1 Basic Set Theory

Set is a fundamental idea in mathematics. It usually refers to a collection of "objects", where an object can be anything. For instance, set of natural numbers, set of individuals, set of preferences, set of curves that can be drawn on a blackboard, set of points on a line segment etc. The objects that a set contains are called **elements** of that set. For instance, if S is any set and a belongs to S, then we write $a \in S$, i.e., a is an element of S. This is the fundamental relation of set theory - something belongs to a set or not. If a does not belong to a set S, then we write $a \notin S$.

If a set S contains finitely many elements, we call it a finite a set. For instance, $S = \{a, b, c\}$ is a finite set because there is a natural number 3 such that we can assign a to 1, b to 2, and c to 3 - in other words, count them finitely. Such counting may not be possible for certain sets. For instance, consider the set of all natural numbers:

$$\mathbb{N} := \{0, 1, 2, \ldots\}.$$

There is no finite natural number n till which we can count all the elements of $\mathbb{N}$. Another example is the set of all points on a line segment. Such sets are **infinite** sets. The number of elements in a finite set is called the **cardinality** of the set. For any finite set S, its cardinality is usually denoted by $|S|$ - but sometimes also denoted by $\#S$. By definition, $|S| \in \mathbb{N}$.

Comparing sets. Many a times, we need to compare a pair of sets: S and T. An important comparison is to check if they are identical. We write $S = T$, i.e., sets S and T are **identical**, if

$$\Big[a \in S\Big] \Leftrightarrow \Big[a \in T\Big].$$

Take $S = \{a, b, c\}$ and $T = \{b, a, c\}$. Verify that $S = T$. Many times, we need to compare sets that are not identical. One common way is to check their cardinality if they are finite sets, if $|S| < |T|$, then we may say T is "larger" than S. Such comparisons overlook the fact that S and T may contain different kinds of elements. For instance S may be a set of elephants and T may be a set of students in a class.

One appropriate way of comparing sets is the subset relation. A set S is called a **subset** of another set T - written as $S \subseteq T$ - if for every $a \in S$, we have $a \in T$. In this case, we will call T a **superset** of S. Note that $T \subseteq T$. A set S is a **proper subset** of T if $S \subseteq T$ and $S \neq T$ - we write this as $S \subsetneq T$. Consider the set

$$S = \{\text{All powers of } 2\} = \{2^n : n \in \mathbb{N}\} = \{1, 2, 4, 8, 16, \ldots\}.$$

We claim that S is a proper subset of $\mathbb{N}$ or $S \subsetneq \mathbb{N}$. This is because every power of 2 is a natural number but not every natural number is a power of 2.

An important set is the **empty set**, denoted by $\emptyset$. It is the set which contains no elements. Mathematically, it serves similar purpose in set theory as natural number 0 for numbers. The cardinality of $\emptyset$ is zero, i.e., $|\emptyset| = 0$. One fundemental property of set theory is that $\emptyset \subseteq S$ for *any* set S.

Take any set S and consider *all* subsets of S. This is called the **power set** of S, and written as 2^S:

$$2^S = \{T : T \subseteq S\}.$$

As discussed, $\emptyset \in 2^S$ and $S \in 2^S$. If $S = \{a, b, c\}$, then 2^S is as follows

$$2^S = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

The subset relation defines relationship between elements (though not between all elements) of 2^S. Mathematically, the element $\{a\}$ in 2^S is a set with element a and the element a in S is just an element a. So, they are different - it is important to realize this.

Operations on sets. Often, we need to do operations on sets. For instance, we take the set of students in a class and take out the set of girls in the class. We may take students of two different classes and merge them together. These will be the *intersection* and *union* operations on sets respectively. The **intersection** of two sets S and T is another set, written $S \cap T$, such that it contains *all* the elements of S that also belong to T:

$$S \cap T = \{x \in S : x \in T\}.$$

Verify that $S \cap T = T \cap S$. Further if $T \in 2^S$, then $S \cap T = T$.

The union operation combines elements of two sets. The **union** of sets S and T is a set, written $S \cup T$, that has elements of S *and* T:

$$S \cup T = \{x : x \in S \text{ or } x \in T\}.$$

So, $S, T \subseteq (S \cup T)$. Notice that $S \cup T = T \cup S$.

