Abstract

A seller is selling multiple objects to a set of agents, who can buy at most one object. Each agent’s preference over (object, payment) pairs need not be quasilinear. The seller considers the following desiderata for her (allocation) rule, which she terms desirable: (1) strategy-proofness, (2) ex-post individual rationality, (3) equal treatment of equals, (4) no wastage (every object is allocated to some agent). The minimum Walrasian equilibrium price (MWEP) rule is desirable. We show that at each preference profile, every MWEP rule generates more revenue for the seller than any desirable rule satisfying no subsidy. Our result works for quasilinear domain, where every MWEP rule is the VCG rule, and for various non-quasilinear domains, some of which incorporate positive income effect of agents. We can relax no subsidy to no bankruptcy in our result for certain domains with positive income effect.

Keywords. multi-object allocation; strategy-proofness; ex-post revenue maximization; minimum Walrasian equilibrium price rule; non-quasilinear preferences; no wastage; equal treatment of equals.

JEL Code. D82, D47, D71, D63.
1 Introduction

One of the most challenging problems in microeconomic theory is the design of revenue maximizing mechanism in multi-object allocation problem. Ever since the seminal work of Myerson (1981) for solving the revenue maximizing mechanism for the single object environment, advances in the mechanism design literature have convinced researchers that it is difficult to precisely describe a revenue maximizing mechanism in multi-object environments.

In the literature on revenue maximizing mechanism design, authors conventionally impose only incentive compatibility and individual rationality conditions, and try to find a mechanism that maximizes the (expected) revenue among mechanisms satisfying those conventional conditions. When allocating public assets, governments are supposed to pursue several goals, such as fairness and efficiency, besides revenue maximization. 1 Since our main focus is on revenue maximization, we impose only moderate desiderata for other goals on mechanisms. 2 In other words, we define several conditions embodying other goals, and maximize revenue in the class of mechanisms satisfying those new conditions along with the conventional incentive and participation constraints.

We study the problem of allocating $m$ indivisible objects to $n > m$ agents, each of whom can be assigned at most one object (unit demand agents) - such unit demand settings are common in allocating houses in public housing schemes (Andersson and Svensson, 2014), selling team franchises in professional sports leagues, and even in selling a small number of spectrum licenses (Binmore and Klemperer, 2002). 3 4 Agents in our model can have non-quasilinear preferences over consumption bundles - (object, payment) pairs.

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1 For example, Klemperer (2002) discusses the list of goals pursued in UK 3G auction conducted in 2000.

2 Allocation rules are sometimes called direct mechanisms in the literature. Since we impose strategy-proofness, we can restrict our attention to direct mechanisms without loss of generality.

3 When a professional cricket league, called the Indian Premier League (IPL) was started in India in 2007, professional teams were sold to interested owners (bidders) by auction. Since it does not make sense for an owner to have two teams, the unit demand assumption is satisfied in this problem. See the Wiki entry of IPL for details: 

4 Although modern spectrum auctions involve sale of of bundles of spectrum licenses, Binmore and Klemperer (2002) report that one of the biggest spectrum auctions in UK involved selling a fixed number of licenses to bidders, each of whom can be assigned at most one license. The unit demand setting is also one of the few computationally tractable model of combinatorial auction studied in the literature (Blumrosen and Nisan, 2007).
We briefly describe the additional axioms that we impose for our revenue maximization exercise. *Equal treatment of equals* is a desideratum for fairness, and requires that two agents having identical preferences be assigned consumption bundles (i.e., (object, payment) pairs) to which they are indifferent. *No wastage* is a desideratum for a mild form of efficiency, and requires that every object be allocated to some agent.

Most of the literature on revenue maximization mechanism design (single or multiple objects) considers only Bayesian incentive compatibility and interim individual rationality. However, since our objective is to provide a robust recommendation of allocation rule in our general setting, we employ a stronger form of incentive and participation constraints: *strategy-proofness* (i.e., dominant strategy incentive compatibility) and *ex-post individual rationality*.

5 We term a rule desirable if it satisfies strategy-proofness, individual rationality, equal treatment of equals, and no wastage.

The mechanism we identify in this paper is based on a market clearing idea. A price vector on objects is called a *Walrasian equilibrium price (WEP) vector* if there is an allocation of objects such that each agent gets an object from his demand set. Demange and Gale (1985) showed that the set of WEP vectors is always a non-empty compact lattice in our model. This means that there is a unique minimum WEP vector. 6 The *minimum Walrasian equilibrium price (MWEP)* rule selects the minimum WEP vector at every profile of preferences and uses a corresponding equilibrium allocation. Every MWEP rule is desirable (Demange and Gale, 1985) and satisfies *no subsidy*. No subsidy requires that payment of each agent be non-negative. In the quasilinear domain of preferences, every MWEP rule coincides with the Vickrey-Clarke-Groves (VCG) rule (Leonard, 1983). However, we emphasize that outside the quasilinear domain, a naive generalization of the VCG rule to non-quasilinear preferences is not strategy-proof (Morimoto and Serizawa, 2015). 7 This also means that for an arbitrary domain of classical preferences, the MWEP rule is very different from the generalized VCG rule.

We show that on a variety of domains (the set of admissible preferences), the MWEP rule is *ex-post revenue optimal* among all desirable rules satisfying *no subsidy*, i.e., for each preference profile, an MWEP rule generates more revenue for the seller than any desirable

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5*Hereafter, whenever we mention individual rationality, it means ex-post individual rationality.*

6*Results of this kind were earlier known for quasilinear preferences (Shapley and Shubik, 1971; Leonard, 1983).*

7*See Section 6.2 in Morimoto and Serizawa (2015).*
rule satisfying no subsidy (Theorem 1). Further, we show that if the domain includes all positive income effect preferences, then an MWEP rule is ex-post revenue optimal in the class of all desirable and \textit{no bankruptcy} rules (Theorem 2). No bankruptcy is a weaker condition than no subsidy and requires the sum of payments of all agents across all profiles be bounded below. This requirement is indispensable since without it, the seller runs the risk of being bankrupt at some profile of preferences.

Our results are robust in the following sense. First, every MWEP rule maximizes ex-post revenue. Hence, we can recommend an MWEP rule without resorting to any prior-based maximization. Notice that ex-post revenue optimality is much stronger than expected (ex-ante) revenue optimality, and mechanisms satisfying ex-post revenue optimality rarely exist. Our results show the existence of ex-post revenue optimal mechanisms in our setting. Second, our results hold on a variety of domains. This is in contrast to many papers in the literature on mechanism design in which results are established only on quasilinear domain. Theorem 1 holds on every domain satisfying a condition called the \textit{richness} condition. The richness condition requires the domain to include enough variety preferences. However it is weak enough to be satisfied by various well known domains, such as the quasilinear domain, the classical domain, the domain of positive income effect preferences, and any domain including one of those domains. Though more restricted than Theorem 1, Theorem 2 also holds on a variety of domains. Indeed, it holds on any domain including the domain of positive income effect preferences.

Ours is the first paper to study revenue maximization in a multi-object allocation problem when preferences of agents are not quasilinear. While quasilinearity is standard and popular in the literature, its practical relevance is debatable in many settings. There are at least two obvious reasons why quasilinearity may fail in practice. First, bidders in auctions usually invest in various supporting products and processes to realize the full value of the object. For instance, cellular companies invest in communication infrastructure development, a sports team owner invests in marketing, and so on. Such ex-post investments cannot be assumed to be independent of the payments in auctions. This hints at an explicit effect of payments in the auction mechanism on the values of objects in these problems. Another source of non-quasilinearity is borrowing costs. Usually, bidders in large auctions (like spectrum auctions, housing auctions, etc.) borrow to pay for objects. The higher interest rates imposed on the larger amount of borrowings make preferences non-quasilinear. \footnote{We will discuss the effect of borrowing cost on preferences in Subsection 4.1.}
We briefly discuss the practical relevance of two of our axioms: equal treatment of equals and no wastage. As we mention above, when allocating public assets using auctions, governments are supposed to pursue several goals other than revenue maximization. One such goal is fairness. As Deb and Pai (2016) report, some laws prohibit governments from favoring particular participants in auctions. Though the literature uses a variety of fairness axioms, each differing from the other in the way they treat different agents, they all agree that equals should be treated equally. In this sense, equal treatment of equals is a minimal requirement of fairness. It is also consistent with some fundamental philosophies of justice. Hence, imposing equal treatment of equals on auction mechanisms requires a weak and indisputable form of fairness.

Efficiency is also an important goal for governments. Indeed, it is widely perceived that the use of auctions to allocate government resources is to enhance efficiency of allocation. Although Pareto efficiency is a standard efficiency desideratum in the literature, since we focus on revenue maximization, we impose no wastage, a much weaker desideratum. Our results show the implication of such a minimal form of efficiency on revenue maximization mechanism design in a multi-object environment.

Unlike Pareto efficiency, when no wastage is violated, most people understand that auction outcomes are inefficient. This is because verifying Pareto efficiency of a mechanism requires knowledge of preferences of agents, but no wastage can be easily verified in practice. Indeed, violation of no wastage in government auctions creates a lot of controversies in the public, and often, the unsold objects are resold. As an example, the Indian spectrum auctions reported a large number of unsold spectrum blocks in 2016, and all of them are supposed to be re-auctioned. In such environments, governments cannot commit to reserve prices even though expected revenue maximization may require them. Indeed, McAfee and McMillan (1987); Jehiel and Lamy (2015); Hu et al. (2017) report that many real-life auctions have zero reserve price - Jehiel and Lamy (2015) and Hu et al. (2017) build theoretical models to

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9See Thomson (2016) for a detailed discussions on other fairness axioms like anonymity in welfare, envy-freeness, egalitarian equivalence, etc.
10Aristotle writes in “Politics” that Justice is considered to mean equality. It does not mean equality - but equality for those who are equal, and not for all.
11For example, see Binmore and Klemperer (2002) and McAfee and McMillan (1996). The latter refers U.S. Congress statement (1993), “The purpose of US Frequency auction is to promote an efficient and intensive use of the electromagnetic spectrum.”
12See the following news article: http://www.livemint.com/Industry/xt5r4Zs5RmxjdwuLUdwJMI/Spectrum-auction-ends-after-lukewarm-response-from-telcos.html
explain it as an equilibrium phenomenon.\textsuperscript{13} While our results do not provide a theory for why the seller should not keep a reserve price, we show that if the seller uses a rule satisfying no wastage and other desirable properties, then the MWEP rule is ex-post revenue optimal. In summary, governments are expected to pursue revenue maximization without wasting resources, and no wastage is an appropriate requirement in such contexts.

