Tullock Contests with Asymmetric Information*

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Abstract

We show that under standard assumptions a Tullock contest with asymmetric information has a pure strategy Bayesian Nash equilibrium. Next we study common-value Tullock contests. We show that in equilibrium the expected payoff of a player is greater or equal to that of any other player with less information, i.e., an information advantage is rewarded. Moreover, if there are only two players and one of them has an information advantage, then in the unique equilibrium both players exert the same expected effort, although the less informed player wins the prize more frequently. These latter properties do not extend to contests with more than two players. Interestingly, players may exert more effort in a Tullock contest than in an all-pay auction.

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1 Introduction

In a Tullock contest – see Tullock (1980) – each player’s probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all players. Baye and Hoppe (2003) have identified a variety of economic settings (rent-seeking, innovation tournaments, patent races) which are strategically equivalent to a Tullock contest. Tullock contests also arise by design, e.g., in sport competition, internal labor markets. A number of studies have provided an axiomatic justification to such contests, see, e.g., Skaperdas (1996) and Clark and Riis (1998)).


In our setting, each player’s value for the prize as well as his cost of effort depend on the state of nature. The set of states of nature is finite. Players have a common prior belief, but upon the realization of the state of nature, and prior to taking action, each player observes some event that contains the realized state of nature. The information of each player at the moment of taking action is described by a partition of the set of states of nature. A contest is therefore formally described by a set of players, a probability space describing players’ uncertainty and their prior belief, a collection of partitions of the state space describing the players’ information, a collection of state-dependent functions describing the players’ values and costs, and a success function specifying the probability distribution that is used to allocate the prize for each profile of efforts. This representation is equivalent to Harsanyi’s model of Bayesian games that uses players’ types – see Jackson (1993) and Vohra (1999). (In a similar framework, Einy et al. (2001, 2002), Forges and Orzach (2011), and Malueg
and Orzach (2009, 2012) study common-value first- and second-price auctions.)

We show that if players’ cost functions are continuously differentiable, strictly increasing and convex with respect to effort, then a Tullock contest has a pure strategy Bayesian Nash equilibrium. Our existence result applies regardless of whether players have private or common values, or whether their costs of effort is the same or different, and makes no assumptions about the players’ private information. Moreover, our result extends to a general class of Tullock-like contests in which success function is given by the ratio between the score given to a player’s effort and the total scores given to all players, provided each player’s score function is strictly increasing and concave.

Our proof of existence of equilibrium involves constructing a sequence of equilibria of contests obtained from the original Tullock contest by truncating the action space so that it is a closed and bounded interval, whose lower bound approaches zero from above. We show that any limit point of a sequence of equilibria of these contests, which have an equilibrium by Nash’s (existence) theorem, is an equilibrium of the original Tullock contest. A key step in the proof is to show that in any such limit point the total effort exerted by players is positive in every state of nature. To our knowledge, there are no previous results in the literature on Tullock contests establishing existence of equilibrium by appealing to Nash’s theorem.

Next we study Tullock contests in which players have a common value for the prize and a common state independent linear cost function, to which we refer simply as common-value Tullock contests. We show that in any equilibrium of a common-value Tullock contests, the payoff of a player is greater or equal to that of any other player with less information. Thus, a common-value Tullock contests rewards any information advantage. This result, which is a direct implication of a theorem of Einy, Moreno and Shitovitz (2002) showing that in any Bayesian Cournot equilibrium of an oligopolistic industry a firm’s information advantage is rewarded, is established by observing the formal equivalence between a common-value Tullock contest and an oligopoly with asymmetric information.

We then proceed to study other properties of the equilibria of common-value

\footnote{The payoff functions in the truncated contests are continuous and concave in players’ own strategies, which allows the use of the Nash’s theorem. However, it cannot be applied to the original, untruncated, contest, in which payoffs have a discontinuity when all efforts are equal to zero.}
Tullock contests. We show that a two-player contest in which one of the players has an information advantage over his opponent (i.e., the partition of one player is finer than that of his opponent) has a unique (pure strategy) Bayesian Nash equilibrium, which we characterize.\(^2\) In equilibrium both players exert the same expected effort, although the player with less information wins the prize more frequently. We also examine how players information affects the effort they exert and their payoffs. Assuming that the distribution of the players’ value for the prize is not too disperse, we show that when one player is better informed than the other the total effort exerted by the players is smaller, and thus the share of the total surplus they capture is larger, than when both players have the same information. However, these properties of equilibrium of two-player contests do not extend to contests with more than two players. Specifically, we construct a three-player contest in which two of the players have symmetric information, which is superior to that of the third player, in which the expected efforts exerted by players differ. We also provide an example of a contest in which a player that has an information advantage over all other players, who have symmetric information, wins the prize with a greater ex-ante probability than that of any other player.

Our results for two-player common-value Tullock contests are akin to those established by Warneryd (2003) in a setting where players’ common value is a continuous random variable, assuming that one player observes the value precisely while the other player does not observe anything. In our framework, our results apply whenever one player’s information partition is finer than that of his opponent, i.e., whenever the players’ information endowments can be ranked. It is unclear whether this is also the case when players’ common value is a continuous random variable.

Finally, we study the relative effectiveness of Tullock contests and all-pay auctions in inducing the players to exert effort. Einy et al. (2013) characterize the unique equilibrium of a two-player common-value all-pay auction, which is in mixed strategies, and provide an explicit formula that allows to compute the players’ total effort. Using the results in Einy et al. (2013) and our results we show that the sign of the difference in the total effort exerted by players in a Tullock contest and an all-pay

\(^2\)Warneryd (2012) establishes existence of equilibrium when there are two types of players, and investigates which players are active, i.e., make a positive effort, in equilibrium.
auction is undetermined, and may be either positive or negative depending on the
distribution of the players’ value for the prize. (Fang (2002) and Epstein, Mealem
and Nitzan (2011) study the outcomes of Tullock contests and all-pay auction under
complete information.)

The rest of the paper is organized as follows: in Section 2 we describe the general
setting. In Section 3 we establish that every Tullock contest has a pure strategy
Bayesian Nash equilibrium. In Section 4 we study common-value Tullock contests.
Section 5 studies the relative effectiveness of common-values Tullock contests and all
pay auction in inducing players to exert effort. Long proofs are given in the Appendix.

2 Tullock Contests

A group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize by choosing a
level of effort in $\mathbb{R}_+$. Players’ uncertainty about the state of nature is described by
a probability space $(\Omega, p)$, where $\Omega$ is a finite set and $p$ is a probability distribution
over $\Omega$ describing the players’ common prior belief about the realized state of nature.
W.l.o.g. we assume that $p(\omega) > 0$ for every $\omega \in \Omega$. The private information about
the state of nature of player $i \in N$ is described by a partition $\Pi_i$ of $\Omega$.

The value for the prize of each player $i$ is given by a random variable $V_i : \Omega \to \mathbb{R}_+$, i.e., if $\omega \in \Omega$ is
realized then player $i$’s (“private”) value for the prize is $V_i(\omega)$. The cost of effort of
each player $i \in N$ is described by a function $c_i : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, which is continuously
differentiable, strictly increasing and convex in effort $x_i$, and such that $c_i(\cdot, 0) = 0$ on
$\Omega$.

