

$$\begin{aligned}
x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\
x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\
x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0.
\end{aligned}$$

The new variables x_4, x_5, x_6 are called **slack variables**, and the old variables x_1, x_2, x_3 are called **decision variables**. Hence our new LP is to

$$\max z \quad \text{s.t.} \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0, \quad (4)$$

where $z = 5x_1 + 4x_2 + 3x_3$. This new LP is equivalent (same set of feasible solutions in terms of decision variables) to (EX-1), given the equations determining the slack variables. The simplex method is an iterative procedure in which having found a feasible solution x_1, \dots, x_6 of (4), we look for another feasible solution $\bar{x}_1, \dots, \bar{x}_6$ of (4) such that

$$5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 \geq 5x_1 + 4x_2 + 3x_3.$$

If an optimal solution exists, we can repeat this finite number of iterations till there is no improvement in the objective function value, at which point we stop. The first step is to find a feasible solution, which is easy in our example: $x_1 = x_2 = x_3 = 0$, which gives $x_4 = 5, x_5 = 11, x_6 = 8$. This gives $z = 0$.

We now need to look for a solution that gives a higher value to z . For this, we look to increase values of any one of the variables x_1, x_2, x_3 . We choose x_1 . Keeping x_2 and x_3 at zero, we notice that we can increase x_1 to $\min(\frac{5}{2}, \frac{11}{4}, \frac{8}{3}) = \frac{5}{2}$ to maintain $x_4, x_5, x_6 \geq 0$. As a result of this, the new solution is

$$x_1 = \frac{5}{2}, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 1, x_6 = \frac{1}{2}, z = \frac{25}{2}.$$

Notice that by increasing the value of x_1 , a variable whose value was positive (x_4) got a value of zero. Now, we have to create system of equation similar to previous iteration. For that we will write the value of z and variables having non-zero values (x_1, x_5, x_6) in terms of variables having zero values (x_4, x_2, x_3).

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4.$$

$$x_5 = 1 + 5x_2 + 2x_4.$$

$$x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4.$$

$$z = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4.$$

Of x_2, x_3, x_4 , the value of z decreases by increasing the values of x_2 and x_4 . So, the only candidate for increasing value in this iteration is x_3 . The amount we can increase the value of x_3 can again be obtained from the feasibility conditions of $x_1, x_5, x_6 \geq 0$, which is equivalent to (given $x_2 = x_4 = 0$) $\frac{5}{2} - \frac{1}{2}x_3 \geq 0$ and $\frac{1}{2} - \frac{1}{2}x_3 \geq 0$. This gives that the maximum possible value of x_3 in this iteration can be $\min(5, 1) = 1$. By setting $x_3 = 1$, we get a new solution as

$$x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0, z = 13. \quad (5)$$

Two things should be noticed here: (a) this solution is also a solution of the previous system of equations and (b) the earlier solution is also a solution of this system of equations. This is precisely because we are just rewriting the system of equations using a different set of decision and slack variables in every iteration, and that is the central theme of the simplex method.

So, the new variable that takes zero value is x_6 . We now write the system of equations in terms of x_2, x_4, x_6 .

$$x_3 = 1 + x_2 + 3x_4 - 2x_6$$

$$x_1 = 2 - 2x_2 - 2x_4 + x_6$$

$$x_5 = 1 + 5x_2 + 2x_4$$

$$z = 13 - 3x_2 - x_4 - x_6.$$

Now, the value of z will decrease by increasing the values of any of the variables x_2, x_4, x_6 . So, we have reached a dead-end. In fact, we have reached an optimal solution. This is clear from the fact that any solution requires $x_2, x_4, x_6 \geq 0$, and by assigning any value not equal to zero to these variables, we will decrease the value of z . Hence, $z = 13$ is an optimal solution. The corresponding values of x_1, x_3, x_5 are 2, 1, 1 respectively.