Finally, every MWEP rule is Pareto efficient and can be implemented as a simultaneous ascending auction (SAA) - for quasilinear domains, see Demange et al. (1986), and for non-quasilinear domains, see Morimoto and Serizawa (2015).\textsuperscript{14} SAAs have distinct advantages of practical implementation and are often used in practice to allocate multiple objects. The efficiency foundations for SAAs have been well-established. Because of their practical importance, it is worth providing alternate foundations for SAAs. Our results provide a revenue maximization foundation for SAAs. This differentiates our results from previous research on an MWEP rule and SAA, most of which focus on efficiency properties (Ausubel and Milgrom, 2002).

\section{Preliminaries}

A seller has $m$ objects to sell, denoted by $M := \{1, \ldots, m\}$. There are $n > m$ agents (buyers), denoted by $N := \{1, \ldots, n\}$. Each agent can receive at most one object (unit demand preference). Let $L \equiv M \cup \{0\}$, where 0 is the null object, which is assigned to any agent who does not receive any object in $M$ - thus, the null object can be assigned to more than one agent. Note that the unit demand restriction can either be a restriction on preferences or an institutional constraint. For instance, objects may be substitutable when houses are being allocated in a public housing scheme (Andersson and Svensson, 2014). The unit demand restriction can also be institutional as was the case in the spectrum license auction in UK in 2000 (Binmore and Klemperer, 2002) or in the Indian Premier League auction. As long as the mechanism designer restricts messages in the mechanisms to only

\textsuperscript{13}McAfee and McMillan (1987) write “While there appear to be no econometric analyses of the use of reserve prices, there is some informal evidence. Practice does not seem to be in accord with the theoretical result that it is in a seller’s interest to announce a reserve price. In practice, reserve prices are often not used; when they are used, their existence is often not announced, and even when their existence is announced, the seller usually keeps the level of the reserve price secret. Thus there appears to be a discrepancy between theory and practice.”

\textsuperscript{14}To be precise, an MWEP rule can be implemented as a simple ascending price auction with a sufficiently small price increment.
use information on preferences over individual objects, our results apply.

The consumption set of every agent is the set $L \times \mathbb{R}$, where a typical (consumption) bundle $z \equiv (a,t)$ corresponds to object $a \in L$ and payment $t \in \mathbb{R}$. Notice that $t$ denotes the amount paid by an agent to the designer. Now, we formally introduce preferences of agents and the notion of a desirable rule.

2.1 The preferences

A preference ordering $R_i$ (of agent $i$) over $L \times \mathbb{R}$, with strict part $P_i$ and indifference part $I_i$, is classical if it satisfies the following assumptions:

1. **Money monotonicity.** for every $t, t' \in \mathbb{R}$ with $t > t'$ and for every $a \in L$, we have $(a,t') P_i (a,t)$.

2. **Desirability of objects.** for every $t \in \mathbb{R}$ and for every $a \in M$, $(a,t) P_i (0,t)$.

3. **Continuity.** for every $z \in L \times \mathbb{R}$, the sets $\{z' \in L \times \mathbb{R} : z' R_i z\}$ and $\{z' \in L \times \mathbb{R} : z R_i z'\}$ are closed.

4. **Possibility of compensation.** for every $z \in L \times \mathbb{R}$ and for every $a \in L$, there exists a pair $t, t' \in \mathbb{R}$ such that $z R_i (a,t)$ and $(a,t') R_i z$.

A classical preference $R_i$ is quasilinear if there exists $v \in \mathbb{R}^{|L|}$ such that for every $a, b \in L$ and $t, t' \in \mathbb{R}$, $(a,t) R_i (b,t')$ if and only if $v_a - t \geq v_b - t'$. We refer to $v$ as the valuation of the agent, and we normalize $v_0$ to 0. The idea of valuation may be generalized as follows for non-quasilinear preferences.

**Definition 1** The valuation at a classical preference $R_i$ for object $a \in L$ with respect to bundle $z \in L \times \mathbb{R}$ is defined as $V^{R_i}(a,z)$, which uniquely solves $(a, V^{R_i}(a,z)) I_i z$.

A straightforward consequence of our assumptions is that for every $a \in L$, for every $z \in L \times \mathbb{R}$, and for every classical preference $R_i$, the valuation $V^{R_i}(a,z)$ exists. For any $R$ and for any $z \in L \times \mathbb{R}$, the valuations at $R$ with respect to $z$ is a vector in $\mathbb{R}^{|L|}$.
An illustration of the valuation is shown in Figure 1. In the figure, the horizontal lines correspond to objects: $L = \{0, a, b, c\}$. The horizontal lines indicate payment levels. Hence, the consumption set consists of the four lines. For example, $z$ denotes the bundle consisting of object $b$ and the payment equal to the distance of $z$ from the vertical dotted line. A preference $R_i$ can be described by drawing (non-intersecting) indifference vectors through these consumption bundles (lines). One such indifference vector passing through $z$ is shown in Figure 1. This indifference vector actually consists of four points: $(0, V^{R_i}(0, z)), (a, V^{R_i}(a, z)), z \equiv (b, t), (c, V^{R_i}(c, z))$ as shown. Parts of the curve in Figure 1 which lie between the consumption bundle lines is useless and has no meaning - it is only displayed for convenience. As we go to the right along the horizontal lines starting from any bundle, we get worse bundles (due to money monotonicity). Similarly, bundles to the left of a particular bundle are better than that bundle. This is shown in Figure 1 with respect to the indifference vector.

Our modeling of preferences captures income effects even though we do not model income explicitly. We explain this point when we introduce positive income effect later.

2.2 Desirable rules

Let $\mathcal{R}^C$ denote the set of all classical preferences and $\mathcal{R}^Q$ denote the set of all quasilinear preferences. We will consider an arbitrary subset of classical preferences $\mathcal{R} \subseteq \mathcal{R}^C$ - we
will put specific restrictions on $\mathcal{R}$ later. A preference of agent $i$ is denoted by $R_i \in \mathcal{R}$. A preference profile is a list of preferences $R \equiv (R_1, \ldots, R_n)$. Given $i \in N$ and $N' \subseteq N$, let $R_{-i} \equiv (R_j)_{j \neq i}$ and $R_{-N'} \equiv (R_j)_{j \in N'}$, respectively.

An object allocation is an $n$-tuple $(a_1, \ldots, a_n) \in L^n$ such that no real (non-null) object is assigned to two agents, i.e., $a_i \neq a_j$ for all $i, j \in N$ with $a_i, a_j \neq 0$. The set of all object allocations is denoted by $A$. A (feasible) allocation is an $n$-tuple $((a_1, t_1), \ldots, (a_n, t_n)) \in (L \times \mathbb{R})^n$ such that $(a_1, \ldots, a_n) \in A$, where $(a_i, t_i)$ is the bundle of agent $i$. Let $Z$ denote the set of all feasible allocations. For every allocation $(z_1, \ldots, z_n) \in Z$, we will denote by $z_i$ the bundle of agent $i$.

An allocation rule or a rule for short is a map $f : \mathcal{R}^n \rightarrow Z$. At a preference profile $R \in \mathcal{R}^n$, we denote the bundle of agent $i$ in rule $f$ as $f_i(R) \equiv (a_i(R), t_i(R))$, where $a_i(R)$ and $t_i(R)$ are respectively the object allocated to agent $i$ and $i$’s payment at preference profile $R$. We call $a(\cdot) \equiv (a_1(\cdot), \ldots, a_n(\cdot))$ and $t(\cdot) \equiv (t_1(\cdot), \ldots, t_n(\cdot))$ the object allocation rule and the payment rule, respectively of $f$.

**Definition 2** A rule $f : \mathcal{R}^n \rightarrow Z$ is desirable if it satisfies the following properties:

1. **Strategy-proofness.** for every $i \in N$, for every $R_{-i} \in \mathcal{R}^{n-1}$, and for every $R_i, R'_i \in \mathcal{R}$, we have
   \[ f_i(R_i, R_{-i}) R_i f_i(R'_i, R_{-i}). \]

2. **(Ex-post) individual rationality (IR).** for every $i \in N$, for every $R \in \mathcal{R}^n$, we have $f_i(R) R_i (0, 0)$.

3. **Equal treatment of equals (ETE).** for every $i, j \in N$, for every $R \in \mathcal{R}^n$ with $R_i = R_j$, we have $f_i(R) I_i f_j(R)$.

4. **No wastage (NW).** for every $R \in \mathcal{R}^n$ and for every $a \in M$, there exists some $i \in N$ such that $a_i(R) = a$.

Out of the four properties of a desirable rule, strategy-proofness and IR are standard constraints imposed on a rule. In Supplementary Appendix A.2, we show that we cannot relax these axioms to Bayesian incentive compatibility and interim individual rationality in our results.\(^{15}\)

\(^{15}\) On a related note, in the single object case, there is strong equivalence between the set of strategy-proof and Bayesian incentive compatible rules (Mookherjee and Reichelstein, 1992; Manelli and Vincent, 2010; Gershkov et al., 2013). But this equivalence is lost in the multi-object problem.
ETE is a very mild form of fairness requirement. It states that two agents with identical preferences must be assigned bundles to which they should be indifferent. As argued in the introduction, such minimal notion of fairness is often required by law. NW states that every object must be allocated to some agent. As also argued in the introduction, governments are often demanded to maximize revenue while satisfying NW. Besides desirability, for some of our results, we will require some form of restrictions on payments.

**Definition 3** A rule \( f : \mathcal{R}^n \rightarrow Z \) satisfies **no subsidy** if for every \( R \in \mathcal{R}^n \) and for every \( i \in N \), we have \( t_i(R) \geq 0 \).

No subsidy can be considered desirable to exclude “fake” agents, who participate in mechanisms just to take away available subsidy. As was discussed earlier, it is an axiom satisfied by most standard mechanisms in practice. No subsidy is motivated by the fact that in many settings, the seller may not have any means to finance any agents.

### 3 The minimum Walrasian equilibrium price rule

It is not clear if we can define a desirable rule in a classical domain. In this section, we define the notion of a Walrasian equilibrium, and use it to define a desirable rule. A price vector \( p \in \mathbb{R}^{\vert L \vert}_+ \) defines a price for every object with \( p_0 = 0 \). At any price vector \( p \in \mathbb{R}^{\vert L \vert}_+ \), let \( D(R_i, p) := \{ a \in L : (a, p_a) \ R_i \ (b, p_b) \ \forall \ b \in L \} \) denote the demand set of agent \( i \) with preference \( R_i \) at price vector \( p \).