The difference operation is another useful operation between sets. The **difference** from S to T is written as $S \setminus T$ and defined as:

$$S \setminus T = \{x \in S : x \notin T\}.$$

So, $S \setminus T$ contains elements of S that are not in T. Note that if $S \cap T = \emptyset$, then $S \setminus T = S$. Further, $S \setminus T$ need not equal $T \setminus S$ - in fact, they are equal if and only if $S = T$. Try proving

$$S \cup T = S \cup (T \setminus S) = (S \setminus T) \cup (S \cap T) \cup (T \setminus S),$$

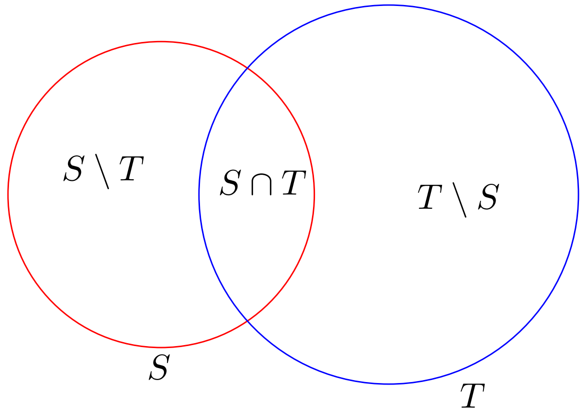

Figure 1: Operations on sets

and each of the sets on the right hand side do not intersect with each other. Figure 1 gives a pictorial description of the sets.

These operations are illustrated using the following example.

$$S = \text{The set of all positive even numbers} = \{2, 4, 6, 8, \ldots\}.$$

$$T = \text{The set of all positive multiples of } 3 = \{3, 6, 9, 12, \ldots\}.$$

Now, the set $S \cap T = \{6, 12, 18, \ldots\}$ contains all even numbers which are multiples of 3, i.e., all numbers which are multiples of 6. The set $S \setminus T = \{2, 4, 8, 10, 14, \ldots\}$ contains all even numbers which are not multiples of 3 - or, all even numbers which are not multiples of 6. Set $T \setminus S = \{3, 9, 15, 21, \ldots\}$ contains all odd numbers which are multiples of 3. Finally, set $S \cup T$ contains all numbers which are either even or multiple of 3.

2 Real Numbers

We encounters integer and rational numbers and they are easy to represent. Formally, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of all integers and $\mathbb{Q}$ is the set of all **rationals**, where a rational number is any number of the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ with $q \neq 0$. Notice that $\mathbb{Z} \subsetneq \mathbb{Q}$. Both the sets $\mathbb{Z}$ and $\mathbb{Q}$ have an important property in common - they are **ordered**. Formally, we can define a "greater than or equal to" relation and say that for any $a, b \in \mathbb{Q}$ either $a \geq b$ or $b \geq a$ - if $a \neq b$, then we can also say $a > b$ or $b > a$.

Besides rational numbers, there are other numbers called irrational numbers. For instance, $\sqrt{2}$ is not a rational number, but it is a number x such that $x^2 = 2$. [1] However, we

[1] To prove why $\sqrt{2}$ is not rational, suppose not and let $\frac{p}{q} = \sqrt{2}$, where we assume without loss of generality

can write down a *sequence* of rational numbers which goes arbitrarily "close" to $\sqrt{2}$:

$$1.4, 1.41, 1.414, 1.4142, \ldots$$

which when squared gives

$$1.96, 1.9881, 1.999396, 1.99996164, \ldots.$$

This is the main idea behind an irrational number is that we can approximate them with arbitrary precision using rational numbers. The above numbers are sequence of rational numbers which are getting closer to $\sqrt{2}$ as the sequence progresses. The set of all such numbers are defined to be irrational numbers. The set of all rational and irrational numbers constitute the set of real numbers - denoted by $\mathbb{R}$. Set of rational numbers can be represented by the **real line** - Figure 2. The real line depicts the ordered relation of the real numbers - as we move to left of a real number, we get smaller numbers and as we go to right, we get bigger numbers.