1.2 Dictionaries

Consider a general LP in standard form:

$$\begin{aligned}
Z &= \max \sum_{j=1}^n c_j x_j \\
\text{s.t.} & \\
\sum_{j=1}^n a_{ij} x_j &\leq b_i \quad \forall i \in \{1, \dots, m\} \\
x_j &\geq 0 \quad \forall j \in \{1, \dots, n\}
\end{aligned} \tag{LP}$$

The first step in the simplex method is to introduce slack variables, $x_{n+1}, \dots, x_{n+m} \geq 0$ corresponding to m constraints, and denote the objective function as z . So,

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j \quad \forall i \in \{1, \dots, m\} \tag{6}$$

$$z = \sum_{j=1}^n c_j x_j. \tag{7}$$

$$x_j \geq 0 \quad \forall j \in \{1, \dots, n, n+1, \dots, n+m\} \tag{8}$$

In simplex method, we search for a feasible solution $\bar{x}_1, \dots, \bar{x}_{m+n}$ given a feasible solution x_1, \dots, x_{m+n} so that the objective function is better, i.e.,

$$\sum_{j=1}^n c_j \bar{x}_j > \sum_{j=1}^n c_j x_j.$$

As we have seen in the example, every feasible solution can be represented with a system of linear equations consisting of *dependent* variables. These system of equations corresponding to a feasible solution is called a **dictionary**. A dictionary will have the following features:

1. Every solution of the system of equations of the dictionary must be a solution of system of equations (6), (7), and (8), and vice versa.
2. The equations of every dictionary must express m of the variables x_1, \dots, x_{m+n} and the objective function z (dependent variables) in terms of the remaining n variables (independent variables).

In the dictionary, the dependent variables are kept on the left hand side (LHS), and they are expressed in terms of the independent variables on the right hand side (RHS). An additional feature of a dictionary is

- setting the RHS variables at zero and evaluating the LHS variables, we arrive at a feasible solution.

A dictionary with this additional property is called a **feasible dictionary**. Hence, every feasible dictionary describes a feasible solution. However, not every feasible solution can be described in terms of a feasible dictionary. Feasible solutions that can be described in terms of a feasible dictionary is called a **basic solution**. The characteristic feature of the simplex method is that it works with basic solutions only.

1.3 Second Example

We conclude the discussion by giving another example.

$$\begin{aligned}
 Z &= \max 5x_1 + 5x_2 + 3x_3 \\
 \text{s.t.} \\
 x_1 + 3x_2 + x_3 &\leq 3 \\
 -x_1 + 3x_3 &\leq 2 \\
 2x_1 - x_2 + 2x_3 &\leq 4 \\
 2x_1 + 3x_2 - x_3 &\leq 2 \\
 x_1, x_2, x_3 &\geq 0.
 \end{aligned}$$

In this case, the initial feasible dictionary has all the slack variables as dependent variables, and it looks as follows:

$$\begin{aligned}
 x_4 &= 3 - x_1 - 3x_2 - x_3 \\
 x_5 &= 2 + x_1 - 3x_3 \\
 x_6 &= 4 - 2x_1 + x_2 - 2x_3 \\
 x_7 &= 2 - 2x_1 - 3x_2 + x_3
 \end{aligned}$$

$$z = 5x_1 + 5x_2 + 3x_3.$$

The feasible dictionary describes the following solution:

$$x_1 = x_2 = x_3 = 0, x_4 = 3, x_5 = 2, x_6 = 4, x_7 = 2.$$

As before, we try to increase the value of z by increasing the value of one of the independent variables as much as we can. Right now, since x_1, x_2, x_3 have all positive coefficients in

the z equation, we randomly choose x_1 . From feasibility of $x_4, \dots, x_7 \geq 0$, we get $x_1 \leq 1$ to be the most stringent constraint. So, we make $x_1 = 1$, which in turn makes $x_7 = 0$. We now write the new dictionary with x_1 leaving the independent variables and x_7 entering the independent variables. First, substitute,

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7.$$

Substituting for x_1 in terms of new set of independent variables in the previous dictionary, we get

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7$$

$$x_4 = 2 - \frac{3}{2}x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_7$$

$$x_5 = 3 - \frac{3}{2}x_2 - \frac{5}{2}x_3 - \frac{1}{2}x_7$$

$$x_6 = 2 + 4x_2 - 3x_3 + x_7$$

$$z = 5 - \frac{5}{2}x_2 + \frac{11}{2}x_3 - \frac{5}{2}x_7.$$

Some comments about terminology are in order:

1. Dependent variables, which appear on the LHS of any dictionary, are called **basic variables**. Independent variables are called **non-basic variables**. In the previous dictionary, x_1, x_4, x_5, x_6 are basic variables, and x_2, x_3, x_7 are non-basic variables.
2. Set of basic and non-basic variables change from iteration to iteration.
3. Choice of **entering** basic variable is motivated by the fact that we want to increase the value of z , and we choose one that does that, and increase its value the maximum possible.
4. Choice of **leaving** basic variable is motivated by the need to maintain feasibility. This is done by identifying the basic variable that poses the most stringent bound on the entering basic variable.
5. The formula for the entering basic variable appears in the **pivot row**, and the process of constructing a new dictionary is called **pivoting**. In the previous dictionary, x_3 is the next entering basic variable, and x_6 is the leaving basic variable. So, the formula for x_3 appears in

$$x_3 = \frac{2}{3} + \frac{4}{3}x_2 - \frac{1}{3}x_6 + \frac{1}{3}x_7,$$

which is the pivot row.

Continuing with our example, the clear choice of entering basic variable is x_3 . Calculations give that x_6 imposes the most stringent bound on x_3 , and should be the leaving basic variable. So, we arrive at the new dictionary.

$$\begin{aligned} x_3 &= \frac{2}{3} + \frac{4}{3}x_2 + \frac{1}{3}x_7 - \frac{1}{3}x_6 \\ x_1 &= \frac{4}{3} - \frac{5}{6}x_2 - \frac{1}{3}x_7 - \frac{1}{6}x_6 \\ x_4 &= 1 - \frac{7}{2}x_2 + \frac{1}{2}x_6 \\ x_5 &= \frac{4}{3} - \frac{29}{6}x_2 - \frac{4}{3}x_7 + \frac{5}{6}x_6 \\ z &= \frac{26}{3} + \frac{29}{6}x_2 - \frac{2}{3}x_7 - \frac{11}{6}x_6. \end{aligned}$$

Now, the entering basic variable is x_2 , and the leaving basic variable is x_5 . Pivoting yields the following dictionary:

$$\begin{aligned} x_2 &= \frac{8}{29} - \frac{8}{29}x_7 + \frac{5}{29}x_6 + \frac{6}{29}x_5 \\ x_3 &= \frac{30}{29} + \frac{1}{29}x_7 - \frac{3}{29}x_6 - \frac{8}{29}x_5 \\ x_1 &= \frac{32}{29} - \frac{3}{29}x_7 - \frac{9}{29}x_6 + \frac{5}{29}x_5 \\ x_4 &= \frac{1}{29} + \frac{28}{29}x_7 - \frac{3}{29}x_6 + \frac{21}{29}x_5 \\ z &= 10 - 2x_7 - x_6 - x_5. \end{aligned}$$

At this point, no more pivoting is possible, and we arrive at the optimal solution described by the last dictionary as:

$$x_1 = \frac{32}{29}, x_2 = \frac{8}{29}, x_3 = \frac{30}{29},$$

and this yields an optimal value of $z = 10$.

2 Pitfalls and How to Avoid Them

Three kind of pitfalls can occur in simplex method:

1. **Initialization:** We may not be able to start. We may not have a feasible dictionary to start.
2. **Iteration:** We may get stuck in some iteration. Can we always choose a new entering and leaving variable?
3. **Termination:** We may not be able to finish. Can the simplex method construct an endless sequence of dictionaries without reaching an optimal solution?

We look at each of these three pitfalls.

2.1 Iteration

2.1.1 Choosing an Entering Variable

The entering variable is a *non-basic variable with a positive coefficient \bar{c}_j in the last row of the current dictionary*. This rule is ambiguous in the sense that it may provide more than one candidate for entering or no candidate at all. The latter alternative implies that the current dictionary has an optimal solution. If there are more than one candidate for entering the basis, then any of these candidates may serve.

2.1.2 Finding a Leaving Variable

The leaving variable is that *basic variable whose non-negativity imposes the most stringent upper bound on the increase of the entering variable*. Again, there may be more than one candidate or no candidate at all. If there are more than one candidate, then we may choose any one of them. If there are no candidate at all, then an interesting conclusion can be drawn. Here is an example of a dictionary:

$$x_2 = 5 + 2x_3 - x_4 - 3x_1$$

$$x_5 = 7 - 3x_4 - 4x_1$$

$$z = 5 + x_3 - x_4 - x_1.$$

The entering variable is x_3 . However, neither of the two basic variables x_2 and x_5 put an upper bound on x_3 . Hence, we can increase x_3 as much as we want without violating feasibility. Set $x_3 = t$ for any positive number t , and we get the solution $x_1 = 0, x_2 =$

$5 + 2t, x_3 = t, x_4 = 0, x_5 = 7,$ and $z = 5 + t.$ Since t can be made arbitrarily large, so can be $z,$ and we conclude that the problem is unbounded. The same conclusion can be reached in general: if there is no candidate for leaving the basis, then we can make the value of the entering variable, and hence the value of the objective function, as large as we wish. In that case, the problem is unbounded.