**Definition 4** An object allocation \((a_1, \ldots, a_n) \in A \) and a price vector \( p \in \mathbb{R}^{\vert L \vert}_+ \) is a **Walrasian equilibrium** at a preference profile \( R \in \mathcal{R}^n \) if

1. \( a_i \in D(R_i, p) \) for all \( i \in N \) and
2. \( p_a = 0 \) for all \( a \in M \setminus \{a_1, \ldots, a_n\} \).

We refer to \( p \) and \(((a_1, p_{a_1}), \ldots, (a_n, p_{a_n})) \) defined above as a **Walrasian equilibrium price vector** and a **Walrasian equilibrium allocation** at \( R \) respectively.

Since we assume \( n > m \) and preferences satisfy desirability of objects, the conditions of Walrasian equilibrium implies that for all \( a \in M \), we have \( a_i = a \) for some \( i \in N \). \(^{16}\)

\(^{16}\)To see this, suppose that there is \( a \in M \) such that \( a_i \neq a \) for each \( i \in N \). Then, by the second condition of Walrasian equilibrium, \( p_a = 0 \). By \( n > m \), \( a_i = 0 \) for some \( i \in N \). By desirability of objects, \((a, 0) \ P_i \ (a_i, 0)\), contradicting the first condition of Walrasian equilibrium.
A Walrasian equilibrium price vector $p$ is a **minimum Walrasian equilibrium price vector** at preference profile $R$ if for every Walrasian equilibrium price vector $p'$ at $R$, we have $p_a \leq p'_a$ for all $a \in L$. At every $R \in (R^C)^n$, a Walrasian equilibrium exists (Alkan and Gale, 1990), the set of Walrasian equilibrium price vectors forms a lattice with a unique minimum and a unique maximum Walrasian equilibrium price vector (Demange and Gale, 1985). We denote the minimum Walrasian equilibrium price vector at $R \in (R^C)^n$ as $p_{\text{min}}(R)$. Notice that by desirability of objects, if $n > m$, then for every $a \in M$, we have $p_{\text{min}}^a(R) > 0$. \(^{17}\)

We give an example to illustrate the notion of minimum Walrasian equilibrium price vector. Suppose $N = \{1, 2, 3\}$ and $M = \{a, b\}$. Figure 2 shows some indifference vectors of a preference profile $R \equiv (R_1, R_2, R_3)$ and the corresponding minimum Walrasian equilibrium price vector $p_{\text{min}}(R) \equiv p_{\text{min}} \equiv (p_0^{\text{min}} = 0, p_a^{\text{min}}, p_b^{\text{min}})$.

![Figure 2: The minimum Walrasian equilibrium price vector](image)

First, note that

$$D(R_1, p_{\text{min}}) = \{a\}, D(R_2, p_{\text{min}}) = \{a, b\}, D(R_3, p_{\text{min}}) = \{0, b\}.$$  

Hence, a Walrasian equilibrium is the allocation where agent 1 gets object $a$, agent 2 gets object $b$, and agent 3 gets the null object at the price vector $p_{\text{min}}$. Also, $p_{\text{min}}$ is the minimum Walrasian equilibrium price vector. To see this, let $p$ be any other Walrasian equilibrium price vector. If $p_a < p_{a}^{\text{min}}$ and $p_b < p_{b}^{\text{min}}$, then no agent demands the null object, contradicting Walrasian equilibrium. Thus, $p_a \geq p_{a}^{\text{min}}$ or $p_b \geq p_{b}^{\text{min}}$. If $p_b < p_{b}^{\text{min}}$, then by $p_a \geq p_{a}^{\text{min}}$,  

\(^{17}\)To see this, suppose $p_{a}^{\text{min}}(R) = 0$ for some $a \in M$. Then any agent $i \in N$ who is not assigned in the Walrasian equilibrium will prefer $(a, 0)$ to $(0, 0)$ contradicting the fact that he is assigned a bundle from his demand set. Indeed, this argument holds for any Walrasian equilibrium price vector.
both agents 2 and 3 will demand only object \( b \), contradicting Walrasian equilibrium. Thus, \( p_b \geq p_b^{\text{min}} \). But, if \( p_a < p_a^{\text{min}} \), both agents 1 and 2 will demand only object \( a \), a contradiction to Walrasian equilibrium. Hence, \( p \geq p^{\text{min}} \).

We now describe a desirable rule satisfying no subsidy. The rule picks a minimum Walrasian equilibrium allocation at every profile of preferences. Although the minimum Walrasian equilibrium price vector is unique at every preference profile, there may be multiple supporting object allocation - all these object allocations must be indifferent to all the agents. To handle this multiplicity problem, we introduce some notation. Let \( Z^{\text{min}}(R) \) denote the set of all allocations at a minimum Walrasian equilibrium at preference profile \( R \). Note that if \( n > m \) and \( ((a_1, p_{a_1}), \ldots, (a_n, p_{a_n})) \in Z^{\text{min}}(R) \) then \( p \equiv (p_a)_{a \in L} = p^{\text{min}}(R) \).

**Definition 5** A rule \( f^{\text{min}} : R^n \rightarrow Z \) is a minimum Walrasian equilibrium price (MWEP) rule if \( f^{\text{min}}(R) \in Z^{\text{min}}(R) \forall R \in R^n \).

Demange and Gale (1985) showed that every MWEP rule is strategy-proof.\(^{18}\) Clearly, it also satisfies IR, ETE, NW, and no subsidy. We document this fact below.

**Fact 1** Every MWEP rule is desirable and satisfies no subsidy.

It is worth comparing the MWEP rule with the VCG rule for quasilinear preferences. Indeed, there is a naive way to generalize the VCG rule to any classical preference domain. Consider a preference profile \( R \). For every agent \( i \in N \) with preference \( R_i \), let \( v^a_i := V^{R_i}(a, (0, 0)) \) for all \( a \in M \). Let \( v^0_i = 0 \) for all \( i \in N \). Now, we compute the allocation and payments according to the VCG rule with respect to this profile of vectors \( (v_1, \ldots, v_n) \).

Such a generalized VCG rule coincides with the MWEP rule if the domain is the quasilinear domain (Leonard, 1983). Else, the generalized VCG rule is very different from the MWEP rule. Further, it is not strategy-proof if the domain is not the quasilinear domain (Morimoto and Serizawa, 2015).

### 4 The results

In this section, we formally state our two results. The proofs of both the results are in Section 5. Before we state the results, we explain the domain richness they use.

\(^{18}\) The MWEP rule satisfies a stronger incentive property called (weak) group-strategy-proofness, which means that for each \( R \in R^n \), there are no coalition \( N' \subseteq N \), of agents and \( R'_{N'} \in R^{|N'|} \) such that for each \( i \in N' \), \( f_i(R'_{N'}, R_{-N'}) P_i f_i(R) \).
4.1 Rich domains

For each pair of price vectors \( p, \hat{p} \in \mathbb{R}_+^{|L|} \), we write \( p > \hat{p} \) if \( p_a > \hat{p}_a \) for all \( a \in M \). The domain of preferences that we consider for our first result requires the following richness.

**Definition 6** A domain of preferences \( \mathcal{R} \) is **rich** if for all \( a \in M \) and for every \( \hat{p} \) with \( \hat{p}_a > 0, \hat{p}_b = 0 \) for all \( b \neq a \) and for every \( p > \hat{p} \) there exists \( R_i \in \mathcal{R} \) such that

\[
D(R_i, p) = \{0\} \text{ and } D(R_i, \hat{p}) = \{a\}.
\]

Figure 3 illustrates this notion of richness with two objects \( a \) and \( b \) - two possible price vectors \( p \) and \( \hat{p} \) are shown and two indifference vectors of a preference \( R_i \) are shown such that \( D(R_i, p) = \{0\} \) and \( D(R_i, \hat{p}) = \{a\} \).

The requirement of the richness condition weak enough to be satisfied by many domains of interest. It is straightforward to see that if a domain of preferences is rich, then any superset of that domain is also rich. We give below some interesting examples of rich domains. Any superset of these domains are also rich.

- Any domain of preferences containing \( \mathcal{R}_Q \) satisfies richness. To see this, fix an object \( a \in M \) and a price vector \( \hat{p} \) with \( \hat{p}_b = 0 \) for all \( b \neq a \) and \( \hat{p}_a > 0 \). Consider any other price vector \( p > \hat{p} \). Now, consider the quasilinear preference \( R_i \) given by the valuation vector \( v \) such that \( v_a = \hat{p}_a + 2\epsilon \), where \( \epsilon > 0 \) is small enough such that \( v_a = \hat{p}_a + 2\epsilon < p_a \) and \( \epsilon < p_b \) for all \( b \in M \setminus \{a\} \). For all \( b \in M \setminus \{a\} \), \( v_b = \epsilon \). This means that \( D_i(R_i, \hat{p}) = \{a\} \) but \( D_i(R_i, p) = \{0\} \).
The set of all positive income effects preferences and the set of all non-negative income effect preferences satisfy richness.

**Definition 7** A preference \( R_i \) satisfies **positive income effect** if for every \( a, b \in L \) and for every \( t, t' \) with \( t < t' \) and \( (b, t') I_i(a, t) \), we have

\[
(b, t' - \delta) \ P_i(a, t - \delta) \quad \forall \delta > 0.
\]

A preference \( R_i \) satisfies **non-negative income effect** if for every \( a, b \in L \) and for every \( t, t' \) with \( t < t' \) and \( (b, t') I_i(a, t) \), we have

\[
(b, t' - \delta) \ R_i(a, t - \delta) \quad \forall \delta > 0.
\]

Let \( R^{++} \) be the set of all positive income effect preferences and \( R^+ \) be the set of all non-negative income effect preferences.

Both \( R^+ \) and \( R^{++} \) are rich domains.

A standard definition of positive income effect will say that a preferred object is more preferred as income increases. We do not model income explicitly, but the zero payment corresponds to the endowed income. Thus, in our model, when income increases by \( \delta > 0 \), the origin of consumption space moves to right by \( \delta \). This movement is equivalent to sliding indifference curves to left. In other words, if the origin is fixed, the increase of income by \( \delta \) is expressed as the decrease of payments of all bundles by \( \delta \). In the above definition, \( (b, t') I_i(a, t) \) and \( t' > t \) imply that object \( b \) is strictly preferred to object \( a \) at any common payment levels \( t'' \in [t, t') \). Then, positive income effect requires that when payments are decreased by \( \delta \), \( b \) will be preferred to \( a \), i.e., \( (b, t' - \delta) P_i(a, t - \delta) \). Hence, our modeling of preferences captures income effects even though we do not model income explicitly.