A contest starts by a move of nature that selects a state $\omega$ from $\Omega$ according
to the distribution $p$. Every player $i \in N$ observes the element $\pi_i(\omega)$ of $\Pi_i$ which
contains $\omega$ – the set of states of nature between which $i$ cannot distinguish given $\omega$.
Then players simultaneously choose their effort levels $(x_1, ..., x_n) \in \mathbb{R}_+^n$. The prize is
awarded in a probabilistic fashion, according to a success function $\rho$, which attributes
to each profile of effort levels $x \in \mathbb{R}_+^n$ a probability distribution $\rho(x)$ in the $n$-simplex
according to which the prize recipient is chosen. Hence, the payoff of player $i \in N$, $u_i : \Omega \times \mathbb{R}_+^n \to \mathbb{R}$, is given for every $\omega \in \Omega$ and $x \in \mathbb{R}_+^n$ by

\[ u_i(\omega, x) = \rho_i(x) V_i(\omega) - c_i(\omega, x_i). \]
Thus, a contest is described by a collection \((N, (\Omega, p), \{\Pi_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho)\).

In a contest, a pure strategy of player \(i \in N\) is a \(\Pi_i\)-measurable function \(X_i : \Omega \to \mathbb{R}_+\) (i.e., \(X_i\) is constant on every element of \(\Pi_i\)), that represents \(i\)'s choice of effort in each state of nature following the observation of his private information. We denote by \(S_i\) the set of strategies of player \(i\), and by \(S = \times_{i=1}^n S_i\) the set of strategy profiles. For any strategy \(X_i \in S_i\) and \(\pi_i \in \Pi_i\), \(X_i(\pi_i)\) stands for the constant value that \(X_i(\cdot)\) takes on \(\pi_i\). Also, given a strategy profile \(X = (X_1, ..., X_n) \in S\), we denote by \(X_{-i}\) the profile obtained from \(X\) by suppressing the strategy of player \(i \in N\). Throughout the paper we restrict attention to pure strategies.

Let \(X = (X_1, ..., X_n)\) be a strategy profile. We denote by \(U_i(X)\) the expected payoff of player \(i\), which is given by

\[
U_i(X) \equiv E[u_i(\cdot, (X_1(\cdot), ..., X_n(\cdot))).
\]

For \(\pi_i \in \Pi_i\), we denote by \(U_i(X | \pi_i)\) the expected payoff of player \(i\) conditional on \(\pi_i\), i.e.,

\[
U_i(X | \pi_i) \equiv E[u_i(\cdot, (X_1(\cdot), ..., X_n(\cdot)) | \pi_i).
\]

An \(N\)-tuple of strategies \(X^* = (X_1^*, ..., X_N^*)\) is a Bayesian Nash equilibrium if for every player \(i \in N\), and every strategy \(X_i \in S_i\)

\[
U_i(X^*) \geq U_i(X_{-i}^*, X_i); \tag{2}
\]

or equivalently,

\[
U_i(X^* | \pi_i) \geq U_i(X_{-i}^*, X_i | \pi_i) \tag{3}
\]

for every \(\pi_i \in \Pi_i\).

## 3 Existence of Equilibrium in Tullock Contests

Tullock contests are identified by a class of success functions \(\rho^T\) such that for \(x \in \mathbb{R}_+^n \setminus \{0\}\) the probability that player \(i \in N\) wins the prize is

\[
\rho_i^T(x) = \frac{x_i}{\bar{x}}, \tag{4}
\]

where \(\bar{x} \equiv \sum_{k=1}^N x_k\) is the total effort exerted by the players. Theorem 1 establishes existence of equilibrium in Tullock contests.
Theorem 1. Every Tullock contest has a (pure strategy) Bayesian Nash equilibrium.

Theorem 1 makes no assumptions about players’ private information, and applies regardless of whether players have private or common values, or whether their costs of effort are the same or different. Theorem 1 also implies existence of a Bayesian Nash equilibrium in a Tullock contest in the Harsanyi types model, where each player’s uncertain type represents his private information, and players have a common prior distribution over all possible realizations of types. (In the discrete case, these two models of incomplete information games are equivalent – see Jackson (1993) and Vohra (1999).)

The proof of Theorem 1, given in the Appendix, relies on Nash’s theorem. Nash’s theorem, however, cannot be applied directly to a Tullock contest because the expected payoff functions have a discontinuity when all efforts are equal to zero in some state of nature. “Truncated” contests, where players are forced to choose efforts from a compact interval with a positive lower bound, need to be considered instead – then the expected payoff functions are continuous and concave in players’ own strategies, which allows the use of the Nash’s theorem to deduce the existence of equilibrium. However, in order for a limit point of the sequence of equilibria of truncated contests with a lower bound on players’ efforts approaching zero to be an equilibrium in the original contest – the upper hemi-continuity property – the expected payoff functions must be continuous at the limit strategy profile, and this holds provided the total effort in the limit profile is positive in all states of nature. Establishing the latter property is the crux of the proof of Theorem 1.

A direct implication of Theorem 1 is Corollary 1 below, which establishes existence of equilibrium for a general class of success functions. The proof of Corollary 1 proceeds by first showing the existence of equilibrium in “scores,” and then in efforts, which is an argument familiar in the literature studying equilibrium under complete information – see Corchon (2007). Existence of equilibrium in contests with success functions belonging to this class was established for the complete information case by Szidarovszky and Okuguchi (1997). Theorem 1 and Corollary 1 extend their result into the incomplete information setting, employing a noticeably different method of proof.
Corollary 1. Every contest in which the success function $\rho$ is given for $x \in \mathbb{R}_+^n \setminus \{0\}$ and $i \in N$ by

$$\rho_i(x) = \frac{g_i(x_i)}{\sum_{j=1}^n g_j(x_j)},$$

where, for every $j \in N$, his score function $g_j : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing and concave bijection, has a Bayesian Nash equilibrium.

Proof. Let $C = (N, (\Omega, p), \{\Pi_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho)$ be a contest satisfying the assumptions of Corollary 1 for $(g_1, \ldots, g_n)$. The Tullock contest $(N, (\Omega, p), \{\Pi_i\}_{i \in N}, \{V_i\}_{i \in N}, \{\bar{c}_i\}_{i \in N}, \rho^T)$ where $\bar{c}_i(\cdot, \cdot) = c_i(\cdot, g_i^{-1}(\cdot))$ for every $i \in N$ and $\rho^T(0) = \rho(0)$ has an equilibrium $X^* = (X_1^*, \ldots, X_n^*)$ by Theorem 1. It is easy to see that $Y^* = (g_1^{-1} \circ X_1^*, \ldots, g_n^{-1} \circ X_n^*)$ is an equilibrium of $C$. ■

4 Common-Value Tullock Contests

Henceforth we study contests in which players have a common value for the prize and a common state-independent linear cost function, i.e., for all $i \in N$, $V_i = V$, and $c_i(\cdot, x) \equiv x$ on $\Omega$. We refer to these contests as common-value contests, and they are described by a collection $(N, (\Omega, p), (\Pi_i)_{i \in N}, V, \rho)$.

We say that player $i \in N$ has an information advantage over player $j \in N$ if partition $\Pi_i$ is finer than partition $\Pi_j$. Thus, if $i$ has an information advantage over $j$, then $\pi_i(\omega) \subset \pi_j(\omega)$ for every $\omega \in \Omega$, i.e., player $i$ knows the realized state of nature with at least the same precision as player $j$.

We begin by establishing in Theorem 2 a general property of common-value Tullock contests: these contests reward information advantage. This is a direct implication of the theorem of Einy, Moreno and Shitovitz (2002), that shows that information advantage is rewarded in any Bayesian Cournot equilibrium of a symmetric oligopolistic industry in which the firms’ cost function is linear.

