > \Pi_i^{f^*}(v_i, a)$. This shows that the utilitarian scf is not ev-ordinal.

STEP 2. Next, we consider a sequence of types $\{v_i^k\}$ for agent i in his type space V converging to v_i. Fix an alternative $a \in A$. This sequence constructs a sequence of polyhedral (convex) sets $\{R(v_i^k, a)\}_k$ in V^{n-1}. Denote by $\bar{R}(v_i^k, a)$ the closure of the set $R(v_i^k, a)$. Hence, $\{\bar{R}(v_i^k, a)\}$ is a sequence of compact convex sets. Moreover, since $\{v_i^k\}$ converges to v_i, the sequence $\{\bar{R}(v_i^k, a)\}$ must converge to $\bar{R}(v_i, a)$ in the Hausdorff metric. Let $\mu(\bar{R}(v_i^k, a))$ be the Lebesgue measure of the set $\bar{R}(v_i^k, a)$. By Beer (1974), we know that if a sequence of compact convex sets converge to a set, then their Lebesgue measures also converge to the

Lebesgue measure of the set. Hence, the sequence of Lebesgue measures $\{\mu(\bar{R}(v_i^k, a))\}$ converges to $\mu(\bar{R}(v_i, a))$. For each k, $\bar{R}(v_i^k, a) \setminus R(v_i^k, a)$ rules out utility vectors which have indifference, and these sets have zero Lebesgue measure. This implies that $\{\mu(R(v_i^k, a))\}$ converges to $\mu(R(v_i, a))$.

Step 3. Since G_i is absolutely continuous with respect to μ, Step 2 implies that as the sequence $\{R(v_i^k, a)\}$ converges to $R(v_i, a)$ in the Hausdorff metric, the sequence $\{\Pi^{f^*}(v_i^k, a)\}$ converges to $\Pi^{f^*}(v_i, a)$. Since every $\{v_i^k\}$ sequence converging to v_i induces a sequence $\{R(v_i^k, a)\}$ converging to $R(v_i, a)$, we conclude that f^* is ec-continuous.

Applying Theorem 5, we conclude that f^* is not Bayesian incentive compatible. ■

With a slight modification in notation, Theorem 7 can also be extended to *weighted utilitarianism* as long as at least two agents receive positive weight.

Definition 12 *An scf* $\tilde{f}$ *is a* **weighted utilitarianism** *if there exists weights* $\lambda_1, \ldots, \lambda_n \geq 0$, *not all equal to zero, such that for every* $\mathbf{v} \in \mathcal{V}$, *we have*

$$\tilde{f}(\mathbf{v}) \in \arg\max_{a \in A} \sum_{i \in N} \lambda_i v_i(a).$$

A weighted utilitarianism with weights $\lambda_1, \ldots, \lambda_n$ *is a* **dictatorship** *if there exists an agent* $i \in N$ *such that* $\lambda_i > 0$ *and* $\lambda_j = 0$ *for all* $j \neq i$. *Agent* i *is the dictator in this case.*

Almost an identical proof to Theorem 5 establishes the following.

Theorem 8 *Suppose Assumption A holds and* $\tilde{f} : V^n \to A$ *is a non-dictatorial weighted utilitarian scf where* V *is as defined in (2). Then,* $\tilde{f}$ *is not Bayesian incentive compatible.*

In the case of dictatorship, the set $R(v_i, a)$ for the dictator agent is either the entire V^{n-1} or the empty set for every a. For any other agent j who is not a dictator, $R(v_j, a)$ does not change by changing v_j for every a. As a result, we cannot reach the conclusion in Step 1 of the proof of Theorem 7.