The set of all quasi-linear preferences with borrowing cost satisfies richness. Imagine a situation in which an agent has a quasilinear preference with valuation \( v \), but has to borrow money from banks at interest rate \( r > 0 \) if his payment for an object exceeds his income \( I > 0 \). Then, given \( t \in \mathbb{R} \), his cost of payment, which we denote by \( c(t, I, r) \), is as follows.

\[
c(t, I, r) = \begin{cases} 
  t & \text{if } t \leq I, \\
  I + (t - I)(1 + r) & \text{if } t > I.
\end{cases}
\]

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Thus, for each pair \((a, t), (b, t') \in L \times \mathbb{R}\), the agent weakly prefers \((a, t)\) to \((b, t')\) if and only if \(v(a) - c(t, I, r) \geq v(b) - c(t', I, r)\). Such preferences are obviously not quasilinear.

Let \(R^B\) be the set of all such preferences. Then, \(R^B\) is rich.

- The set of all single-peaked preferences satisfies richness. Imagine a condominium in which each floor has one room. Some agents prefer the highest floor because of good views, some prefer the lowest to avoid walking up stairs, and some prefer middle floors. Then, it is natural that each agent has a single-peaked preference - an ideal floor, and as we go away from the ideal floor, we go down our preference.

Formally, there is a strict order \(\succ\) over \(L\) such that for each \(a \in M, a \succ 0\). A preference \(R_i\) is single-peaked if there is a unique object \(\tau(R_i)\) such that for all \(t \in \mathbb{R}\)

- \((\tau(R_i), t) \ R_i (a, t)\) for all \(a \in M \setminus \{\tau(R_i)\}\) and
- if \(\tau(R_i) \succ a \succ b\) or \(b \succ a \succ \tau(R_i)\), then \((a, t) \ R_i (b, t)\).

In other words, an agent with preference \(R_i\) has a “peak” floor, say \(\tau(R_i)\), such that when the prices of all floors are the same, he prefers \(\tau(R_i)\) to other floors, and for any two floors \(a\) and \(b\), if \(b \succ a \succ \tau(R_i)\) or \(\tau(R_i) \succ a \succ b\), he prefers \(a\) to \(b\). Let \(R^S\) be the set of all single-peaked preferences. Then, \(R^S\) is rich.

We can summarize the above discussions in this claim.

**Claim 1** The following domains are rich: \(R^Q, R^B, R^+, R^{++}, R^S, R^C\).

We omit a formal proof for the above claim. However, the intuition for its proof is similar to the quasilinear domain proof outlined above. Given \(a \in M\) and two price vectors \(p, \hat{p} \in \mathbb{R}_+^{\lfloor L \rfloor}\) with \(\hat{p} < p\), in those domains, we can find a preference \(R_i\) that satisfies the following: 
\(V^{R_i}(a, (0, 0)) < p_a\) and for each \(b \in M \setminus \{a\}\), \(V^{R_i}(b, (0, 0))\) is close to zero and \(V^{R_i}(b, (a, \hat{p}_a)) < 0\). Then, for preferences satisfying these conditions, the demand sets at \(p\) and \(\hat{p}\) contain only 0 and \(a\), respectively.

### 4.2 Ex-post revenue maximization of desirable mechanisms

We now formally state our first main result. For any rule \(f : \mathcal{R}^n \rightarrow Z\), we define the revenue at preference profile \(R \in \mathcal{R}^n\) as

\[
\text{Rev}^f(R) := \sum_{i \in N} t_i(R).
\]
Definition 8 A rule $f : \mathcal{R}^n \rightarrow Z$ is **ex-post revenue optimal** among a class of rules defined on $\mathcal{R}^n$ if for every rule $g$ in this class, we have

$$\text{Rev}^f(R) \geq \text{Rev}^g(R) \quad \forall R \in \mathcal{R}^n.$$  

It is not clear that an ex-post revenue optimal rule exists. Our main result shows that every MWEP rule is ex-post revenue optimal among the class of desirable rules satisfying no subsidy.

**Theorem 1** Suppose $\mathcal{R}$ is a rich domain of preferences. Every MWEP rule is ex-post revenue optimal among the class of desirable rules satisfying no subsidy defined on $\mathcal{R}^n$.

Theorem 1 clearly implies that even if we do expected revenue maximization with respect to any prior on the preferences of agents, we will only get an MWEP rule among the class of desirable and no subsidy rules. We emphasize that it is not clear that an ex-post revenue optimal desirable rule satisfying no subsidy exists and our result establishes that any MWEP rule is one such rule.

We use Claim 1 to spell out our result in specific domains.

**Corollary 1** Suppose $\mathcal{R} \in \{\mathcal{R}^Q, \mathcal{R}^+, \mathcal{R}^{++}, \mathcal{R}^B, \mathcal{R}^S, \mathcal{R}^C\}$. Every MWEP rule is ex-post revenue optimal among the class of desirable rules satisfying no subsidy defined on $\mathcal{R}^n$.

In the quasilinear domain, the outcome of an MWEP rule coincides with the Vickrey-Clarke-Groves (VCG) mechanism. Hence, the VCG mechanism is ex-post revenue optimal in the quasilinear domain among the class of desirable rules satisfying no subsidy. We now give some remarks about Theorem 1.

**Remark 1.** Although it is difficult to describe the set of desirable rules satisfying no subsidy which is different from an MWEP rule, such rules exist even in the domain of quasilinear preference (which is a rich domain). We include an example of Tierney (2016) in the Supplementary Appendix A.1 at the end of this manuscript for completeness. Indeed, the set of all desirable rules satisfying no subsidy is quite complicated to describe in the quasilinear domain of preferences.

**Remark 2.** A closer inspection of the richness reveals that if $p$ is too small, then richness requires the existence of a preference where the valuations (with respect to $(0,0)$) for real objects is very small. We can weaken this richness to a weaker condition which requires
that valuations lie in an interval of the form $(v_{\text{min}}, v_{\text{max}})$, where $v_{\text{min}}$ and $v_{\text{max}}$ are any lower and upper bounds on the valuation of the objects such that $v_{\text{max}} > v_{\text{min}} \geq 0$ and $v_{\text{max}} \in \mathbb{R}_+ \cup \{+\infty\}$. A formal definition and proof is available upon request.

**Remark 3.** All the axioms used in Theorem 1 are necessary. Supplementary Appendix A.2 contains examples of rules which violate one of the axioms in Theorem 1 but generates more revenue than the MWEP rule at some profile of preferences.

We now show how Theorem 1 can be strengthened in some specific rich domains. In particular, if the domain contains all the positive income effect preferences, then our result can be strengthened - we can replace no subsidy in Theorem 1 by the following no bankruptcy condition.

**Definition 9** A rule $f : \mathcal{R}^n \rightarrow \mathbb{Z}$ satisfies **no bankruptcy** if there exists $\ell \leq 0$ such that for every $R \in \mathcal{R}^n$, we have $\sum_{i \in N} t_i(R) \geq \ell$.

Obviously, no bankruptcy is a weaker property than no subsidy. 19 No bankruptcy is motivated by settings where the seller has limited means to finance the auction participants. Theorem 1 can now be strengthened in the positive income effect domain.

**Theorem 2** Suppose $\mathcal{R} \supseteq \mathcal{R}^{++}$. Every MWEP rule is ex-post revenue optimal among the class of desirable rules satisfying no bankruptcy defined on $\mathcal{R}^n$.

Analogous to Corollary 1, the following is a corollary of Theorem 2.

**Corollary 2** Suppose $\mathcal{R} \in \{\mathcal{R}^+, \mathcal{R}^{++}, \mathcal{R}^C\}$. Every MWEP rule is ex-post revenue optimal among the class of desirable rules satisfying no subsidy defined on $\mathcal{R}^n$.

### 4.3 Pareto efficiency

Since no wastage is a minimal form of efficiency axiom, it is natural to explore the implications of stronger forms of efficiency. We now discuss the implications of Pareto efficiency in our problem and relate it to our results. Before we formally define it, we must state the obvious fact that no wastage is a much weaker but more testable axiom in practice than Pareto efficiency.

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19In the literature, the no-deficit condition is sometimes imposed instead of no subsidy. A rule $f : \mathcal{R}^n \rightarrow \mathbb{Z}$ satisfies **no deficit** if for each $R \in \mathcal{R}^n$, $\sum_{i \in N} t_i(R) \geq 0$. It is clear that no bankruptcy is weaker than no deficit.
Definition 10 A rule \( f : \mathcal{R}^n \to Z \) is **Pareto efficient** if at every preference profile \( R \in \mathcal{R}^n \), there exists no allocation \( ((\hat{a}_1, \hat{t}_1), \ldots, (\hat{a}_n, \hat{t}_n)) \in Z \) such that

\[
(\hat{a}_i, \hat{t}_i) R_i f_i(R) \quad \forall i \in N
\]

\[
\sum_{i \in N} \hat{t}_i \geq \text{Rev}^f(R),
\]

with either the second inequality holding strictly or some agent \( i \) strictly preferring \((\hat{a}_i, \hat{t}_i)\) to \( f_i(R_i) \).

The above definition is the appropriate notion of Pareto efficiency in this setting. Notice that by distributing some money among all the agents, we can always make each agent better off than the allocation in any rule. Hence, the above definition requires that there should not exist another allocation where the auctioneer’s revenue is not less and every agent is weakly better off.

Every MWEP rule is Pareto efficient (Morimoto and Serizawa, 2015). Our results establish that even if a seller maximizes her revenue with this weak form of efficiency, it will be forced to use a Pareto efficient rule. We state this as corollaries below.

**Corollary 3** Let \( \mathcal{R} \) be rich and \( f : \mathcal{R}^n \to Z \) be ex-post revenue optimal among desirable mechanisms satisfying no subsidy. Then, \( f \) is efficient.

**Corollary 4** Let \( \mathcal{R} \supseteq \mathcal{R}^{++} \) and \( f : \mathcal{R}^n \to Z \) be ex-post revenue optimal among desirable mechanisms satisfying no bankruptcy. Then, \( f \) is efficient.

In other words, even if the seller maximizes her revenue among the set of all desirable rules satisfying no subsidy (or no bankruptcy in the positive income effect domain), it will be forced to use a Pareto efficient rule. Hence, we get Pareto efficiency as a corollary without imposing it explicitly.