2.1.3 Degeneracy

The presence of more than one candidate for leaving the basis has interesting consequences. For example, consider the dictionary

$$\begin{aligned}x_4 &= 1 - 2x_3 \\x_5 &= 3 - 2x_1 + 4x_2 - 6x_3 \\x_6 &= 2 + x_1 - 3x_2 - 4x_3 \\ \\z &= 2x_1 - x_2 + 8x_3.\end{aligned}$$

Having chosen x_3 as the entering variable, we see that $x_4, x_5,$ and x_6 are all candidates for leaving variable. Choosing $x_4,$ and pivoting, we get the new dictionary as

$$\begin{aligned}x_3 &= 0.5 - 0.5x_4 \\x_5 &= -2x_1 + 4x_2 + 3x_4 \\x_6 &= x_1 - 3x_2 + 2x_4 \\ \\z &= 4 + 2x_1 - x_2 - 4x_4.\end{aligned}$$

This dictionary is different from others in one important aspect: along with the non-basic variables, two of the basic variables, x_5 and x_6 have value of zero. Basic solutions with one or more basic variables at zero are called **degenerate**.

Although harmless, degeneracy has annoying side effects. In the next iteration, we have x_1 as the entering variable, and x_5 as the leaving variable. But the value of x_1 can be increased by a maximum of zero. Hence, the objective function value does not change. Pivoting changes the dictionary to:

$$\begin{aligned}
x_1 &= 2x_2 + 1.5x_4 - 0.5x_5 \\
x_3 &= 0.5 - 0.5x_4 \\
x_6 &= -x_2 + 3.5x_4 - 0.5x_5
\end{aligned}$$

$$z = 4 + 3x_2 - x_4 - x_5.$$

but the solution remains the same. Simplex iterations that do not change the basic solution are called **degenerate**. One can verify that the next iteration is also degenerate, but the one after that is not - in fact, it is the optimal solution.

Degeneracy is an accident. Many practical problems face degeneracy, and when it happens the simplex goes through few (many a times quite a few) degenerate iterations before coming up with a non-degenerate solution. But there are occasions when this may not happen.

2.2 Cycling

Sometimes, a sequence of dictionary can appear again and again. This phenomenon is called **cycling**. To understand cycling let us look at a series of dictionaries.

$$\begin{aligned}
x_5 &= -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}$$

$$z = 10x_1 - 57x_2 - 9x_3 - 24x_4.$$

The following rule for selecting the entering and leaving variable is the following:

- The entering variable will always be the nonbasic variable that the largest coefficient in the z -row of the dictionary.
- If two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript will be made to leave.

Now, the sequence of dictionaries constructed in the first six iterations goes as follows. After the first iteration:

$$\begin{aligned}
x_1 &= 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= -4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 \\
z &= 53x_2 + 41x_3 - 20x_4 - 20x_5.
\end{aligned}$$

After the second iteration:

$$\begin{aligned}
x_2 &= -0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_1 &= -0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_7 &= 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6 \\
z &= 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6.
\end{aligned}$$

After the third iteration:

$$\begin{aligned}
x_3 &= 8x_4 + 1.5x_5 - 5.5x_6 - 2x_1 \\
x_2 &= -2x_4 - 0.5x_5 + 2.5x_6 + x_1 \\
x_7 &= 1 - x_1 \\
z &= 18x_4 + 15x_5 - 93x_6 - 29x_1.
\end{aligned}$$

After the fourth iteration:

$$\begin{aligned}
x_4 &= -0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2 \\
x_3 &= -0.5x_5 + 4.5x_6 + 2x_1 - 4x_2 \\
x_7 &= 1 - x_1 \\
z &= 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2.
\end{aligned}$$

After the fifth iteration:

$$\begin{aligned}
x_5 &= 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_4 &= -x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_7 &= 1 - x_1 \\
z &= 24x_6 + 22x_1 - 93x_2 - 21x_3.
\end{aligned}$$

After the sixth iteration:

$$x_5 = -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4$$

$$x_6 = -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4$$

$$x_7 = 1 - x_1$$

$$z = 10x_1 - 57x_2 - 9x_3 - 24x_4.$$

Since the dictionary after the sixth iteration is identical with the initial dictionary, the simplex method will go through the same set of dictionaries again and again without ever finding the optimal solution (which is $z = 1$ in this example).