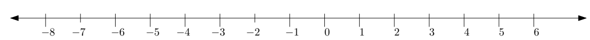

Figure 2: Real line

Real numbers also admit important types of subsets. An **interval** I is a subset of real numbers satisfying the property that for any $x, y \in I$ with $x < y$, every $z \in \mathbb{R}$ with $x < z < y$ is also in I. This allows two types of intervals. To illustrate, consider the interval I which are all the real numbers between 0 and 1, and including 0 and 1. Trivially, if we pick $x < y$, then any z between x and y will also be between 0 and 1, and hence, an element of I. Such intervals are called **closed intervals**. A closed interval can also be written as a set described by two real number $a < b$ as

$$I = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

Now, consider another interval I' which are all the real numbers between 0 and 1, but excluding 0 and 1. Again, it is routine to check that if $x < y$, any real number z with $x < z < y$ will be *strictly* between 0 and 1, and hence, belong to I'. So, I' is also an interval. Such intervals are called **open intervals**. An open interval can also be written as a set described by two real number $a < b$ as

$$I' = \{x \in \mathbb{R} : a < x < b\}.$$

that the highest common factor of p and q is 1. Then, $p^2 = 2q^2$, which implies that p^2 is even. But then, p is even, which in turn implies that q is even, contradicting the fact that the highest common factor of p and q is 1.

A natural open interval is the whole of $\mathbb{R}$, and it is often written as $(-\infty, \infty)$.

The idea of open intervals can be defined more rigorously by observing that the set of real numbers allows for a notion of **distance** between any pair of elements. In particular, for any pair of real numbers $x, y \in \mathbb{R}$, we can define the distance between them in the usual way:

$$d(x, y) = |x - y|,$$

where for any real number $|z| = z$ if $z \geq 0$ and $|z| = -z$ if $z < 0$.

Now, consider an open interval $(0, 1)$. Now, take the real number 0.0001. Even though it is close to zero (than one), we can find a number less than 0.0001 which lies in $(0, 1)$ - for instance, $0.0001 - 0.00001$. Also, $0.0001 + 0.00001$ lies in $(0, 1)$. Moreover, everything between these two numbers lie in $(0, 1)$ - this follows from the interval set definition. More generally, if we take any point $x \in (0, 1)$, we can always find *small* enough $\epsilon > 0$ such that any real number y satisfying $x - \epsilon < y < x + \epsilon$ lies in $(0, 1)$. This is an artifact of the open property of the interval $(0, 1)$. A formal definition of open sets will involve something like this. Notice that a closed interval $[0, 1]$ violates this property because if we consider the element $0 \in [0, 1]$, we cannot find any ϵ such that $0 - \epsilon \in [0, 1]$.

Figure 3 illustrates the open and closed intervals.

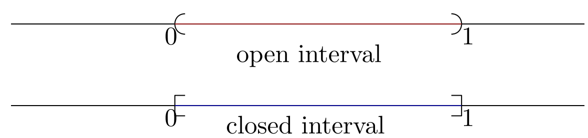

Figure 3: Open and closed intervals

Distances in other sets. The idea of distance between two elements come very naturally in a set like $\mathbb{R}$. There are many such sets. where such a notion of distance can be defined. This allows us to define natural notions of openness and continuity on such sets.

Boundedness. Consider the set of real numbers defined as below:

$$P = \{2^n : n \in \mathbb{N}\} = \{1, 2, 4, 8, 16, \ldots\}.$$

Consider another set of real numbers defined as below:

$$P' = \{\frac{1}{2^n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}.$$

Both P and P' are infinite sets of real numbers. However, we notice that if we pick any two elements in P', the distance between them always less than 1. Such a property is not true for P - we cannot find a real number $\bar{U}$ such that for every $x, y \in P$, $d(x, y) < \bar{U}$. For this reason, P is an **unbounded** set but P' is a **bounded** set.

More formally, a set of real numbers X is bounded if there is a real number $\bar{U}$ such that for any $x, y \in X$, we have $d(x, y) < \bar{U}$. Any set that is not bounded is unbounded. As can be seen, the notion of boundedness is quite distinct from the notion of finiteness. A finite set of real numbers is always bounded. But an infinite set of real numbers may or may not be bounded.

3 Functions, Continuity, and Differentiability

After sets, the idea of functions is probably the most fundamental concept in mathematics. Functions are "mappings" between two sets - meaning, it relates two sets. Take any pair of sets D and X. A function f that assigns every element in D an element in X is denoted as

$$f : D \to X.$$

Here, D is called the **domain** of f and X is called the **co-domain** of f. It is possible that f assigns every element of D the same element of X - a **constant** function: $f(y) = x$ for all $y \in D$, where x is some element of X. But in general, f may assign some values of X to elements of D. The subset of X that gets assigned values to elements in D by f is called the **range** of f. Formally,

$$\text{Range}(f) = \{f(y) : y \in D\}.$$

Notice that if $|\text{Range}(f)| = 1$, then f is a constant function.