An alternative proof of Theorem 7 (and Theorem 8) can be provided along the following lines. We know that utilitarianism can be implemented by the Vickrey-Clarke-Groves (VCG) mechanisms in dominant strategies by using transfers in a quasilinear environment. Typically, the expected transfer of an agent in a VCG mechanism is non-zero. If the utilitarianism is Bayesian incentive compatible without transfers, then we have two mechanism implementing the same scf in Bayesian equilibrium, but one with zero expected transfers and one with non-zero expected transfers. Standard revenue equivalence results in these settings (Rockafellar, 1970; Krishna and Maenner, 2001; Milgrom and Segal, 2002) imply that this is not possible. [10]

6 Relationship to the Literature

The primary focus of the literaure on mechanism design without monetary transfers has been on deterministic models and dominant strategies. As we have noted earlier, the issue of cardinal information has no bearing in these cases and is of relevance only on (a) models involving randomization (b) models using Bayes-Nash rather than dominant strategies as a solution concept. We comment briefly on the literature in these areas and their relationship with our work.

The seminal paper in randomized mechanism design for voting models is Gibbard (1977). The paper considered an explicitly ordinal problem in a dominant strategy framework when the preferences of agents is unrestricted. It proposed a demanding notion of incentive-compatibility where the truth-telling lottery was required to stochastically dominate the lottery arising from any manipulation. This approach has been extended to several restricted domains of ordinal preferences in voting models (Chatterji et al. (2012), Peters et al. (2014), Chatterji et al. (2014) and Pycia and Unver (2015)) and in ordinal matching models (Bogomolnaia and Moulin (2001), and Erdil (2014)).

Random mechanism design with cardinal utilities and dominant strategies has received far less attention. Hylland (1980), Barbera et al. (1998); Dutta et al. (2007); Nandeibam (2013) revisit Gibbard's voting model with cardinal utilities in the unrestricted ordinal domain while Zhou (1990) considers the one-sided matching problem. Significantly, all the papers use the vNM domain except Nandeibam (2013). While Barbera et al. (1998); Dutta et al. (2007) show that every randomized DSIC scf in the vNM domain with unrestricted ordinal preferences must be a random dictatorship under unanimity and other additional conditions, Nandeibam (2013) shows a weaker version of this result without assuming vNM domain but still requiring unrestricted ordinal preferences. Zhou (1990) shows incompatibility of Pareto efficiency, DSIC, and *symmetry* in the one-sided matching problem.

Majumdar and Sen (2004) consider deterministic Bayesian incentive compatible mechanisms in an ordinal model employing a solution concept developed for this framework in d'Aspremont and Peleg (1988). Borgers and Postl (2009) consider a model with three al-

[10] Showing the fact that a VCG mechanism generates non-zero expected transfers will require some work, and needs some assumptions on priors.

ternatives and two agents in the vNM domain. The two agents have fixed but completely opposed ordinal preferences. The type of each agent is the utility of the common second-ranked or "compromise" alternative. In the vNM domain, this model gives rise to a special one-dimensional mechanism design problem with transfers. They characterize the set of cardinal Bayesian incentive compatible scfs using Myersonian techniques. The characterization is further extended in Postl (2011). Miralles (2012) considers a model of allocating two objects to agents without monetary transfers and Bayesian incentive compatibility. A recent paper by Kim (2014) considers the vNM domain with Bayesian incentive compatibility. He shows that every ordinal mechanism is *dominated* (in terms of utilitarian social welfare) by a suitable cardinal mechanism in the vNM domain.

Almost all the papers on cardinal mechanisms cited above, consider vNM domains. Many of them highlight the fact that there is an expansion in the set of incentive compatible scfs relative to the strongly ordinal model. Our paper provides a foundation for the use of vNM domains - in a cardinal model, c-continuity and incentive compatibility implies vNM-ordinality. However, strengthening c-continuity to uc-continuity brings us to a completely ordinal model giving a precise description of the boundary between vNM domain and the strongly ordinal model. Moreover, our conclusions are completely independent of the underlying ordinal structure of preferences.

Gershkov et al. (2014) considers the design of expected welfare maximizing mechanism by considering cardinal utilities when agents have particular ordinal preferences. Since they consider deterministic DSIC mechanisms, these mechanisms must be ordinal. But they still consider cardinal utilities to compute expected welfare maximizing mechanism. Similarly, Ashlagi and Shi (2015) consider the *optimal* design of randomized BIC mechanisms for matching problems by assuming that agents have cardinal utilities.