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3Functions $g_j$ do not, in fact, need to be bijections, for our claim to hold. This can be shown using the same argument as in the proof below, provided Theorem 1 is extended to hold for contests where the levels of effort are restricted to be in a set $[a, b]$ for $0 \leq a \leq b$ and $b \in \mathbb{R}_+ \cup \{\infty\}$ (under the additional assumption that $\lim_{x \to b} c_i(\cdot, x) = \infty$). This extension of Theorem 1 can be obtained by essentially the same proof as the one given in the Appendix.
Theorem 2. Let $X^* = (X_1^*, ..., X_n^*)$ be any equilibrium of an $n$-player common-value Tullock contest. If player $i$ has an information advantage over player $j$, then $U_i(X^*) \geq U_j(X^*)$.

**Proof.** An $n$-player common-value Tullock contest $(N, (\Omega, p), (\Pi_i)_{i \in N}, V)$ is formally identical to an oligopolist industry $(N, (\Omega, p), P, c, (\Pi_i)_{i \in N})$, where the market demand $P$ and the cost function $c$ are defined for $(\omega, x) \in \Omega \times \mathbb{R}_+$ as

$$P(\omega, x) = \frac{V(\omega)}{x},$$

and

$$c(\omega, x) = x,$$

respectively. With this convention, the state-dependent profit of firm $i \in N$ in the industry coincides with the payoff of player $i \in N$ in the contest, i.e., for $\omega \in \Omega$ and $X \in S$,

$$u_i(\omega, X) = \frac{V(\omega)}{\sum_{s=1}^{n} X_s(\omega) - X_i(\omega)} - \sum_{s=1}^{n} X_s(\omega)X_i(\omega) - c(\omega, X_i(\omega)).$$

Theorem 2 then follows from the theorem of Einy, Moreno and Shitovitz (2002).

Next we study other properties of common-value Tullock contests. Let us index the set of states of nature as

$$\Omega = \{\omega_1, ..., \omega_m\}.$$

For $k = 1, ..., m$, write

$$p(\omega_k) = p_k \text{ and } V(\omega_k) = v_k,$$

and w.l.o.g. assume that

$$0 < v_1 \leq v_2 \leq ... \leq v_m.$$

We begin by studying two-player contests in which player 2 has an information advantage over player 1. Thus, we may assume w.l.o.g. that the only information player

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4 The demand function $P(\omega, x)$ is not differentiable at $x = 0$ – it is not even defined – and therefore does not formally satisfy the assumptions of Einy, Moreno and Shitovitz (2002). However, it is easy to see that in any equilibrium $X$ of a common-value Tullock contest the total effort is positive in all states of nature, i.e., $X(\cdot) > 0$. Thus the non-differentiability at 0 is irrelevant, and the proof of the theorem in Einy, Moreno and Shitovitz (2002) applies in this case with no change.
1 has about the state is the common prior belief, i.e., \( \Pi_1 = \{\Omega\} \), whereas player 2 has perfect information about the state of nature, i.e., \( \Pi_2 = \{\omega_1, \ldots, \omega_m\} \). In such contests a strategy profile is a pair \((X, Y)\), where \( X \) can be identified with a number \( x \in \mathbb{R}_+ \) specifying player 1’s unconditional effort, and \( Y \) can be identified with a vector \((y_1, \ldots, y_m) \in \mathbb{R}_+^m \) specifying the effort of player 2 in each of the \( m \) states of nature. Thus, abusing notation, we shall write \( X = x \) and \( Y = (y_1, \ldots, y_m) \) whenever appropriate.

The following notation will be useful in characterizing the pure strategy Bayesian Nash equilibria of a contest. For \( k \in \{1, \ldots, m\} \) write

\[
A_k = \left( \sum_{s=k}^{m} p_s \sqrt{v_s} \right) \left( 1 + \sum_{s=k}^{m} p_s \right)^{-1}.
\]

(5)

Note that \( A_1 = \frac{E(\sqrt{V})}{2} \).

Lemma 1 establishes a key property of the sequence \( \{A_k\}_{k=1}^m \).

**Lemma 1.** If \( \sqrt{v_k} > A_k \) for some \( \tilde{k} < m \), then \( \sqrt{v_k} > A_{\tilde{k}} \) and \( A_k > A_{\tilde{k}} \) for all \( k > \tilde{k} \).

Proposition 1 shows that a two-player common-value Tullock contest in which player 2 has an information advantage has a unique pure strategy equilibrium, which is calculated explicitly. Let \( k^* \in \{1, \ldots, m\} \) be the smallest index such that \( \sqrt{v_k} > A_k \). Since

\[
\sqrt{v_m} > \frac{p_m}{1 + p_m} \sqrt{v_m} = A_m,
\]

\( k^* \) is well defined.

**Proposition 1.** A two-player common-value Tullock contest in which player 2 has an information advantage has a unique Bayesian Nash equilibrium \((X^*, Y^*)\) given by

\[
x^* = A_{k^*}^2,
\]

and

\[
y^*_k = \begin{cases} 
0 & \text{if } k < k^* \\
A_{k^*} (\sqrt{v_k} - A_{k^*}) & \text{otherwise.}
\end{cases}
\]
Proposition 1 in particular implies uniqueness and symmetry of equilibrium in the complete information case, i.e., when \( m = 1 \). (Note that in this case \( k^* = 1 \), and therefore \( y_1^* = A_1(\sqrt{v_1} - A_1) = v_1/2 - v_1/4 = A_1^2 = x^* \). This result is well known in the literature.) When \( m > 1 \), we have \( \sqrt{v_1} > A_1 = E(\sqrt{V})/2 \) (and hence \( k^* = 1 \)) whenever the distribution of values is not too disperse; e.g., this inequality holds when \( v_m < 4v_1 \). When this is the case, the unique equilibrium is interior. For future references we state this observation in Remark 2.

**Remark 2.** Consider a two-player common-value Tullock contest in which player 2 has an information advantage. The unique Bayesian Nash equilibrium is interior if and only if \( \sqrt{v_1} > E(\sqrt{V})/2 \), i.e., the distribution of values is not too disperse.

Interestingly, when one player has superior information the expected effort exerted by players in the equilibrium of the contest is the same.

**Proposition 2.** In a two-player common-value Tullock contest in which player 2 has an information advantage both players exert the same (expected) effort, i.e.,

\[
E(Y^*) = A_{k^*}^2 = x^* = X^*.
\]

Hence the expected total effort is

\[
TE = X^* + E(Y^*) = 2A_{k^*}^2.
\]

**Proof.** By Proposition 1,

\[
E(Y^*) = \sum_{s=1}^{m} p_s y_s^* = \sum_{s=k^*}^{m} p_s A_{k^*} (\sqrt{v_k} - A_{k^*}) = A_{k^*} \sum_{s=k^*}^{m} p_s \sqrt{v_k} - A_{k^*}^2 \sum_{s=k^*}^{m} p_s = A_{k^*}^2 \left( 1 + \sum_{s=k^*}^{m} p_s \right) - A_{k^*}^2 \sum_{s=k^*}^{m} p_s = A_{k^*}^2.
\]
In a two-player common-value Tullock contest in which player 2 has an information advantage the equilibrium probability that player 1 wins the prize when the state is $\omega_k$ is

$$
\rho^*_1 := \frac{A^*_{k^*} \sqrt{\omega_k}}{A^*_{k^*} \sqrt{\omega_k} - A_{k^*}} \right\}
$$

when $k \geq k^*$, whereas the probability that player 2 wins the prize is $\rho^*_2 = 1 - \rho^*_1$. Thus, the larger is the realized value of the prize, the smaller (larger) is the probability that player 1 (player 2) wins the prize, i.e., $\rho^*_1 \geq \rho^*_2$ and $\rho^*_2 \geq \rho^*_2$ for $k' > k \geq k^*$, with a strict inequality if $v_{k'} > v_k$. Of course, the larger is the realized value of the prize, the larger is the effort of player 2, i.e.,

$$
y^*_k = A_{k^*} \left( \sqrt{v_k} - A_{k^*} \right) \geq A_{k^*} \left( \sqrt{v_k} - A_{k^*} \right) = y^*_k.
$$

for $k' > k \geq k^*$ (with a strict inequality if $v_{k'} > v_k$). Additionally, for $k' > k \geq k^*$,

$$
\rho^*_1 v_{k'} = A_{k^*} \sqrt{v_{k'}} \geq A_{k^*} \sqrt{v_k} = \rho^*_1 v_k
$$