Notice that cycling means, we have a series of degenerate solutions, else we will have increase in objective function, and cannot have the same dictionaries repeating. It is important to note that **cycling implies that we get the same solution in every iteration, even though the set of basic variables change**. It is not possible that we are changing the value of some variable without changing the objective function value (because we always choose an entering variable that changes the objective function value when its value is changed).

Theorem 1 *If the simplex method fails to terminate, then it must cycle.*

Proof: There are a total of $m + n$ variables. Since in every iteration of the simplex method we choose m basic variables, there are finite number of ways to choose them. Hence, if the simplex method does not terminate, then there will be two dictionaries with the same set of basic variables. Represent the two dictionaries as:

$$x_i = b_i - \sum_{j \notin B} a_{ij} x_j \quad \forall i \in B$$

$$z = v + \sum_{j \notin B} c_j x_j.$$

and

$$x_i = b_i^* - \sum_{j \notin B} a_{ij}^* x_j \quad \forall i \in B$$

$$z = v^* + \sum_{j \notin B} c_j^* x_j.$$

with the same set of basic variables $x_i (i \in B)$.

But there is a unique way of representing a (basic) variable in terms of a set of non-basic variable. Hence the two dictionaries must be exactly equal.

Alternatively, it is a defining property of the dictionaries that every solution x_1, \dots, x_{n+m}, z of one dictionary is also a solution of any other dictionary. In particular, if x_k is a non-basic variable and t is a number, then the numbers

$$x_k = t, x_j = 0 (j \notin B, j \neq k), x_i = b_i - a_{ik}t (i \in B), z = v + c_k t$$

constitute a solution of the second dictionary. Hence the following must hold.

$$b_i - a_{ik}t = b_i^* - a_{ik}^*t \quad \forall i \in B, \quad v + c_k t = v^* + c_k^*t.$$

Setting $t = 0$, we get $b_i = b_i^*$ for all $i \in B$, and $v = v^*$. Then set $t = 1$, to get $a_{ik} = a_{ik}^*$ for all $i \in B$ and $c_k = c_k^*$. Since x_k was chosen arbitrarily, the two dictionaries are identical. ■

Cycling is a rare phenomena, but sometimes they do occur in practice. In fact, constructing an LP problem on which the simplex method may cycle is difficult. [Marshall and Suurballe \(1969\)](#) proved that if the simplex method cycles off-optimum on a problem that has an optimal solution, then the dictionaries must involve at least six variables and at least three equations. In practice, cycling occurs very rarely.

Two popular rules for avoiding cycling are: (a) perturbation method and lexicographic ordering (b) smallest subscript rule. We describe the smallest subscript rule here, first proposed by Bland in his Math. of Operations Research paper in 1977 ([Bland, 1977](#)). The former requires extra computation to choose the entering and leaving variables while the latter leaves no choice in the hands of users to choose entering variables, which we can get in the former one.

To avoid cycling, we introduce a rule called (Bland's) **smallest subscript rule**. This refers to breaking ties in the choice of the entering and leaving variables by always choosing the candidate x_k that has the smallest subscript k .

Theorem 2 *The simplex method terminates as long as the entering and leaving variables are selected by the smallest subscript rule (SSR) in each iteration.*

Proof: By virtue of previous theorem, we need to show that cycling is impossible when the SSR is used. Assume for contradiction that the simplex method with SSR generates a sequence of dictionaries D_0, D_1, \dots, D_k such that $D_k = D_0$.