From a methodological standpoint, our paper is related to (Borgers and Postl, 2009; Goswami et al., 2014; Gershkov et al., 2014; Hafalir and Miralles, 2015). These papers either explicitly (Goswami et al., 2014; Gershkov et al., 2014) or indirectly (Borgers and Postl, 2009; Hafalir and Miralles, 2015) work in a model with one dimensional types. As a result, they can use the machinery developed in Myerson (1981) for one-dimensional type spaces. On the other hand, agents in our model have multidimensional types, and we use results from the multidimensional mechanism design literature - see Vohra (2011) for a comprehensive treatment of this topic.

Appendix: Proof of Claim 1

Fix a $P \in \mathcal{D}$ and consider a sequence of types $\{v_k\}_{k\in\mathbb{N}}$ in V^P such that it converges to $\mathbf{0}$. Since $\{v_k\}_{k\in\mathbb{N}}$ is convergent in the closure of V^P, it must be a Cauchy sequence. We argue that since f_P is uniformly continuous, the sequence $\{f_P(v_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence. To see this, for every $\delta > 0$, since $\{v_k\}_{k\in\mathbb{N}}$ is Cauchy sequence, there exists a number J such that for all $j, j' > J$ we have $\|v_j - v_{j'}\| < \delta$. But by uniform continuity, for every $\epsilon > 0$, there exists a number $\delta > 0$ such that if $\|v_j - v_{j'}\| < \delta$ then $\|f_P(v_j) - f_P(v_{j'})\| < \epsilon$. This shows that the sequence $\{f_P(v_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence.

As a consequence, $\{f_P(v_k)\}_{k\in\mathbb{N}}$ must converge. Denote this limit point as $L_1 \in [0,1]^{|A|}$. Similarly, pick another sequence of types $\{v'_k\}_{k\in\mathbb{N}}$ such that it converges to $\mathbf{0}$. By the same argument, $\{f_P(v'_k)\}_{k\in\mathbb{N}}$ must also converge to $L_2 \in [0,1]^{|A|}$. We will show that $L_1 = L_2$.

To do this, pick an arbitrary $\epsilon > 0$. Now, using the definition of convergence, since $\{f_P(v_k)\}_{k\in\mathbb{N}}$ converges to L_1, there must exist a number n_1 such that for all $k > n_1$, we have

$$\|f_P(v_k) - L_1\| < \frac{\epsilon}{3}. \tag{3}$$

Similarly, since $\{f_P(v'_k)\}_{k\in\mathbb{N}}$ converges to L_2, there must exist a number n_2 such that for all $k > n_2$, we have

$$\|f_P(v'_k) - L_2\| < \frac{\epsilon}{3}. \tag{4}$$

By uniform continuity of f_P, we get that there exists $\delta > 0$ such that for all $v, v' \in V^P$ with $\|v - v'\| < \delta$, we have

$$\|f_P(v) - f_P(v')\| < \frac{\epsilon}{3}. \tag{5}$$

Since both the sequences $\{v_k\}_{k\in\mathbb{N}}$ and $\{v'_k\}_{k\in\mathbb{N}}$ are converging to $\mathbf{0}$, there must exist a number n_3 such that for all $k > n_3$ such that $\|v_k\| < \frac{\delta}{2}$ and $\|v'_k\| < \frac{\delta}{2}$, and hence, $\|v_k - v'_k\| < \delta$. By Inequality 5, we get for all $k > n_3$,

$$\|f_P(v_k) - f_P(v'_k)\| < \frac{\epsilon}{3}. \tag{6}$$

Now, pick a number $K > \max(n_1, n_2, n_3)$ and note that

$$\|L_1 - L_2\| \le \|L_1 - f_P(v_K)\| + \|f_P(v_K) - f_P(v'_K)\| + \|f_P(v'_K) - L_2\| < \epsilon,$$

where the first inequality followed from the Euclidean norm property and the second one followed from Inequalities 3, 4, and 6. Since ϵ can be chosen arbitrarily small, we conclude that $L_1 = L_2$.

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