5 **The proofs**

In this section, we present all the proofs. The proofs use the following fact very crucially: every MWEP rule chooses a Walrasian equilibrium outcome. Before diving into the proofs, we want to stress here that a greedy approach of proving our results would be to first prove that any desirable rule satisfying no subsidy and maximizing revenue must be Pareto efficient. In the quasilinear domain, using revenue equivalence will then pin down the MWEP (VCG) rule.
This approach will fail in our setting because our results work even without quasilinearity and revenue equivalence does not hold in such domains. Further, it is not obvious even in quasilinear domain that the desirability, no subsidy, and the revenue optimality implies Pareto efficiency. Our proofs work by showing various implications of desirability and no subsidy on consumption bundles of agents. It uses richness of the domain to derive these implications. In that sense, it departs from traditional Myersonian techniques, where revenue maximization is a programming problem with object allocation rules as decision variables.

It is worth discussing how our proofs are different from Morimoto and Serizawa (2015), who characterize the MWEP mechanisms. Their focus is on Pareto efficiency and their proofs depend on this. Since we use only no wastage as an efficiency desideratum, which is much weaker than Pareto-efficiency, we need to develop our own proof techniques to establish our results.

We start off by showing an elementary lemma which shows that at every preference profile, if a rule gives every agent weakly better consumption bundles than an MWEP rule, then its revenue is no more than any MWEP rule. This lemma will be used to prove both our results.

**Lemma 1** For every rule \( f : \mathcal{R}^n \rightarrow Z \) and for every \( R \in \mathcal{R}^n \), the following holds:

\[
[f_i(R) \ R_i f^{\text{min}}_i(R) \ \forall \ i \in N] \Rightarrow [\text{Rev}^{f^{\text{min}}}(R) \geq \text{Rev}^{f}(R)],
\]

where \( f^{\text{min}} \) is an MWEP rule.

**Proof:** Fix a profile of preferences \( R \in \mathcal{R}^n \) and denote \( f^{\text{min}}_i(R) = (a_i, p^{\text{min}}_{a_i}(R)) \) for each \( i \in N \). Now, for every \( i \in N \), we have \( f_i(R) \equiv (a_i(R), t_i(R)) \ R_i (a_i, p^{\text{min}}_{a_i}(R)) \) and by the Walrasian equilibrium property, \( (a_i, p^{\text{min}}_{a_i}(R)) \ R_i (a_i(R), p^{\text{min}}_{a_i(R)}(R)) \). This gives us \( t_i(R) \leq p^{\text{min}}_{a_i(R)}(R) \) for each \( i \in N \). Hence,

\[
\text{Rev}^{f}(R) = \sum_{i \in N} t_i(R) \leq \sum_{i \in N} p^{\text{min}}_{a_i(R)}(R) \leq \text{Rev}^{f^{\text{min}}}(R),
\]

where the last inequality follows from \( p^{\text{min}}(R) \in \mathbb{R}_{+}^{L} \).

5.1 Proof of Theorem 1

We start with a series of Lemmas before providing the main proof. Throughout, we assume that \( \mathcal{R} \) is a rich domain of preferences and \( f \) is a desirable rule satisfying no subsidy on \( \mathcal{R}^n \). For the proofs, we need the following definition.
**Definition 11** A preference $R_i$ is $(a,t)$-favoring for $t \geq 0$ and $a \in M$ if for price vector $p$ with $p_a = t, p_b = 0$ for all $b \neq a$, we have $D(R_i, p) = \{a\}$.

An equivalent way to state this is that $R_i$ is $(a,t)$-favoring for $t > 0$ and $a \in M$ if $V_{R_i}^p(b, (a, t)) < 0$ for all $b \neq a$. A slightly stronger version of $(a, t)$-favoring preference is the following.

**Definition 12** For every bundle $(a, t) \in M \times \mathbb{R}_+$ with $t > 0$ and for every $\epsilon > 0$, a preference $R_i \in \mathcal{R}$ is a $(a, t)^\epsilon$-favoring preference if it is a $(a, t)$-favoring preference and

$$V_{R_i}^p(a, (0, 0)) < t + \epsilon$$
$$V_{R_i}^p(b, (0, 0)) < \epsilon \quad \forall \ b \in M \setminus \{a\}.$$ 

The following lemma shows that if $\mathcal{R}$ is rich, then $(a, t)^\epsilon$-favoring preferences exist for every $(a, t) \in M \times \mathbb{R}_+$ and $\epsilon > 0$.

**Lemma 2** Suppose $\mathcal{R}$ is rich. Then, for every bundle $(a, t) \in M \times \mathbb{R}_+$ and for every $\epsilon > 0$, there exists a preference $R_i \in \mathcal{R}$ such that it is $(a, t)^\epsilon$-favoring.

**Proof:** Define $\hat{p}$ as follows: $\hat{p}_a = t, \hat{p}_b = 0 \ \forall \ b \neq a$.

Define $p$ as follows: $p_a = t + \epsilon, \ p_0 = 0, \ p_b = \epsilon \ \forall \ b \in M \setminus \{a\}$.

By richness, there exists $R_i$ such that $D(R_i, \hat{p}) = \{a\}$ and $D(R_i, p) = \{0\}$. But this implies that $R_i$ is $(a, t)$-favoring. Further, $V_{R_i}^p(a, (0, 0)) < t + \epsilon$ and $V_{R_i}^p(b, (0, 0)) < \epsilon \ \forall \ b \in M \setminus \{a\}$.

Hence, $R_i$ is $(a, t)^\epsilon$-favoring. \[ \square \]

Using this, we prove the following lemma which will be used in the proof.

**Lemma 3** For every preference profile $R \in \mathcal{R}^n$, for every $i \in N$, for every $t \in \mathbb{R}_+$, if there exists $j \neq i$ such that $R_j$ is $(a_i(R), t)$-favoring, then $t_i(R) > t$.

**Proof:** Suppose $t_i(R) \leq t$. Since $R_j$ is $(a_i(R), t)$-favoring, $t_i(R) \leq t$ implies that $R_j$ is also $f_i(R) \equiv (a_i(R), t_i(R))$-favoring. Consider a preference profile $R' \equiv (R'_i = R_j, R'_{-i} = R_{-i})$.

By equal treatment of equals (since $R'_i = R'_j = R_j$),

$$f_i(R') \equiv f_j(R').$$ \hfill (1)

We argue that $f_i(R') = f_i(R)$. If $a_i(R') = a_i(R)$, then strategy-proofness implies that $t_i(R') = t_i(R)$ and we are done. Assume for contradiction that $a_i(R) = a \neq b = a_i(R')$. By strategy-proofness, $(b, t_i(R')) R'_i (a, t_i(R))$, which implies that $t_i(R') \leq V_{R_i}^p(b, (a, t_i(R)))$.
Since \( R'_j = R_j \) is \((a, t_i(R))\)-favoring, we have \( V_{R'_j}(b, (a, t_i(R))) < 0 \). This implies that \( t_i(R') < 0 \), which is a contradiction to no subsidy. Hence, we have

\[
f_i(R') = f_i(R). \tag{2}
\]

Combining Inequality (1) and Equation (2), we get that \( f_i(R) I_j f_j(R') \). Hence, \( t_j(R) = V_{R'_j}(a_j(R'), f_i(R)) < 0 \), where the strict inequality followed from the fact \( R_j \) is \( f_i(R) \)-favoring and \( a_i(R) = a_i(R') \neq a_j(R') \). This is a contradiction to no subsidy. \( \blacksquare \)

We will now prove Theorem 1 using these lemmas.

**Proof of Theorem 1**

**Proof:** Fix a desirable rule \( f : \mathcal{R}^n \rightarrow Z \) satisfying no subsidy, where \( \mathcal{R} \) is a rich domain of preferences. Fix a preference profile \( R \in \mathcal{R}^n \). Let \( (z_1, \ldots, z_n) \equiv f^{\text{min}}(R) \) be the allocation chosen by an MWEP rule \( f^{\text{min}} \) at \( R \). Let \( p \equiv \min_{a \in M} p_{a_i}^{\text{min}}(R) \). Clearly, \( p > 0 \). For simplicity of notation, we will denote \( z_i \equiv (a_i, p_i) \), where \( p_i \equiv p_{a_i}^{\text{min}}(R) \) for all \( i \in N \). We prove that \( f_i(R) R_i z_i \) for all \( i \in N \), and by Lemma 1, we will be done.

Assume for contradiction that there is some agent \( i \in N \) such that \( z_i P_i f_i(R) \). We first construct a finite sequence of agents and preferences, without loss of generality \((1, R'_1), \ldots, (n, R'_n)\), satisfying certain properties. Let \( N_0 \equiv \emptyset, N_k \equiv \{1, \ldots, k\} \) for each \( k \geq 1 \), and \( (R'_{N_0}, R_{-N_0}) \equiv R \). This sequence satisfies the properties that for every \( k \in \{1, \ldots, n\} \),

1. \( z_k P_k f_k(R'_{N_{k-1}}, R_{-N_{k-1}}) \) for each \( k \geq 1 \),

2. \( a_k \neq 0 \),

3. \( R'_k \) is \((z_k)^{\epsilon_k}\)-favoring for some \( \epsilon_k > 0 \) with \( \epsilon_k < \min\{V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) - p_k, p\} \).

Now, we construct this sequence inductively.

**Step 1 - Constructing** \((1, R'_1)\). Let \( i = 1 \). By our assumption, \( z_1 P_1 f_1(R) \). This implies \( p_1 - V^{R_1}(a_1, f_1(R)) < 0 \). Thus, there is \( \epsilon_1 > 0 \) such that \( \epsilon_1 < \min\{V^{R_1}(a_1, f_1(R)) - p_1, p\} \).

By Lemma 2, there is a \((z_1)^{\epsilon_1}\)-favoring preference \( R'_1 \). Suppose \( a_1 = 0 \). Then, \((0, 0) = z_1 P_1 f_1(R)\), which contradicts individual rationality. Hence, \( a_1 \neq 0 \).

**Step 2 - Constructing** \((k, R'_k)\) for \( k > 1 \). We proceed inductively - suppose, we have already constructed \((1, R'_1), \ldots, (k-1, R'_{k-1})\) satisfying Properties 1, 2, and 3. By no wastage and the fact that \( a_{k-1} \neq 0 \), there is agent \( j \in N \) such that \( a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1} \).
If \( j = k - 1 \), then individual rationality implies that
\[
t_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_{k-1}}(a_{k-1}, (0, 0)) < p_{k-1} + \epsilon_{k-1} < V^{R_{k-1}}(a_{k-1}, f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}})),
\]
where the second inequality followed from the fact that \( R'_{k-1} \) is \((z_{k-1})^{\epsilon_{k-1}}\)-favoring, and the last inequality followed from the definition of \( \epsilon_{k-1} \). Thus,
\[
f_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}}) P_{k-1} f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}}),
\]
which contradicts strategy-proofness. Hence, \( j \neq k - 1 \).