(with a strict inequality if $v_{k'} > v_k$), i.e., the larger is the realized value of the prize, the larger is the conditional expected payoff of player 1; also,

$$
\rho^*_2 v_{k'} \geq \rho^*_2 v_{k'} \geq \rho^*_2 v_k
$$

(with a strict inequality if $v_{k'} > v_k$), i.e., the larger is the realized value of the prize, the larger is the conditional expected payoff of player 2. Write $\overline{\rho}^*_i = E(\rho^*_i)$ for the ex-ante probability that player $i$ wins the prize. Proposition 3 establishes another interesting property of equilibrium.

**Proposition 3.** Consider a two-player common-value Tullock contest in which player 2 has an information advantage. If $v_1 < v_2 < \ldots < v_m$, then the ex-ante probability that player 1 wins the prize is greater than that of player 2, i.e., $\overline{\rho}^*_1 > \overline{\rho}^*_2$.

The next Remark 3 is a corollary of propositions 1 and 2, and its straightforward proof if omitted.

**Remark 3.** A two-player common-value Tullock contest in which players have symmetric information has a unique pure strategy equilibrium, in which each player exerts the same expected effort, equal to $E(V)/4$. 

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The surplus captured by the players in a contest is the difference between the expected (total) surplus $E(V)$ and the expected total effort they exert. In Proposition 4 below we show that asymmetry of information leads, in an interior equilibrium, to exertion of less effort, and therefore to the capture a greater share of surplus, compared to the symmetric information case.

**Proposition 4.** Consider a two-player common-value Tullock contest in which player 2 has an information advantage. If $v_1 < v_m$ and the distribution of values is not too disperse, i.e., $\sqrt{v_1} > E(\sqrt{V})/2$, then the players’ exert less effort and hence capture a greater share of the surplus than when both players have symmetric information.

**Proof.** When player 2 has an information advantage, then $\sqrt{v_1} > E(\sqrt{V})/2$ implies that the equilibrium is interior by Remark 2, and therefore the expected total effort is $TE = 2A_1^2 = \left( E(\sqrt{V}) \right)^2 / 2$ by Proposition 2. When players have symmetric information the expected total effort $\overline{TE}$ is $\overline{TE} = E(V)/2$ by Remark 3. Then $v_1 < v_m$ together with Jensen’s inequality imply

$$\overline{TE} - TE = \frac{E(V)}{2} - \frac{\left( E(\sqrt{V}) \right)^2}{2} > 0. \blacksquare$$

The following example illustrates our findings for two-player contests.

**Example 1.** Let $m = 2$, $p_1 = 1 - p$, $v_1 = 1$, and $v_2 = v$, where $p \in (0,1)$ and $v \in (1, \infty)$. Then $E(V) = 1 - p(1 - v)$, $E(\sqrt{V}) = 1 - p(1 - \sqrt{v})$, $A_1 = E(\sqrt{V})/2$, and $A_2 = p\sqrt{v}/(1 + p)$. If $v < (1 + p)^2/p^2$, then $\sqrt{v_1} = 1 > A_1$ and $k^* = 1$; otherwise $k^* = 2$. In a Tullock contest in which player 2 observes the value but player 1 does not, the unique equilibrium is

$$X^* = A_1^2, \quad Y^* = (A_1 (1 - A_1), A_1 (\sqrt{v} - A_1)),$$

and the total effort is $TE = 2A_1^2 = [1 - p(1 - \sqrt{v})]^2/2$ when $v < (1 + p)^2/p^2$. Otherwise, the unique equilibrium is

$$X^* = A_2^2, \quad Y^* = (0, A_2 (\sqrt{v} - A_2)),$$

and the total effort is $TE = 2A_2^2 = 2p^2v/(1 + p)^2$. If $v < (1 + p)^2/p^2$, then the ex-ante probability that player 1 wins the prize is

$$\tilde{p}_1 = (1 - p) A_1 + p \frac{A_1}{\sqrt{v}} = \frac{1}{2} \left( p + (1 - p)\sqrt{v} \right) \frac{1 - p + p\sqrt{v}}{\sqrt{v}} \geq \frac{1}{1 + p} > \frac{1}{2},$$

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Otherwise, this probability is
\[ \tilde{p}_1^* = (1 - p) + \frac{A_2}{\sqrt{v}} = (1 - p) + \frac{p^2}{1 + p} = 1 + p > \frac{1}{2}. \]
Hence, consistent with Proposition 3 the uninformed player wins the prize more frequently than the informed player. Further, if \( v < (1 + p)^2 / p^2 \), then
\[
2 [U_2(X^*, Y^*) - U_1(X^*, Y^*)] = (1 - p) \frac{A_1 (1 - A_1) - A_1^2}{A_1^2 + A_1 (1 - A_1)} + p^2 \frac{A_1 (\sqrt{v} - A_1) - A_1^2}{A_1^2 + A_1 (\sqrt{v} - A_1)}
\]
\[ = (1 - p) p \left( 1 - \frac{1}{p} \right)^2 \]
\[ > 0. \]
And if \( v \geq (1 + p)^2 / p^2 \), then
\[
2 [U_2(X^*, Y^*) - U_1(X^*, Y^*)] = - (1 - p) + p^2 \frac{A_2 (\sqrt{v} - A_2) - A_2^2}{A_2^2 + A_2 (\sqrt{v} - A_2)}
\]
\[ = \frac{1 - p}{p + 1} (p(v - 1) - 1)
\]
\[ > \frac{1 - p}{p}
\]
\[ > 0. \]
That is, consistent with Theorem 2, the payoff of the informed player is greater or equal to that of the uninformed player. Under symmetric information the equilibrium total effort in a Tullock contest is \( E(V) / 2 > \max \{2A_1^2, 2A_2^2\} \), i.e., the total effort when player 2 has an information advantage is less than when both players have the same information.

Warneryd (2003) establishes counterparts to propositions 1 to 4 when the players’ common-value \( V \) is a continuous random variable, and also shows that the fully informed player obtains a greater payoff than the uninformed player. This latter result also holds when \( V \) is a discrete random variable, according to Theorem 2, and in fact extends to contests with three or more players in a very general way – when a player has an information advantage (not necessarily an extreme one) over some other player, then in any equilibrium the expected payoff of the former is greater or equal to that of the latter.

The following examples show that the properties established in propositions 2 and 3 do not extend to common-value Tullock contests with more than two players.
In Example 2 player 1 has only the prior information whereas players 2 and 3 have complete information. In equilibrium the expected effort of the uninformed player is below that of each of the informed players.