Consider $D = \{a, b, c, d\}$ and $X = \{p, q, x, y\}$. Suppose the function f is defined as: $f(a) = p, f(b) = f(c) = q, f(d) = y$. The $\text{Range}(f) = \{p, q, y\} \subsetneq X$. Now, suppose $X' = \{p, q, y\}$. Then f is also a function $f : D \to X'$ and notice that $\text{Range}(f) = X'$. This is the idea of a **surjective** function - a function $f : D \to X$ is surjective if $\text{Range}(f) = X$.

Also, notice that in the above example, $f(b) = f(c) = q$. Compare this with another function $g : D \to X$ such that $g(a) = p, g(b) = x, g(c) = q, g(d) = y$. Here, every element of D is mapped to a unique element in X. Such functions are called **injective** function. A function which is both injective and surjective is called a **bijection** or a **one-to-one** mapping - every element of D is assigned a unique element of X and every element of X is assigned to a unique element of D. Consider the following function on the set of real numbers - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$ for all $x \in \mathbb{R}$. Clearly, the domain and range of this function is the same - $\mathbb{R}$. Since f assigns every member of $\mathbb{R}$ to itself, it is a bijection.

Functions on real numbers. While functions are defined on arbitrary domains to co-domains, function whose domains and codomains are sets of real numbers have special properties. This is because the set of real numbers have some natural properties - for instance, the distance is a well-defined concept on set of real numbers. We will talk about two elementary properties - continuity and differentiability - for functions $f : \mathbb{R} \to \mathbb{R}$.

The main idea behind continuity is that changes in $f(x)$ is "gradual" with changes in x. In other words, if we pick an arbitrary point $x \in \mathbb{R}$ and look at a point y "close" to x, then the value of $f(x)$ is a good approximation of $f(y)$, i.e., $f(y)$ will also lie very close to $f(x)$. For concreteness, take the example of $f(x) = x^2$. Take a "sequence" of points as follows:

$$1.9, 1.99, 1.999, 1.9999, \ldots$$

Notice that the value of $f(x) = x^2$ for these points are

$$3.61, 3.9601, 3.996001, 3.99960001, \ldots$$

So, as the value of x approaches close to 2, the value of $f(x)$ approaches $f(2) = 4$. Here, we are approaching 2 through a sequence of numbers which are less than 2. We could have taken a sequence of numbers which are larger than 2 as follows:

$$2.1, 2.01, 2.001, 2.0001, \ldots$$

and verified a similar "convergence". Such functions are called **continuous**. Informally, if we take a "sequence" of points which are becoming closer and closer to some point x, then the value of f along those points must also become closer and closer to $f(x)$ - in this case, we say that f is continuous at x. If a function is continuous at every point in its domain, then it is a continuous function. If a function is not continuous at some x, then it is **discontinuous**.

Figure 4 illustrates the idea of a continuous and discontinuous functions.

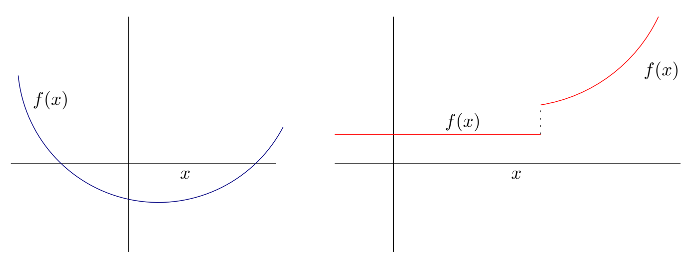

Figure 4: Continuous and discontinuous functions

Here is an example of a discontinuous function. Consider the function f defined as $f(x) = 1$ for $x < 2$ and $f(x) = x^2$ for $x \geq 2$. Now, consider the sequence of points

$$1.9, 1.99, 1.999, 1.9999, \ldots$$

These are points which are approaching 2 - meaning that they are becoming closer to 2. Notice that the value of f at these points is constant and equal to 1. Hence, $f(x)$ is approaching 1 along this sequence of points. However, $f(2) = 4$. In other words, if we take a point "very close" to 2 but smaller than 2, its value will not be a "good approximation" of $f(2)$. Hence, this function is discontinuous - not continuous at $x = 2$.