If \( j \in N_{k-2} \), then by individual rationality, we get
\[
t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_{j}}(a_{k-1}, (0, 0)) < \epsilon_j < p_{k-1}, \tag{3}
\]
where the second inequality followed from the fact that \( R'_{j} \) is \((z_{j})^{\epsilon_{j}}\)-favoring and \( j \neq (k - 1) \), and the last inequality followed from the definition of \( \epsilon_j \). But, notice that agent \((k - 1) \neq j \) and \( R'_{k-1} \) is \( z_{k-1} \)-favoring (since it is \((z_{k-1})^{\epsilon_{k-1}}\)-favoring). Further \( a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1} \). Then, Lemma 3 implies that \( t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) > p_{k-1} \), which contradict Inequality (3).

Thus, we have established \( j \notin N_{k-1} \). Hence, we denote \( j \equiv k \), and note that
\[
z_k R_k z_{k-1} P_k f_k(R'_{N_{k-1}}, R_{-N_{k-1}}),
\]
where the first preference relation follows from the Walrasian equilibrium property and the second follows from the fact that \( a_k(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1} \) and \( p_{k-1} < t_k(R'_{N_{k-1}}, R_{-N_{k-1}}) \) (Lemma 3). Hence Property 1 is satisfied for agent \( k \). Next, if \( a_k = 0 \), then \((0, 0) = z_k P_k f_k(R'_{N_{k-1}}, R_{-N_{k-1}}) \) contradicts individual rationality. Hence, Property 2 also holds. By \( z_k P_k f_k(R'_{N_{k-1}}, R_{-N_{k-1}}), p_k - V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) > 0 \). Thus, there is \( \epsilon_k > 0 \) such that \( \epsilon_k < \min\{V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) - p_k, p_\bot\} \). Hence, by Lemma 2, there is a \( z_k^{\epsilon_k}\)-favoring \( R'_k \).

Thus, we have constructed a sequence \( (1, R'_1), \ldots, (n, R'_n) \) such that \( a_k \neq 0 \) for all \( k \in N \). This is impossible since \( n > m \), giving us the required contradiction.

\[\blacksquare\]

### 5.2 Proof of Theorem 2

We now fix a desirable rule \( f : \mathcal{R}^n \to Z \), where \( \mathcal{R} \supseteq \mathcal{R}^{++} \). Further, we assume that \( f \) satisfies no bankruptcy, where the corresponding bound as \( \ell \leq 0 \). We start by proving an analogue of Lemma 3.
LEMMA 4 For every preference profile $R \in \mathcal{R}^n$, for every $i \in N$, and every $(a, t) \in M \times \mathbb{R}_+$ with $a = a_i(R)$, if there exists $j \neq i$ such that for each $b \in L \setminus \{a\}$,

$$V^{R_i}(b, (a, t)) < -n \left( \max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0)) \right) + \ell,$$

then $t_i(R) > t$.

Proof: Assume for contradiction $t_i(R) \leq t$. Consider $R'_i = R_j$. By strategy-proofness, $f_i(R'_i, R_{-i})$ $f_i(R, (a, t_i(R)))$. By equal treatment of equals,

$$f_j(R'_i, R_{-i}) I_j f_i(R'_i, R_{-i}) R_j (a, t_i(R)).$$

Note that either $a_i(R'_i, R_{-i}) \neq a$ or $a_j(R'_i, R_{-i}) \neq a$. Without loss of generality, assume that $a_j(R'_i, R_{-i}) = b \neq a$. Then, using the fact that $(b, t_j(R'_i, R_{-i})) R_j (a, t_i(R))$ and $t_i(R) \leq t$, we get

$$t_j(R'_i, R_{-i}) \leq V^{R_j}(b, (a, t_i(R))) \leq V^{R_j}(b, (a, t)) \leq -n \left( \max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0)) \right) + \ell.$$

By individual rationality

$$t_i(R'_i, R_{-i}) \leq V^{R_i}(a_i(R'_i, R_{-i}), (0, 0)) \leq \max_{c \in M} V^{R'_i}(c, (0, 0)).$$

Further, individual rationality also implies that for all $k \notin \{i, j\}$,

$$t_k(R'_i, R_{-i}) \leq V^{R_k}(a_k(R'_i, R_{-i}), (0, 0)) \leq \max_{c \in M} V^{R_k}(c, (0, 0)).$$

Adding these three sets of inequalities above, we get

$$\sum_{k \in N} t_k(R'_i, R_{-i}) \leq -n \left( \max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0)) \right) + \ell + \max_{c \in M} V^{R'_i}(c, (0, 0)) + \sum_{k \in N \setminus \{i, j\}} \max_{c \in M} V^{R_k}(c, (0, 0)) \leq -n \left( \max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0)) \right) + \ell + \left( n - 1 \right) \left( \max_{k \in N \setminus \{i, j\}} \max_{c \in M} V^{R_k}(c, (0, 0)) \right) + \ell \leq \ell.$$

This contradicts no bankruptcy.
Using Lemma 4, we can mimic the proof of Theorem 1 to complete the proof of Theorem 2. We start by defining a class of positive income effect preferences by strengthening the notion of \((a,t)\epsilon\)-favoring preference. For every \((a,t)\in M \times \mathbb{R}_+\), for each \(\epsilon > 0\), and for each \(\delta > 0\), let \(\mathcal{R}((a,t),\epsilon,\delta)\) be the set of preferences such that for each \(\hat{R}_i \in \mathcal{R}((a,t),\epsilon,\delta)\), the following holds:

1. \(\hat{R}_i\) is \((a,t)\epsilon\)-favoring and
2. \(V^{\hat{R}_i}(b,(a,t)) < -\delta\) for all \(b \neq a\).

A graphical illustration of \(\hat{R}_i\) is provided in Figure 4. Since \(\delta > 0\), it is clear that a \(\hat{R}_i\) can be constructed in \(\mathcal{R}((a,t),\epsilon,\delta)\) such that it exhibits positive income effect. Hence, \(\mathcal{R}^+ \cap \mathcal{R}((a,t),\epsilon,\delta) \neq \emptyset\).

![Figure 4: Illustration of \(\hat{R}_i\)](image)

**Proof of Theorem 2**

**Proof:** Now, we can mimic the proof of Theorem 1. We only show parts of the proof that requires some change. As in the proof of Theorem 1, by Lemma 1, we only need to show that for every profile of preferences \(R \in \mathcal{R}^n\) and for every \(i \in N\), \(f_{i}^{\text{min}}(R) R_i f(R)\), where \(f_{i}^{\text{min}}\) is an MWEP rule. Assume for contradiction that there is some profile of preferences \(R \in \mathcal{R}^n\) and some agent \(i \in N\) such that \(z_i P_i f_i(R)\), where \((z_1, \ldots, z_n) \equiv f_{\text{min}}(R)\) be the allocation.
chosen by an MWEP rule at \( R \). Let \( p \equiv \min_{a \in M} p^{\text{min}}_a(R) \). For simplicity of notation, we will denote \( z_j \equiv (a_j, p_j) \), where \( p_j \equiv p^{\text{min}}_{a_j}(R) \), for all \( j \in N \).

Define \( \tilde{\delta} > 0 \) as follows:

\[
\tilde{\delta} := n \left( \max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0)) \right) - \ell.
\]

We first construct a finite sequence of agents and preferences, without loss of generality \((1, R'_1), \ldots, (n, R'_n)\), satisfying certain properties. Let \( N_0 \equiv \emptyset \), \( N_k \equiv \{1, \ldots, k\} \) for each \( k \geq 1 \), and \((R'_{N_0}, R_{-N_0}) \equiv R\). This sequence satisfies the properties that for every \( k \in \{1, \ldots, n\} \),

1. \( z_k \overset{P_k}{\sim} f_k(R'_{N_{k-1}}, R_{-N_{k-1}}) \) for each \( k \geq 1 \),

2. \( a_k \neq 0 \),

3. \( R'_k \in \mathcal{R}^+ \cap \mathcal{R}(z_k, \epsilon, \tilde{\delta}) \) for some \( \epsilon_k > 0 \) with \( \epsilon_k < \min \{V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) - p_k, p\} \).

Now, we can complete the construction of this sequence inductively as in the proof of Theorem 1 (using Lemma 4 instead of Lemma 3), giving us the desired contradiction.

6 Relation to the literature

Ever since the work of Myerson (1981), various extensions of his work to multi-object case have been attempted in quasilinear domain. Many of these extensions focus on the single agent (or, screening problem of a monopolist) with additive valuations (value for a bundle of objects is the sum of values of objects) and our model is different in that respect. Further, ours is an axiomatic analysis and these papers mostly rely on convex analysis - for a recent treatment, interested readers may consider Manelli and Vincent (2007); Daskalakis et al. (2016); Hart and Reny (2015), and references therein. The complexity of revenue maximization in our model is apparent in the work of Thirumulanathan et al. (2017), who show that the analysis becomes intractable even for one buyer (with unit demand) and two objects (with uniformly distributed values). Finally, it is not clear if revenue maximization using Myersonian techniques is feasible if preferences are not quasilinear. In a companion paper (Kazumura et al., 2017), we investigate mechanism design without quasilinearity more abstractly and illustrate the difficulty of solving the single object optimal mechanism design problem. Hence, solving for full optimality without imposing the additional axioms that we put seems to be even more challenging in our model.

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Our work can be connected to a result by Bulow and Klemperer (1996) and its extension by Roughgarden et al. (2015). In Bulow and Klemperer (1996), it was shown that (under standard independent and identical agent assumption with regular distribution) a single object optimal mechanism (with quasilinear preferences) for \( n \) agents generates less expected revenue than a single object Vickrey rule for \((n+1)\) agents. Hence, if the cost of recruiting an agent is small, then the Vickrey rule can be recommended. \(^{20}\) This result has been extended to our multi-object unit-demand agent setting with quasilinear preferences: the expected revenue maximizing mechanism for \( n \) agents generates less expected revenue than the VCG rule for \((m+n)\) agents, where \( m \) is the number of objects (Roughgarden et al., 2015) - note that an MWEP rule is the VCG rule in the quasilinear domain. Our results complement these results by establishing an axiomatic revenue maximizing foundation of the MWEP rule (even when preferences are not quasilinear). \(^{21}\)

There is a short but important literature on object allocation problem with non-quasilinear preferences. Baisa (2016a) considers the single object model and allows for randomization with non-quasilinear preferences. He introduces a novel rule in his setting and studies its optimality properties (in terms of revenue maximization). We do not consider randomization and our solution concept is different from his. Further, ours is a model with multiple objects.