Example 2. Consider a 3-player common-value Tullock contest in which $m = 2$, $p_1 = p_2 = 1/2$, $v_1 = 1$ and $v_2 = 2$. Player 1 has no information, i.e., his information partition is $\Pi_1 = \{\omega_1, \omega_2\}$, and players 2 and 3 have complete information, i.e., their information partitions are $\Pi_2 = \Pi_3 = \{\{\omega_1\}, \{\omega_2\}\}$. In the interior equilibrium of this contest, which is readily calculated by solving the system of equations formed by the players’ reaction functions, the effort of player 1 is $X_1^* = 0.30899$ while the efforts of players 2 and 3 are $X_2^* = X_3^* = (0.20342, 0.46933)$. Note that

$$X_1^* = 0.30899 < \frac{1}{2}(0.20342 + 0.46933) = E(X_2^*) = E(X_3^*),$$

i.e., the effort of player 1 is less than the expected effort of players 2 and 3.

In Example 3 there is an informed player and a number of uninformed players. In equilibrium, the ex-ante probability that the informed player wins the prize is above that of the uninformed players. Thus, the natural extension of Proposition 3 to contests with more than two players does not hold.

Example 3. Consider an eight player common-value Tullock contest in which $m = 2$, $p_1 = p_2 = 1/2$, $v_1 = 1$ and $v_2 = 2$. Players 1 to 7 have no information, i.e., their information partition is $\Pi_i = \{\omega_1, \omega_2\}$ for $i \in \{1, \ldots, 7\}$, and player 8 is completely informed, i.e., his information partitions is $\Pi_8 = \{\{\omega_1\}, \{\omega_2\}\}$. This contest has a (corner) equilibrium given by

$$X_1^* = \ldots = X_7^* = 0.15551, \quad X_8^* = (0, 0.38694).$$

In equilibrium, the ex-ante probability that player $i \in \{1, 2, \ldots, 7\}$ wins the prize is

$$\tilde{p}_i^* = \frac{1}{2} \left( \frac{1}{7} + \frac{0.15551}{7(0.15551) + 0.38694} \right) = 0.12413,$$

whereas the ex-ante probability that player 8 win the prize is

$$\tilde{p}_8^* = 1 - 7(0.12413) = 0.13109.$$

Thus, the informed player wins the prize more frequently than an uninformed player.
5 Common-Value Tullock and All-Pay Auction Contests

Contests that arise in many economic and political applications are effectively all-pay auctions either by design (e.g., sports or political competition) or by the nature of the problem (e.g., patent races). Here we study whether the players’ expected total effort in all pay auctions and Tullock contests can be ranked.

A common-value all-pay auction is a common-value contest in which the success function is given for \( x \in \mathbb{R}_+^n \) by

\[
AP_A(x) = \frac{1}{m(x)} \text{ if } x_i = \max\{x_j\}_{j \in N}, \quad \text{and } \quad AP_A(x) = 0 \text{ otherwise,}
\]

where \( m(x) = \{k \in N : x_k = \max\{x_j\}_{j \in N}\} \). Einy et al. (2013) show that in the unique equilibrium of a two-player common-value all-pay auction in which \( v_1 < ... < v_m \) and player 2 observes the value while player 1 does not, the players’ total expected effort is

\[
TE^{AP_A} = 2 \sum_{s=1}^{m} p_s \left( \sum_{k=1}^{s-1} p_k v_k + \frac{1}{2} p_s v_s \right) = 2 \sum_{s=1}^{m} p_s \sum_{k=1}^{s-1} p_k v_k + \sum_{s=1}^{m} p_s^2 v_s.
\]

Hence the difference between total efforts in an all-pay auction and a Tullock contest is

\[
\Delta := TE^{AP_A} - TE = 2 \sum_{s=1}^{m} p_s \sum_{k=1}^{s-1} p_k v_k + \sum_{s=1}^{m} p_s^2 v_s - 2 A_k^2.
\]

For simplicity, consider the case where there are only two states of nature, i.e., \( m = 2 \). If the equilibrium of the Tullock contest is interior, then

\[
\Delta = 2 p_1 p_2 v_1 + (p_1^2 v_1 + p_2^2 v_2) - 2 A_1^2
\]

\[
= 2 p_1 p_2 v_1 + p_1^2 v_1 + p_2^2 v_2 - 2 \left( p_1 \sqrt{v_1} + p_2 \sqrt{v_2} \right)^2 / 4
\]

\[
= 2 p_1 p_2 v_1 + \frac{1}{2} (p_1 \sqrt{v_1} - p_2 \sqrt{v_2})^2
\]

\[
> 0.
\]

Hence an all-pay auction generates more effort that a Tullock contest. However, if the Tullock contest has a corner equilibrium, then

\[
\Delta = 2 p_1 p_2 v_1 + (p_1^2 v_1 + p_2^2 v_2) - 2 A_2^2
\]

\[
= 2 p_1 p_2 v_1 + p_1^2 v_1 + p_2^2 v_2 - 2 \left( p_2 \sqrt{v_2} \right)^2 / (1 + p_2)^2
\]

\[
= p_1 v_1 (1 + p_2) - p_2^2 v_2 \left( \frac{2}{(1 + p_2)^2} - 1 \right).
\]
We show that $\Delta$ may be either positive or negative depending on the distribution of the players’ common value. Consider the environment described in Example 1. In an all-pay auction in which player 2 observes the value but player 1 does not, the equilibrium total effort is

$$TE^\text{APA} = 2(1 - p)p + (1 - p)^2 + p^2v = (1 - p)(1 + p) + p^2v.$$ 

As we have shown above, if $v < (1 + p)^2/p^2$, then the expected total effort in the unique equilibrium of the Tullock contest, which is interior, is $TE = 2A_2^2 = [1 - p(1 - \sqrt{v})]^2/2$. Hence

$$TE^\text{APA} - TE = (1 - p)(1 + p) + p^2v - \frac{(1 - p(1 - \sqrt{v}))^2}{2} > 2p > 0.$$ 

However, if $v \geq (1 + p)^2/p^2$, then the expected total effort in the unique equilibrium of the Tullock contest, which is a corner equilibrium, is $TE = 2p^2v/(1 + p)^2$. Assume that $p = 1/4$. Then

$$TE^\text{APA} - TE = \frac{15}{16} - \frac{7}{400}v.$$ 

Hence $TE^\text{APA} < TE$ for $v > 375/7$. Therefore the level of effort generated by these two contests cannot be ranked in general.

6 Appendix

Proof of Theorem 1. Let $C = (N, (\Omega, p), \{\Pi_i\}_{i \in N}, \{V_i\}_{i \in N}, \{c_i\}_{i \in N}, \rho^T)$ be a Tullock contest. Since the cost function of each player is strictly increasing and convex in the player’s effort, it follows from (1) that there exists $Q > 0$ such that $u_i(\cdot, x) < 0$ for every $i \in N$ and every $x \in \mathbb{R}^n_+$, provided $x_i \geq Q$. For any $0 < \varepsilon < Q$ consider a variant of the contest, denoted by $C_\varepsilon$, in which the effort set of each player $i$ is restricted to be the bounded interval $[\varepsilon, Q]$. In $C_\varepsilon$, the set of strategies of player $i$, $S_{i,\varepsilon}$, is identifiable with the compact set $[\varepsilon, Q]^{\Pi_i}$ via the the bijection $x_i \leftrightarrow (x_i(\pi_i))_{\pi_i \in \Pi_i}$. Player $i$’s expected payoff function $U_i$ is continuous on $S_\varepsilon = \times_{i=1}^n S_{i,\varepsilon}$ (since the success function $\rho$ in (4) is continuous if efforts are restricted to $[\varepsilon, Q]$), and it is concave in $i$’s own strategy (as the state-dependent payoff function $u_i(\cdot, x)$ is concave in the variable $x_i$ if efforts are restricted to $[\varepsilon, Q]$). Nash’s theorem thus guarantees...
existence of a Bayesian Nash equilibrium in $C_\epsilon$; pick one such equilibrium and denote it by $X_\epsilon^* = (X_1^*_{1,\epsilon}, ..., X_n^*_{n,\epsilon})$.