Call a variable **fickle** if it is nonbasic in some iteration and basic in some other. Among all fickle variables, let x_t have the largest subscript. Due to cycling, there is a dictionary D in the sequence D_0, \dots, D_k with x_t leaving (basic in D but nonbasic in the next dictionary),

and some other fickle variable x_s entering (nonbasic in D but basic in the next iteration). Further along in the sequence $D_0, D_1, \dots, D_k, D_1, \dots, D_k$, there is a dictionary D^* with x_t entering. Let us record D as

$$x_i = b_i - \sum_{j \notin B} a_{ij} x_j \quad \forall i \in B$$

$$z = v + \sum_{j \notin B} c_j x_j.$$

Since all iterations leading from D to D^* are degenerate, the objective function z must have the same value v in both D and D^* . Thus, the last row of D^* may be recorded as

$$z = v + \sum_{j=1}^{m+n} c_j^* x_j,$$

where $c_j^* = 0$ wherever x_j is basic in D^* . Since this equation has been obtained from D by algebraic manipulations, it must satisfy every solution of D . In particular, it must be satisfied by

$$x_s = y, x_j = 0 \ (j \notin B, j \neq s), x_i = b_i - a_{is} y \ (i \in B), z = v + c_s y \quad \forall y.$$

Thus, we have

$$v + c_s y = v + c_s^* y + \sum_{i \in B} c_i^* (b_i - a_{is} y).$$

and, after simplification,

$$\left(c_s - c_s^* + \sum_{i \in B} c_i^* a_{is} \right) y = \sum_{i \in B} c_i^* b_i$$

for every choice of y . Since the RHS of the previous equation is a constant independent of y , we have

$$c_s - c_s^* + \sum_{i \in B} c_i^* a_{is} = 0.$$

But x_s is entering in D , implying $c_s > 0$. Since x_s is not entering in D^* and yet $s < t$, we have $c_s^* \leq 0$. Hence for some $r \in B$, $c_r^* a_{rs} < 0$. Note two points now:

- Since $r \in B$, the variable x_r is basic in D ; since $c_r^* \neq 0$, the same variable is nonbasic in D^* . Hence, x^r is fickle, and we have $r \leq t$.

- $r \neq t$: since x_t is leaving in D , we have $a_{ts} > 0$ and so $c_t^* a_{ts} > 0$ (since $c_t^* > 0$ with x_t entering in D^*).

This shows that $r < t$ and yet x_r is not entering in D^* . Thus, $c_r^* \leq 0$, and $a_{rs} > 0$. Since all iterations from D to D^* are degenerate, the two dictionaries describe the same solution. Since x_r is non-basic in D^* , its value is zero in both D and D^* , meaning $b_r = 0$. Hence, x_r was a candidate for leaving the basis of D - yet we picked x_t , even though $r < t$. This is a contradiction. ■

2.3 Initialization

The only remaining point that needs to be explained is getting hold of the initial feasible dictionary in a problem

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \forall i \in \{1, \dots, m\} \\ & x_j \geq 0 \quad \forall j \in \{1, \dots, n\}. \end{aligned}$$

with an infeasible origin. The problem with infeasible origin is that we may not know whether a feasible solution exists at all, and even we know what a feasible dictionary will be for that solution. One way of getting around these two solutions is the so called **auxiliary problem**:

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & \\ & \sum_{j=1}^n a_{ij} x_j - x_0 \leq b_i \quad \forall i \in \{1, \dots, m\} \\ & x_j \geq 0 \quad \forall j \in \{0, 1, \dots, n\}. \end{aligned}$$

A feasible solution of the auxiliary problem is readily available: set $x_j = 0$ for $j \neq 0$ and x_0 equals the $\min(0, \min_i b_i)$. If the original problem has a feasible solution than the auxiliary problem has the same feasible solution with $x_0 = 0$. Further if the auxiliary problem has a feasible solution with $x_0 = 0$, then the original problem has the same feasible solution.