The literature with non-quasilinear preferences and multiple objects have traditionally looked at Pareto efficient rules. As discussed earlier, the closest paper is Morimoto and Serizawa (2015) who consider the same model as ours. They characterize the MWEP rule using Pareto efficiency, individual rationality, incentive compatibility, and no subsidy when the domain includes all classical preferences - see an extension of this characterization in a smaller domain in Zhou and Serizawa (2018). Similar characterizations are also available for other settings: Sakai (2008, 2013b,a) provide such characterizations in the single object auction model; Saitoh and Serizawa (2008); Ashlagi and Serizawa (2012); Adachi (2014) in the homogeneous object auction model with unit demand preferences. Pareto efficiency and the complete class of classical preferences play a critical role in pinning down the MWEP rule.

\(^{20}\) Of course, one can argue that if we have \((n+1)\) agents, then the seller must use the expected revenue maximizing rule for \((n+1)\) agents. The main point in Bulow and Klemperer (1996) is that the Vickrey rule is a prior-free robust rule, whereas the expected revenue maximizing mechanism requires knowledge of priors.

\(^{21}\) The computer science literature is interested in such prior-free bounds on optimal multidimensional rules (which is hard to compute) - a recent paper by Eden et al. (2017) provide further extensions of Bulow-Klemperer results in multi-object environments where buyers can consume more than one object but have additive valuations.
in these papers. As Tierney (2016) points out, even in the quasilinear domain of preferences, there are desirable rules satisfying no subsidy which are different from the MWEP rule. By imposing revenue maximization as an objective instead of Pareto efficiency, we get the MWEP rule in our model. Pareto efficiency is obtained as an implication (Corollaries 3 and 4). Finally, our results work for not only the complete class of classical preferences, but for a large variety of domains, such as the class of all quasilinear preferences, one including all non-quasilinear preferences, one including all preferences exhibiting positive income effects, etc.

Tierney (2016) considers axioms like no discrimination, welfare continuity, and some stronger form of strategy-proofness to give various characterizations of the MWEP rule with reserve prices in the quasilinear domain. Using our result, he shows that in the quasilinear domain, the MWEP rule is the unique rule satisfying strategy-proofness, no-discrimination, individual rationality, no wastage, and welfare continuity.

When the set of preferences include all or a very rich class of non-quasilinear preferences strategy-proofness and Pareto efficiency (along with other axioms) have been shown to be incompatible if the unit demand assumption is violated - (Kazumura and Serizawa, 2016) show this for multi-object allocation problems where agents can be allocated more than one object; (Baisa, 2016b) shows this for homogeneous object allocation problems. Hence, it is not clear how our result can be extended to such models.

7 Conclusion

We challenge the design of revenue maximizing mechanisms in multi-object allocation problem. Since governments are expected to pursue several goals other than revenue maximization in public auctions, we characterize mechanisms that maximize revenue while satisfying desiderata relating to such goals. As a result, we provide robust recommendations on revenue maximizing mechanisms: every MWEP rule is revenue-maximal profile-by-profile, and the preferences of agents need not be quasilinear. Our proofs are elementary and without any convex analysis techniques used in the literature.
REFERENCES


A Supplementary Appendix

A.1 A non-MWEP desirable rule

In this appendix, we reproduce an example of a desirable rule for quasilinear preferences from Tierney (2016). This example demonstrates that there is a desirable rule satisfying no subsidy on the quasilinear domain that is not an MWEP rule. It also illustrates that the space of desirable rules satisfying no subsidy may be complex to describe.

The example has three objects: $M := \{a, b, c\}$ and requires the following four quasilinear preferences. To remind, a quasilinear preference $R_i$ of agent $i$ can be described by a valuation function $v_i : M \rightarrow \mathbb{R}_+$. Hence, we report the valuation functions of these four preferences in Table 1 to describe the respective preferences. Denote the quasilinear preference corresponding to valuation functions $v^\alpha, v^\beta, v^\gamma, v^\lambda$ as $R^\alpha, R^\beta, R^\gamma, R^\lambda$ respectively.

<table>
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<td>$v^\lambda$</td>
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Table 1: Four quasilinear preferences - $\epsilon > 0$ but arbitrarily close to zero

The example has five agents: $N := \{1, 2, 3, 4, 5\}$. The rule we describe works in the class of all quasilinear preferences, and we denote this domain by $Q$. For any $i \in N$, we say a profile of preferences $R \equiv (R_1, \ldots, R_5) \in Q^5$ is **special** for agent $i$ if there exists a bijective map

$$\rho : (N \setminus \{i\}) \rightarrow \{\alpha, \beta, \gamma, \lambda\}$$

such that for each $j \in (N \setminus \{i\})$, $R_j = R^{\rho(j)}$. We say a preference profile $R$ is **special** if there is some agent $i$ such that $R$ is special for $i$.

Before describing the rule, we make a comment about special preference profiles.

**Claim 2** For every special preference profile $R \in Q^5$, $p^\text{min}_a(R) = p^\text{min}_b(R) = p^\text{min}_c(R) = 2$. 

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**Proof:** Suppose $R$ is special for agent $i$. Let $p^*$ be the price vector: $p^*_a = p^*_b = p^*_c = 2$. Then for all $j \neq i$, $0 \in D(R_j, p^*)$ and for each $x \in M$, there is $j \neq i$ such that $x \in D(R_j, p^*)$. These properties ensure that $p^*$ is a Walrasian equilibrium price vector at $R$. To see that it is the minimum Walrasian equilibrium price vector at $R$, assume for contradiction $p < p^*$ is the minimum Walrasian equilibrium price vector. If price of at least two objects are less than 2 in $p$, then $0 \notin D(R_j, p)$ for all $j \neq i$. This is impossible since at any Walrasian equilibrium, at least two agents must be allocated the null object. So, assume without loss of generality, $p_a = p_b = 2$ and $p_c < 2$. But then, $|\{j \in N \setminus \{i\} : \{c\} = D(R_j, p)\} \geq 3$, which contradicts the fact that $p$ is a Walrasian equilibrium price vector. Hence, $p^* = p^{\text{min}}(R)$. \hfill $\blacksquare$

Now, the rule $f^* : Q^5 \to Z$ is defined as follows. Let $p$ be a price vector with $p_a = p_b = p_c = 1$. For every preference profile $R \in Q^5$ and for every $i \in N$, let $f^*_i(R) \equiv (a^*_i(R), p^*_i(R))$ be such that

$$a^*_i(R) \in \begin{cases} D(R_i, p) & \text{if } R \text{ is special for } i, \\ D(R_i, p^{\text{min}}(R)) & \text{otherwise}, \end{cases}$$

$$p^*_i(R) = \begin{cases} p_{a^*_i(R)} & \text{if } R \text{ is special for } i, \\ p^{\text{min}}_{a^*_i(R)}(R) & \text{otherwise}. \end{cases}$$

Further, $f^*$ must allocate all the objects at $R$, i.e., for every $x \in M$, there exists $i \in N$ such that $a^*_i(R) = x$.

A clarification regarding the feasibility of $f^*$ is in order. It is not clear that $a^*(R)$ is an object allocation. If $R$ is not special, then by the definition of Walrasian equilibrium, a feasible object allocation can be chosen by $a^*(R)$ such that all the objects are allocated. If $R$ is special, then it is special for either (a) one agent or (b) for two agents. We consider both the cases. Note here that by Claim 2, $p^{\text{min}}(R) \equiv (2, 2, 2)$.

**Case 1.** If it is special for some agent $i$ only, then agent $i$ can be assigned any object in $D(R_i, p)$. Since each agent $j \neq i$ has $0 \notin D(R_j, p^{\text{min}}(R))$ (due to Claim 2), $a^*(R)$ can be chosen as a feasible object allocation. Moreover, since for each $S \subseteq M$, $|S| \leq |\{j \neq i : D(R_j, p^{\text{min}}(R)) \cap S \neq \emptyset\}|$, Hall’s marriage theorem implies that $a^*(R)$ can allocate objects in $M \setminus \{a^*_i(R)\}$ to agents in $N \setminus \{i\}$. This implies that $a^*(R)$ can be constructed such that all the objects are assigned at $R$.

**Case 2.** If it is special for two agents $\{i, j\}$, then $R_i = R_j \in \{R^a, R^b, R^c, R^\lambda\}$. In that case, by the definition of $p$, $0 \notin D(R_i, p) = D(R_j, p)$ and $|D(R_i, p)| = |D(R_j, p)| \geq 2$. Hence, we
can assign \( a^*_i(R) \in D(R_i, p) \) and \( a^*_j(R) \in D(R_j, p) \) such that \( a^*_i(R) \neq a^*_j(R) \). Notice that \( a^*_i(R), a^*_j(R) \in M \). Without loss of generality assume that \( a^*_i(R) = a, a^*_j(R) = b \). Note that there is some \( k \notin \{i, j\} \) such that \( c \in D(R_k, p_{\min}(R)) \). Hence, \( a^*(R) \) can be constructed such that all the objects are assigned at \( R \). Also, \( 0 \in D(R_k, p_{\min}(R)) \) for all \( k \notin \{i, j\} \) (due to Claim 2). As a result, \( a^*(R) \) can be constructed as a feasible object allocation.

In principle, \( f^* \) is not defined uniquely since \( a^*(R) \) can be chosen in various ways at some \( R \) by breaking the ties in the demand sets differently. Here, we refer to \( f^* \) as any one such selection of object allocation. Our next claim argues that \( f^* \) is strategy-proof.

**Claim 3** \( f^* \) is strategy-proof.

**Proof:** Fix \( R \in Q^5 \) and \( i \in N \). If \( R \) is not special for \( i \), then by changing his preference to \( R_i', (R_i', R_{-i}) \) is not special for \( i \). In both the preference profiles, we pick the respective minimum Walrasian equilibrium allocation, and by Demange and Gale (1985), \( i \) cannot manipulate to \( R_i' \).