We show that

$$\lim \inf_{\epsilon \to 0^+} X_\epsilon^*(\cdot) > 0.$$ 

Indeed, suppose to the contrary that there is a vanishing positive sequence $\{\epsilon_k\}_{k=1}^\infty$ such that

$$\lim_{k \to \infty} \min_{\omega \in \Omega} X_{\epsilon_k}^* (\omega) = 0,$$

and fix $\omega^* \in \Omega$ such that

$$X_{\epsilon_k}^* (\omega^*) = \min_{\omega \in \Omega} X_{\epsilon_k}^* (\omega)$$

for infinitely many $k$ (and thus, w.l.o.g., for every $k$). Since the expected payoff of player $i$ is negative in every state of nature when $x_i = Q$, for any sufficiently small $\epsilon_k$ the equilibrium strategy $X_{i,\epsilon_k}^*$ satisfies $X_{i,\epsilon_k}^* (\cdot) < Q$. Thus, for a given $\pi_i \in \Pi_i$, $X_i^* (\pi_i) \in [\epsilon_k, Q)$. Additionally, $X_i^* (\pi_i)$ and $X_{i,\epsilon_k}^*$ can both be viewed as the argument of the function $U_i(X_{i,\epsilon_k}^* (\cdot) \mid \pi_i)$, since $X_i^* (\pi_i)$ is the only numerical input needed to determine the conditional expected payoff of player $i$ given $\pi_i$, when the equilibrium strategies of players other than $i$ are $X_{i,\epsilon_k}^*$. Since the equilibrium strategy $X_{i,\epsilon_k}^*$ is a (local) maximizer of $U_i(X_{i,\epsilon_k}^* (\cdot) \mid \pi_i)$ by (3),

$$\frac{dU_i(X_{i,\epsilon_k}^* (\cdot), X_i (\pi_i))}{dX_i (\pi_i)} \bigg|_{X_i (\pi_i) = X_{i,\epsilon_k}^* (\pi_i)} \leq 0.$$

That is,

$$\frac{dE[u_i(\cdot, X_{i,\epsilon_k}^* (\cdot), X_i (\pi_i)) \mid \pi_i]}{dX_i (\pi_i)} \bigg|_{X_i (\pi_i) = X_{i,\epsilon_k}^* (\pi_i)} \leq 0;$$

or, equivalently,

$$E \left[ \frac{du_i(\cdot, X_{i,\epsilon_k}^* (\cdot), X_{i,\epsilon_k}^* (\pi_i))}{dx_i} \mid \pi_i \right] \leq 0.$$

Using (4) and (1) we calculate the derivative explicitly,

$$E \left[ \frac{V_i(\cdot)}{X_{\epsilon_k}^* (\cdot)} - \frac{X_{i,\epsilon_k}^* (\pi_i) V_i(\cdot)}{X_{\epsilon_k}^* (\cdot)^2} - \frac{d}{dx_i} c_i(\cdot, X_{i,\epsilon_k}^* (\pi_i)) \mid \pi_i \right] \leq 0.$$ 

Thus

$$E \left[ \frac{V_i(\cdot)}{X_{\epsilon_k}^* (\cdot)} - \frac{d}{dx_i} c_i(\cdot, X_{i,\epsilon_k}^* (\pi_i)) \mid \pi_i \right] - X_{i,\epsilon_k}^* (\pi_i) E \left[ \frac{V_i(\cdot)}{X_{\epsilon_k}^* (\cdot)^2} \mid \pi_i \right] \leq 0,$$
which leads to
\[
X_{i,\varepsilon k}^* (\pi_i) \geq \frac{E \left[ \frac{V_i(\cdot)}{X_{\varepsilon k}^* (\cdot)} - \frac{d}{dx_i} c_i (\cdot, X_{i,\varepsilon k}^* (\pi_i)) \right]}{E \left[ \frac{V_i(\cdot)}{X_{\varepsilon k}^* (\cdot)^2} | \pi_i \right]}.
\] (9)

Inequality (9) holds, in particular, for \( \pi_i = \pi_i (\omega^*) \). Since \( X_{i,\varepsilon k}^* (\omega^*) = X_{i,\varepsilon k}^* (\pi_i (\omega^*)) \) (as, by definition, \( \omega^* \in \pi_i (\omega^*) \)), (9) yields
\[
X_{i,\varepsilon k}^* (\omega^*) \geq \frac{E \left[ \frac{V_i(\cdot)}{X_{\varepsilon k}^* (\cdot)} - \frac{d}{dx_i} c_i (\cdot, X_{i,\varepsilon k}^* (\omega^*)) \right]}{E \left[ \frac{V_i(\cdot)}{X_{\varepsilon k}^* (\cdot)^2} | \pi_i (\omega^*) \right]}.
\] (10)

Summing over \( i \in N \) we obtain
\[
\bar{X}_{\varepsilon k}^* (\omega^*) \geq \sum_{i=1}^{n} \frac{E \left[ \frac{V_i(\cdot)}{X_{\varepsilon k}^* (\cdot)} - \frac{d}{dx_i} c_i (\cdot, X_{i,\varepsilon k}^* (\omega^*)) \right]}{E \left[ \frac{V_i(\cdot)}{X_{\varepsilon k}^* (\cdot)^2} | \pi_i (\omega^*) \right]},
\]
or (since \( \bar{X}_{\varepsilon k}^* (\omega^*) \geq n \varepsilon > 0 \)
\[
1 \geq \sum_{i=1}^{n} \frac{E \left[ \frac{\bar{X}_{\varepsilon k}^* (\omega^*)}{X_{\varepsilon k}^* (\cdot)} V_i(\cdot) - \bar{X}_{\varepsilon k}^* (\omega^*) \frac{d}{dx_i} c_i (\cdot, X_{i,\varepsilon k}^* (\omega^*)) \right]}{E \left[ \frac{\bar{X}_{\varepsilon k}^* (\omega^*)^2}{X_{\varepsilon k}^* (\cdot)^2} V_i(\cdot) | \pi_i (\omega^*) \right]}.
\] (11)

By the definition of \( \bar{X}_{\varepsilon k}^* (\omega^*) \) (see (8)),
\[
0 \leq \frac{\bar{X}_{\varepsilon k}^* (\omega^*)}{X_{\varepsilon k}^* (\omega^*)} \leq 1
\]
for every \( \omega \in \Omega \). Hence we assume w.l.o.g. (by moving to a subsequence if necessary) that the limit
\[
a (\omega) = \lim_{k \to \infty} \frac{\bar{X}_{\varepsilon k}^* (\omega^*)}{X_{\varepsilon k}^* (\omega^*)}
\]
exists for every \( \omega \in \Omega \). Note also that \( a (\omega) = 1 \) for \( \omega = \omega^* \), which occurs with positive probability by our assumption on \( p \), and thus
\[
\lim_{k \to \infty} E \left[ \frac{\bar{X}_{\varepsilon k}^* (\omega^*)^2}{X_{\varepsilon k}^* (\cdot)^2} V_i(\cdot) | \pi_i (\omega^*) \right] = E \left[ a (\cdot)^2 V_i(\cdot) | \pi_i (\omega^*) \right] > 0.
\] (12)
Also, (7) and (8) imply
\[
\lim_{k \to \infty} E \left[ X_{\varepsilon k}^* (\omega^*) \frac{dc_i (\cdot, X_{i,\varepsilon k}^* (\omega^*))}{dx_i} \mid \pi_i (\omega^*) \right] = 0. \tag{13}
\]