An alternate approach to define continuity is the "neighborhood" idea. Take any real number $x \in \mathbb{R}$. We can define a set, called the **neighborhood** of x with distance ϵ as follows: take all the points which are at distance less than ϵ from x:

$$N_\epsilon(x) = \{y \in \mathbb{R} : d(x, y) = |x - y| < \epsilon.\}.$$

For instance, if we take $x = 2$ and $\epsilon = 0.1$, then the open interval $(1.9, 2.1)$ is the set $N_{0.1}(2)$. We can define a neighborhood of the function f too. Consider the discontinuous function f defined above. We can think of $N_{0.1}(f(2) = 4) = (3.9, 4.1)$. Now, if we pick values of 2 very close to 2 but lower than 2, then f assigns a value of 1, which lies outside $(3.9, 4.1)$. Indeed, there is *no* neighborhood around 2 such that values of f in that neighborhood lie in $(3.9, 4.1)$. In that sense, the function has a discontinuity at 2. This is the main idea behind the neighborhood definition of continuity: a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at x if for every $\delta > 0$, there is an $\epsilon > 0$ such that for every point $y \in N_\epsilon(x)$, we have $f(y) \in N_\delta(f(x))$.

Both the approaches of defining continuity is the same. The neighborhood idea hints at ways of generalizing this notion to functions defined on arbitrary sets. All we need is a coherent measure of distance between elements of sets. Such a measure of distance naturally allows a definition of neighborhood, which in turn defines continuity.

Derivatives of functions on real numbers. Often, we are interested to measure the rate at which the value of a function changes. Consider the function $f(x) = x^2$. Suppose we are interested at evaluating the rate at which $f(x) = x^2$ changes at $x = 2$. We could take some value 2.1 and observe that $f(2.1) = 4.41$. From this we can derive a rate:

$$\frac{4.41 - 4}{2.1 - 2} = 4.1.$$

We take even closer values $2.01, 2.001$ and compute the rates

$$\frac{4.0401 - 4}{2.01 - 2} = 4.01, 4.001.$$

We can do similar calculations for $1.9, 1.99, 1.999$ and compute rates as

$$\frac{4 - 3.61}{0.1} = 3.9, \frac{4 - 3.9601}{0.01} = 3.99, 3.999.$$

So, we notice that as we take closer and closer values of x to 2, the value of "rate of change" or "slope" of $f(x)$ at $x = 2$ approaches 4. In this case, we will say a **derivative** of f exists at $x = 2$ and its value is equal to 4.

Consider the function $f(x) = x$ for all $x \leq 1$ and $f(x) = x^2$ for $x > 1$. Now, since the function is constant below 1, its rate of change as it "approaches" $x = 1$ from below is 1. But consider the points

$$1.1, 1.01, 1.001, \ldots$$

and note that the rate of change of f is computed as

$$\frac{1.21 - 1}{0.1} = 2.1, \frac{1.0201 - 1}{0.01} = 2.01, 2.001.$$

Hence, as x approaches 2 from above the value of $f(x)$ approaches 2. In this case, we will say that the f is not **differentiable** at $x = 2$.

Figure 4 illustrates the idea of a continuous and discontinuous functions.

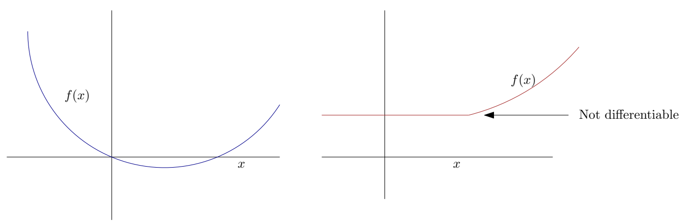

Figure 5: Differentiable and non-differentiable functions

Formally, the rate is measured by measuring $\frac{f(x+h)-f(x)}{h}$ for h arbitrarily close to zero. If this ratio approaches a particular number as h becomes arbitrarily close to zero, then the function f is said to be differentiable at x with the value of the derivative equal to that number.

Monotone functions. Consider a function $f(x) = x^2$ for all $x \in \mathbb{R}$. As the value of x increases, $f(x) = x^2$ also increases. Such functions are called **monotone** functions: for every $x, y \in \mathbb{R}$, $x > y$ implies $f(x) \geq f(y)$. Monotonicity is a property that can be extended to other functions too. For instance, consider a function defined on 2^S for some set S to 2^T for some set T. More precisely, $f : 2^S \to 2^T$. Here, we could say that f is monotone if for all $X, Y \in 2^S$ with $X \subseteq Y$, we have $f(X) \subseteq f(Y)$.