Written differently, **the original problem has a feasible solution if and only if the optimal value of the auxiliary problem is zero.**

Hence, our objective is to solve the auxiliary problem. Consider the following example.

$$\begin{aligned} \max x_1 - x_2 + x_3 \\ \text{s.t.} \\ 2x_1 - x_2 + 2x_3 \leq 4 \\ 2x_1 - 3x_2 + x_3 \leq -5 \\ -x_1 + x_2 - 2x_3 \leq -1 \\ x_1, x_2, x_3 \geq 0. \end{aligned}$$

To avoid unnecessary confusion, we write the auxiliary problem in its maximization form, and construct the dictionary as

$$\begin{aligned} x_4 &= 4 - 2x_1 + x_2 - 2x_3 + x_0 \\ x_5 &= -5 - 2x_1 + 3x_2 - x_3 + x_0 \\ x_6 &= -1 + x_1 - x_2 + 2x_3 + x_0 \end{aligned}$$

$$w = -x_0,$$

which is an infeasible dictionary. But it can be made feasible by pivoting on the most negative b_i row, i.e., x_5 in this case, and choosing x_0 as the entering variable. The new (feasible) dictionary is:

$$\begin{aligned} x_0 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\ x_4 &= 9 - 2x_2 - x_3 + x_5 \\ x_6 &= 4 + 3x_1 - 4x_2 + 3x_3 + x_5 \end{aligned}$$

$$z = -5 - 2x_1 + 3x_2 - x_3 - x_5.$$

In general, the dictionary corresponding to the auxiliary problem is:

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij}x_j + x_0 \quad \forall i \in \{1, \dots, m\}$$

$$w = -x_0.$$

which is infeasible. However, this can be transformed into a feasible dictionary by a single pivot in which x_0 enters and the “most infeasible” x_{n+i} leaves. More precisely, the leaving variable is that x_{n+k} whose b_k has the largest negative value among all negative b_i s. After pivoting, x_0 assumes the positive value $-b_k$, and each basic x_{n+i} assumes the non-negative value $b_i - b_k$. Now, we can continue with our simplex method. In our example, two more iterations yield the following dictionary:

$$\begin{aligned}x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 - 0.8x_0 \\x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 - 0.6x_0 \\x_4 &= 3 - x_1 - x_6 + 2x_0\end{aligned}$$

$$w = -x_0.$$

This dictionary is an optimal solution of the auxiliary problem with $x_0 = 0$. Further, this points to a feasible dictionary of the original problem.

$$\begin{aligned}x_3 &= 1.6 - 0.2x_1 + 0.2x_5 + 0.6x_6 \\x_2 &= 2.2 + 0.6x_1 + 0.4x_5 + 0.2x_6 \\x_4 &= 3 - x_1 - x_6\end{aligned}$$

$$z = -0.6 + 0.2x_1 - 0.2x_5 + 0.4x_6.$$

So, we learned how to construct the auxiliary problem, and its first feasible dictionary. In the process of solving the auxiliary problem, it may be possible that x_0 may be a candidate for the leaving variable in which case it is natural to pick x_0 . Immediately, after pivoting we get

- x_0 as a non-basic variable, in which case $w = 0$.

Clearly, this is an optimal solution. However, we may also reach an optimal dictionary of auxiliary problem with x_0 basic. If value of w is non-zero in that, then we simply conclude that the original problem is infeasible. Else, x_0 is basic and the optimal value of w is zero. We argue that this is not possible. Since the next to last dictionary was not optimal, the value of $w = -x_0$ must have changed from some negative value to zero in the last iteration. To put it differently, the value of x_0 must have changed from some positive level to zero in this iteration. This means, x_0 was also a candidate for leaving the basis, and we should have picked it according to our policy. This is a contradiction.

Hence, we either construct an optimal solution of the auxiliary problem where x_0 is a non-basic variable, and we proceed to the original problem by constructing a new feasible dictionary, or we conclude that the original problem is infeasible.

This strategy of solving an LP is known as the **two phase simplex method**. In the first phase, we set up and solve the auxiliary problem; if we find an optimal solution of the auxiliary problem, then we proceed to the **second phase**, solving the original problem.

Theorem 3 (Fundamental Theorem of Linear Programming) *Every LP problem in the standard form has the following three properties:*

1. *If it has no optimal solution, then it is either unbounded or infeasible.*
2. *If it has a feasible solution, then it has a basic feasible solution.*
3. *If it has an optimal solution, then it has a basic optimal solution.*

Proof: The first phase of the two phase simplex method either discovers that the problem is infeasible or else it delivers a basic feasible solution. The second phase either discovers that the problem is unbounded or gives a basic optimal solution. ■

Note that if the problem is not in standard form then the theorem does not hold, e.g., $\max x$ s.t. $x < 0$ has no optimal solution even though it is neither infeasible nor unbounded.

3 Disclaimer

We have followed [Chvátal \(2002\)](#) for these notes.

References

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