Taking limit of the right-hand side of (11), which exists by (12) and (13), we get
\[
1 \geq \sum_{i=1}^{n} \frac{E [a (\cdot) V_i (\cdot) \mid \pi_i (\omega^*)]}{E [a (\cdot)^2 V_i (\cdot) \mid \pi_i (\omega^*)]}. \tag{16}
\]

Furthermore, as \(0 \leq a (\cdot)^2 \leq a (\cdot) \leq 1\), we obtain
\[
1 \geq \sum_{i=1}^{n} \frac{E [a (\cdot) V_i (\cdot) \mid \pi_i (\omega^*)]}{E [a (\cdot)^2 V_i (\cdot) \mid \pi_i (\omega^*)]} \geq n. \tag{16}
\]

Since by assumption \(n \geq 2\), we have reached a contradiction. This proves that, indeed,
\[
\lim \inf_{\varepsilon \to 0^+} \bar{X}_\varepsilon^* (\cdot) > 0. \tag{14}
\]

Now let \(\{\varepsilon_k\}_{k=1}^{\infty}\) be a vanishing positive sequence such that the limit

\[
X_i^* (\omega) \equiv \lim_{k \to \infty} X_{i,\varepsilon_k}^* (\omega)
\]

exists for every \(i \in N\) and \(\omega \in \Omega\). (Such a sequence exists since all \(X_{i,\varepsilon}^* (\omega)\) belong to the compact interval \([0, Q]\).) Obviously, \(X^* = (X_1^*, ..., X_n^*)\) constitutes a strategy profile in the contest \(C\), and it follows from (14) that
\[
\bar{X}^* (\cdot) > 0. \tag{15}
\]

We show that \(X^*\) is a Bayesian Nash equilibrium of \(C\).

Since the state-dependent payoff function \(u_i (\cdot, x)\) is continuous at any point \(x\) with \(x > 0\), for every \(i \in N\), every \(\pi_i \in \Pi_i\), and every sequence \(\{Y_{k}\}_{k=0}^{\infty}\) of strategy profiles such that \(Y_{0} (\cdot) > 0\) and \(Y_{i,0} (\omega) = \lim_{k \to \infty} Y_{i,k} (\omega)\) for every \(i\) and \(\omega\), we have
\[
\lim_{k \to \infty} U_i (Y_{1,k}, ..., Y_{n,k} \mid \pi_i) = U_i (Y_{1,0}, ..., Y_{n,0} \mid \pi_i). \tag{16}
\]

Since every \(X_{\varepsilon}^*\) is a Bayesian Nash equilibrium in \(C_{\varepsilon}\), for every sufficiently large \(k\) and every strategy \(X_i\) of player \(i\) satisfying \(0 < X_i (\cdot) \leq Q\) we have
\[
U_i (X_{\varepsilon_k}^* \mid \pi_i) \geq U_i (X_{i,\varepsilon_k}^*, X_i \mid \pi_i). \tag{17}
\]
Applying the limit as $k \to \infty$ to both sides of inequality (17), it follows from (16) (and the fact (15)) that

$$U_i(X^* \mid \pi_i) \geq U_i(X^*_{-i}, X_i \mid \pi_i)$$ (18)

for every strategy $X_i$ of player $i$ satisfying $0 < X_i(\cdot) \leq Q$ and every $\pi_i \in \Pi_i$.

It is easy to see that

$$\lim_{x_i \to 0^+} \inf U_i(X^*_i, x_i \mid \pi_i) \geq U_i(X^*_i, 0 \mid \pi_i),$$

where $x_i > 0$ (respectively, $x_i = 0$) is identified with a strategy of $i$ for which $X_i(\pi_i) = x_i$ (respectively, $X_i(\pi_i) = 0$). Thus (18) in fact holds for every strategy $X_i$ satisfying $0 \leq X_i(\cdot) \leq Q$ (i.e., the deviations of $i$ may be zero at some states of nature).

Finally, note that player $i$ can improve upon any strategy $X_i$ for which $X_i(\cdot) > Q$ at some $\omega$ by lowering the effort on $\pi_i(\omega)$ to zero and thus receiving non-negative expected payoff conditional on $\pi_i(\omega)$. Thus, in contemplating a unilateral deviation from $X^*_i$, player $i$ is never worse off by limiting himself to strategies $X_i$ satisfying $0 \leq X_i(\cdot) \leq Q$. But this implies that (18) holds for every strategy $X_i \in S_i$. Since this is the case for every $i \in N$, we have shown that $X^*$ is a Bayesian Nash equilibrium of $C$.

**Proof of Lemma 1.** Assume that $\sqrt{v_k} > A_k$ for some $\bar{k} < m$.

We show that $\sqrt{v_k} > A_k$ for all $k > \bar{k}$. Suppose not; let $\bar{k} > \bar{k}$ be the first index $k > \bar{k}$ such that for $\sqrt{v_k} \leq A_k$. Note that $v_k \geq v_{k-1}$ and $\sqrt{v_{k-1}} > A_{k-1}$ imply

$$\left(1 + \sum_{s=\bar{k}}^{m} p_s \right) \sqrt{v_k} \geq \left(1 + \sum_{s=\bar{k}}^{m} p_s \right) \sqrt{v_{k-1}} - p_{k-1} \sqrt{v_{k-1}}$$

$$> \left(1 + \sum_{s=\bar{k}}^{m} p_s \right) A_{k-1} - p_{k-1} \sqrt{v_{k-1}}$$

$$= \sum_{s=\bar{k}}^{m} p_s \sqrt{v_s} - p_{k-1} \sqrt{v_{k-1}}$$

$$= \left(1 + \sum_{s=\bar{k}}^{m} p_s \right) A_k,$$

which contradicts the assumption that $\sqrt{v_k} \leq A_k$. 

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Now we show that $A_k > A_{\tilde{k}}$ for all $k > \tilde{k}$. Suppose not; let $\tilde{k} > \tilde{k}$ be the first index $k > \tilde{k}$ such that $A_k \leq A_{\tilde{k}}$. Since $\sqrt{v_{k-1}} > A_{\tilde{k}-1}$ (as we have just shown),

$$
\left(1 + \sum_{s=k-1}^{m} p_s\right) A_{k-1} - \sum_{s=k-1}^{m} p_s \sqrt{v_s} = p_{k-1} \sqrt{v_{k-1}} + \sum_{s=k}^{m} p_s \sqrt{v_s} > p_{k-1} A_{k-1} + \left(1 + \sum_{s=k}^{m} p_s\right) A_{\tilde{k}}.
$$

Hence

$$
\left(1 + \sum_{s=k-1}^{m} p_s\right) A_{k-1} - p_{k-1} A_{k-1} > \left(1 + \sum_{s=k}^{m} p_s\right) A_{\tilde{k}},
$$

i.e.,

$$
\left(1 + \sum_{s=k}^{m} p_s\right) A_{\tilde{k}} > \left(1 + \sum_{s=k}^{m} p_s\right) A_{\tilde{k}}.
$$

Thus, $A_k > A_{k-1} > A_{\tilde{k}}$, which contradicts the choice of $\tilde{k}$. \(\blacksquare\)

**Proof of Proposition 1.** Let $(X, Y)$, where $X = x$ and $Y = (y_1, \ldots, y_m)$, be a Bayesian Nash equilibrium, whose existence is guaranteed by Theorem 1. We show that $x > 0$. If $x = 0$, then $\rho_{2}^{Y}(0) = 1$, since otherwise player 2 does not have a best response against $x = 0$. But then $y_1 = y_2 = \ldots = y_m = 0$, and therefore player 1 can profitably deviate by exerting an arbitrarily small effort $\varepsilon > 0$. Hence $x > 0$. Moreover, $y_k > 0$ for some $k \in \{1, \ldots, m\}$ since otherwise $x > 0$ is not a best response of player 1.