If \( R \) is special for \( i \), then by changing his preference to \( R_i', (R_i', R_{-i}) \) is also special for \( i \). Hence, \( a^*_i(R) \in D(R_i, p) \) and \( a^*_i(R_i', R_{-i}) \in D(R_i', p) \). Clearly, agent \( i \) cannot manipulate to \( R_i' \).

Since \( f^* \) does not discriminate between agents, it satisfies equal treatment of equals. By construction, it satisfies no subsidy and ex-post individual rationality. It also allocates all the object at every profile of preferences, and hence, satisfies no wastage. As a result, \( f^* \) is a desirable rule satisfying no subsidy in the domain of preferences \( Q^5 \). However, if \( R \) is a special preference profile, revenue from \( f^* \) at \( R \) can be lower than the revenue from the MWEP rule. In particular, if \( R \) is special for \( i \) and \( i \) is assigned a (real) object in \( f^* \), then the payment of \( i \) in \( f^* \) is strictly lower than the corresponding payment in the MWEP rule. Thus, \( f^* \) is not an MWEP rule.

### A.2 Necessity of our axioms

In this section, we give some examples to illustrate the implications of our axioms on the result.

**Notion of incentive compatibility and IR.** Consider a rule that chooses the maximum Walrasian equilibrium allocation at every profile. Such a rule will satisfy no subsidy and all the properties of desirability except strategy-proofness. Similarly, an MWEP rule
supplemented by a participation fee satisfies no subsidy and all the properties of desirability except ex-post IR. Both these rules generate more revenue than an MWEP rule. Hence, strategy-proofness and ex-post IR are necessary for our results to hold.

What is less clear is if we can relax the notion of incentive compatibility to Bayesian incentive compatibility in our results. For this, consider an example with a single object and quasilinear preferences. With symmetric agents (i.e., agents having independent and identical distribution of values), a symmetric Bayesian Nash equilibrium strategy of the first price auction is increasing and continuous function $b(\cdot)$ of valuations - for an exact expression of this function, see Krishna (2009). Consider the rule such that for each valuation profile $v = (v_1, \ldots, v_n)$, the outcome of the bid profile $(b(v_1), \ldots, b(v_n))$ of the first price auction is chosen. Call this rule the first-price based rule. It is Bayesian incentive compatible. Though, the first-price based rule satisfies no subsidy, ex-post individual rationality, and no wastage, it fails to satisfy ETE (unless, we break ties using uniform randomization). To see this, if two agents have same value, they bid the same amount in the first-price based rule. If there is no randomization to break ties, only one of those agents wins the object at his bid amount, whereas the other agent gets zero payoff. Since bid amount is less than the value in the first-price based rule, the winner gets positive payoff, and this violates ETE.

However, this can be rectified in two ways. First, whenever there is tie for the winning bid, all the winning agents get the object with equal probability. This introduces uniform randomization, and ETE is now satisfied. Hence, the randomized first-price based rule is Bayesian incentive compatible, satisfies ex-post IR, ETE, no wastage, and no subsidy. Obviously, there are profiles of values where such a first-price based rule generates more revenue than the Vickrey rule - winning bid in the first-price auction may be higher than the second highest value.\(^\text{22}\)

An alternate approach to restoring ETE in the first-price based rule is to modify it in a deterministic manner whenever there is a tie in the winning bids. Consider a profile of values $(v_1, \ldots, v_n)$ such that more than one agent has bid the highest amount, say, $B$. Note that this bid $B$ corresponds to value $b^{-1}(B)$. In such a case, we break the winning agent tie deterministically by giving the object (with probability 1) to one of the winning agents. Further, we ask him to pay his value $b^{-1}(B)$. This ensures that the winner and

\(^{22}\text{It is well known that the expected revenue from both the auctions is the same. Also, as we discussed earlier, interim equivalence of strategy-proof and Bayesian incentive compatible rules are known for single object quasilinear models.}\)
the losing agents all get a payoff of zero, and thus, it restores ETE. More formally, the rule corresponding to this modified first-price based rule is the following.

1. Agents submit their values \((v_1, \ldots, v_n)\).

2. If there is a unique highest valued agent \(i\), he is given the object and he pays \(b(v_i)\), where \(b\) is the unique symmetric Bayesian equilibrium bidding function of the first-price auction.

3. If there are more than one highest valued agents, then \text{any} one of them is given the object and is asked to pay his value.

Notice that this only modifies the rule corresponding to the first-price auction at zero measure profiles of values. Hence, the modified first-price based rule is Bayesian incentive compatible. Further, it is deterministic, satisfies ETE, no wastage, no subsidy, and ex-post IR. Because of the same reasons given for first-price auction, there are profiles of values where such a modified first-price based rule generates more revenue than the Vickrey rule.

This illustrates that we cannot relax strategy-proofness to Bayesian incentive compatibility in our results.

No wastage. It is easy to see that no wastage is required for our result - in the quasilinear domain of preferences with one object, Myerson (1981) shows that the Vickrey rule with an optimally chosen reserve price maximizes expected revenue for independent and identically distributed values of agents. Such a rule wastes the object and generates more revenue than the Vickrey rule, which is also an MWEP rule, at some profiles of preferences.

No wastage is also necessary in a more indirect manner. Consider the domain of quasilinear preferences with two objects \(M \equiv \{a, b\}\) and \(N = \{1, 2, 3\}\). We show that the seller may increase her revenue by \textit{not} selling all the objects. Consider a profile of valuations as follows:

\[
\begin{align*}
v_1(a) &= v_1(b) = 5 \\
v_2(a) &= v_2(b) = 4 \\
v_3(a) &= v_3(b) = 1.
\end{align*}
\]

The MWEP price at this profile is \(p^m_{a} = p^m_{b} = 1\), which generates a revenue of 2 to the seller. On the other hand, suppose the seller conducts a Vickrey rule of object \(a\) only. Then,
he generates a revenue of 4. Hence, the seller can increase her revenue at some profiles of valuations by withholding objects. Notice that withholding objects is a stronger violation of efficiency, and is easier to detect than misallocating the objects among agents.

In allocating public assets, governments are supposed to pursue several goals such as revenue and efficiency. Usually, revenue and efficiency are not compatible. No wastage is a mild requirement on efficiency and our result shows how revenue maximization can be reconciled with efficiency using no wastage.

**Equal treatment of equals.** Consider an example with one object and two agents in the quasilinear domain of preferences. Hence, the preference of each agent \(i \in \{1, 2\}\) can be described by his valuation for the object \(v_i\). Note that every MWEP rule collapses to the Vickrey rule for this problem.

We define the following rule: the object is first offered to agent 1 at price \(p > 0\); if agent 1 accepts the offer, then he gets the object at price \(p\) and agent 2 does not get anything and does not pay anything; else, agent 2 is given the object for free.

This rule generates a revenue of \(p\) whenever \(v_1 \geq p\) (but generates zero revenue otherwise). However, note that the Vickrey rule generates a revenue of \(v_2\) when \(v_1 > v_2\). Hence, if \(p > v_2\), then this rule generates more revenue than the Vickrey rule. Also, notice that this rule satisfies no subsidy and all the properties of desirability except equal treatment of equals.

**No subsidy.** It is tempting to conjecture that no subsidy can be relaxed in quasilinear domain of preferences. A natural approach to prove this is to use Theorem 1, which applies to the quasilinear domain, in the following way: (1) For every desirable rule, we construct another desirable rule which satisfies no subsidy and generates more revenue; (2) Use Theorem 1 to arrive at the conclusion that the MWEP rule is revenue-optimal in the class of desirable rule. The first step does not quite work. In the quasilinear domain, every desirable rule can be converted to a strategy-proof, individually rational, and no subsidy rule using “multidimensional” versions of revenue equivalence formula (Chung and Olszewski, 2007). But such a transformation may not preserve equal treatment of equals. As a result, we cannot apply Step (2) any more. We now give a concrete example to illustrate that our result does not hold without no subsidy.
For the example, consider one object and two agents in the quasilinear domain - hence, preferences of agents can be represented by their valuations \( v_1 \) and \( v_2 \). Further, assume that valuations lie in \( \mathbb{R}_{++} \). Choose \( k \in (0, 1) \) and define the rule \( f \equiv (a, t) \) as follows: for every \((v_1, v_2)\)

\[
a(v_1, v_2) = \begin{cases} 
(1, 0) & \text{if } kv_1 > v_2, \\
(0, 1) & \text{otherwise},
\end{cases}
\]

\[
t_1(v_1, v_2) = \begin{cases} 
-(v_2 - kv_2) & \text{if } a_1(v_1, v_2) = 0, \\
\frac{v_2}{k} - (v_2 - kv_2) & \text{if } a_1(v_1, v_2) = 1,
\end{cases}
\]

\[
t_2(v_1, v_2) = \begin{cases} 
0 & \text{if } a_2(v_1, v_2) = 0, \\
kv_1 & \text{if } a_2(v_1, v_2) = 1.
\end{cases}
\]

It is straightforward to check that the object allocation rule \( a \) is monotone (i.e., fixing the valuation of one agent, if valuation of the other agent is increased, his allocation probability increases) and payments satisfy the revenue equivalence formula, and hence, the rule is strategy-proof (a more direct proof is also possible). It is also not difficult to see that utilities of the agents are always non-negative, and hence, individual rationality holds. Finally, if \( v_1 = v_2 \), we have

\[
a_1(v_1, v_2) = 0, a_2(v_1, v_2) = 1, \quad t_1(v_1, v_2) = -(v_2 - kv_2), t_2(v_1, v_2) = kv_1.
\]

Hence, net utility of agent 1 is \( v_2 - kv_2 \) and that of agent 2 is \( v_1 - kv_1 \), which are equal since \( v_1 = v_2 \). This shows that the rule satisfies equal treatment of equals.

However, the rule pays agent 1 when he does not get the object. Thus, it violates no subsidy. The revenue from this rule when \( kv_1 > v_2 \) is

\[
v_2 \left( \frac{1}{k} + k - 1 \right) \geq v_2.
\]

The Vickrey rule generates a revenue of \( v_2 \) when \( kv_1 > v_2 \). Hence, this rule generates more revenue than the Vickrey rule when \( kv_1 > v_2 \). This shows that we cannot drop no subsidy from Theorem 1.\(^{23}\)

\(^{23}\)Further inspection reveals that the revenue from this rule when \( v_1 = v_2 = v \) is \( kv - v(1 - k) = v(2k - 1) \). So, if \( k < \frac{1}{2} \), this revenue approaches \(-\infty\) as \( v \to \infty \). Hence, this rule even violates no bankruptcy.