Since $x > 0$ maximizes player 1’s payoff given $Y$,

$$
\frac{\partial}{\partial x} \left(\sum_{s=1}^{m} p_s \left(v_s \frac{x}{x + y_s} - x\right)\right) = \sum_{s=1}^{m} p_s v_s \frac{y_s}{(x + y_s)^2} - 1 = 0.
$$

(19)

And since $y_s$ maximizes player 2’s payoff in state $\omega_s$ given $x$,

$$
\frac{\partial}{\partial y_s} \left(v_s \frac{y_s}{x + y_s} - y_s\right) = v_s \frac{x}{(x + y_s)^2} - 1 \leq 0,
$$

(20)

(with equality if $y_s > 0$) for each $s = 1, \ldots, m$. 21
Notice next that if \( y_k > 0 \) for some \( k < m \), then \( y_{k'} > 0 \) for all \( k' > k \). Since \( x > 0 \), if \( y_k > 0 \) then \( y_k = \sqrt{x} \left( \sqrt{v_k} - \sqrt{x} \right) \) by (20), and since \( v_{k'} \geq v_k \) for all \( k' > k \), \( \sqrt{x} \left( \sqrt{v_{k'}} - \sqrt{x} \right) > 0 \), i.e.,

\[
v_{k'} \frac{x}{x^2} - 1 > 0,
\]

for all \( k' > k \). Then \( y_{k'} = 0 \) would violate inequality (20) for \( s = k' \). Hence \( y_{k'} > 0 \).

Let \( k^o \) be the smallest index such that \( y_k > 0 \). Thus, (19) implies

\[
\sum_{s=1}^{m} p_s v_s \frac{y_s}{(x + y_s)^2} = \sum_{s=k^o}^{m} p_s v_s \frac{y_s}{(x + y_s)^2} = 1,
\]

and (20) implies \( y_{k^o} = \sqrt{x} \left( \sqrt{v_{k^o}} - \sqrt{x} \right) > 0 \) for all \( k' \geq k^o \). Hence \( x = A^2_{k^o} \), \( y_k = A^2_{k^o} \left( \sqrt{v_k} - A_{k^o} \right) \) for all \( k \geq k^o \), and \( y_k = 0 \) for all \( k < k^o \).

We now show that \( k^o = k^* \), which establishes that the profile \((x^*, y_1^*, \ldots, y_m^*)\) identified in Proposition 1 is the unique equilibrium. Assume first that \( k^o < k^* \). Then \( \sqrt{v_{k^o}} \leq A_{k^o} \) since \( k^* \) is the smallest index such that \( \sqrt{v_k} > A_k \), and hence \( y_{k^o} = \sqrt{x} \left( \sqrt{v_{k^o}} - \sqrt{x} \right) = A^2_{k^o} \left( \sqrt{v_{k^o}} - A_{k^o} \right) \leq 0 \), a contradiction as \( y_{k^o} > 0 \) by the definition of \( k^o \). Assume next that \( k^o > k^* \). In this case, \( y_k^* = 0 \). Since \( \sqrt{v_{k^*}} > A_{k^*} \), by Lemma 1

\[
A^2_{k^*} > A^2_{k^o} = x,
\]

and therefore

\[
v_{k^*} \frac{x}{x^2} - 1 = \frac{A^2_{k^o}}{A^2_{k^o}} \left( v_{k^*} - A^2_{k^o} \right) > 0.
\]

This stands in contradiction to (20), as \( y_{k^*} = 0 \) by the definition of \( k^o (\geq k^*) \). We conclude that indeed \( k^o = k^* \). ■

**Proof of Proposition 3.** Let us be given a two-player common-value Tullock contest in which player 2 has an information advantage over player 1. Given \((y_{k^*}, \ldots, y_m) \in \mathbb{R}^{m-k*+1}_+\) define the function

\[
\tilde{p}_2 (y_{k^*}, \ldots, y_m) := \sum_{k=k^*}^{m} \frac{p_k y_k}{y_k + \sum_{s=k^*}^{m} p_s y_s}.
\]

Hence, recalling (6), \( \tilde{p}_2 = \tilde{p}_2 (y_{k^*}, \ldots, y_m) \). We show that a maximum point \( \overline{y} \) of \( \tilde{p}_2 \) on \( K = \{(y_{k^*}, \ldots, y_m) \in \mathbb{R}^{m-k^*+1}_+ \mid y_{k^*} \leq y_{k^*+1} \leq \ldots \leq y_m\} \) must satisfy \( \overline{y}_{k^*} = \ldots = \overline{y}_m \). Hence

\[
\max_{K} \tilde{p}_2 = \frac{\sum_{s=k^*}^{m} p_s}{1 + \sum_{s=k^*}^{m} p_s} \leq \frac{1}{2}.
\]
Since $y_k^* < \ldots < y_m^*$ (the inequalities are strict, which follows from our assumption that $v_1 < v_2 < \ldots < v_m$ and the expressions for $(y_k^*)_{k=k^*}^m$ given in Proposition 1), (22) implies

$$\tilde{p}_2 = \tilde{p}_2(y_k^*, \ldots, y_m^*) < \max_{K} \tilde{p}_2 \leq 1/2,$$

which establishes Proposition 3.

Differentiating $\tilde{p}_2$ with respect to $y_k$ for $k \in \{k^*, \ldots, m\}$ we get

$$\frac{\partial \tilde{p}_2}{\partial y_k} = p_k \left( \sum_{t=k^*, t \neq k}^{m} \frac{p_t y_t}{(y_k + \sum_{s=k^*}^{m} p_s y_s)^2} - \sum_{t=k^*, t \neq k}^{m} \frac{p_t y_t}{(y_l + \sum_{s=k^*}^{m} p_s y_s)^2} \right). \quad (23)$$

For every $(y_k^*, \ldots, y_m) \in K$ such that $y_k^* < y_{k^*+1} \leq \ldots \leq y_m$, $\partial \tilde{p}_2/\partial y_k^*(y) > 0$, and therefore necessarily $\bar{y}_k^* = \bar{y}_{k^*+1}$. Suppose now that it has already been shown that $\bar{y}_k^* = \bar{y}_{k^*+1} = \ldots = \bar{y}_k$, $m - 1 \geq k > 1$. We show that $\bar{y}_{k+1} = \bar{y}_k$ as well. Indeed, if $\bar{y}_k^* = \bar{y}_{k^*+1} = \ldots = \bar{y}_k < \bar{y}_{k+1} \leq \ldots \leq \bar{y}_m$, then by (23) we obtain that $\partial \tilde{p}_2/\partial y_k(\bar{y}) > 0$, a contradiction. Thus $\bar{y}_k^* = \ldots = \bar{y}_m$. \(\blacksquare\)
References


